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# ON A GENERALIZATION OF LAPLACE INTEGRALS

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## Introduction

Let $\boldsymbol{R}^n$ be the Euclidean space of dimension $n \geqq 1$ with the standard inner product $\langle x, y\rangle = \sum x_i y_i$ and the norm $|x| = \langle x, x\rangle^{1/2}$, $S^{n-1}$ be the unit sphere $\{x \in \boldsymbol{R}^n; |x| = 1\}$ and $d\omega_{n-1}$ be the volume element of $S^{n-1}$ such that $S^{n-1}$ gets the volume 1. Let $\Omega$ be an open set of $\boldsymbol{R}^n$ containing $S^{n-1}$ and let $f\colon \Omega \to \boldsymbol{R}^m$ be a smooth map. With each integer $\nu \geqq 0$, we shall associate a form $f_\nu$ of degree $\nu$ on $\boldsymbol{R}^m$ defined by

$$(0.1)\qquad f_\nu(\xi) = \int_{S^{n-1}} \langle \xi, f(x)\rangle^\nu d\omega_{n-1}\,, \qquad \xi \in \boldsymbol{R}^m\,.$$

We then consider the number $\sigma_\nu(f)$ which is the mean value of the form $f_\nu$ on the sphere $S^{m-1}$:

$$(0.2)\qquad \sigma_\nu(f) = \int_{S^{m-1}} f_\nu(\xi) d\omega_{m-1}\,, \qquad \nu \in \boldsymbol{Z}_+\,.$$

When $f$ is an affine map: $\boldsymbol{R}^n \to \boldsymbol{R}^m$, the function $f_\nu$ is substantially the Legendre polynomial of order $\nu$ and (0.1) is the Laplace integral for it[1]. Therefore it is natural to ask questions about forms $f_\nu$ associated with more general map $f$.

In this paper, we shall focus our attention on the determination of the number $\sigma_\nu(f)$ for any smooth map $f$. It will turn out that the main ingredient of the number $\sigma_\nu(f)$ is the number:

$$(0.3)\qquad N_k(f) = \int_{S^{n-1}} |f(x)|^{2k}\, d\omega_{n-1}\,, \qquad \nu = 2k^{2)}\,.$$

Since $N_k(f) = 1$ whenever $f$ maps $S^{n-1}$ into $S^{m-1}$, we see that all these "spherical" maps share the same numbers $\sigma_\nu(f)$ for all $\nu \in \boldsymbol{Z}_+$; hence these numbers measure a deviation of $f$ from being spherical. We shall consider

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1) See Appendix for a detailed discussion on this matter.

2) As is easily seen, $\sigma_\nu(f) = 0$ if $\nu$ is odd.

examples of a family of maps $\{f_\rho\}_{\rho \in \boldsymbol{R}}$ for which $\{f_{\pm 1}\}$ are Hopf maps and show that the number $N_k(f_\rho)$ can be written as a hypergeometric polynomial.

The author would like to mention here that the idea of associating the number like $\sigma_\nu(f)$ with a map $f$ came from his earlier work [7] on functions over finite fields.

Notation and conventions

The symbols $\boldsymbol{Z}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{C}$ denote the set of integers, rational numbers, real numbers and complex numbers. The set of non-negative real numbers is denoted by $\boldsymbol{R}_+$. We put $\boldsymbol{Z}_+ = \boldsymbol{Z} \cap \boldsymbol{R}_+$, $\boldsymbol{Q}_+ = \boldsymbol{Q} \cap \boldsymbol{R}_+$. The set of all multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$ is $\boldsymbol{Z}_+^n$. We denote by $1_n$ the multi-index $(1, \cdots, 1) \in \boldsymbol{Z}_+^n$. For $\alpha, \beta \in \boldsymbol{Z}_+^n$ and $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $m\alpha = (m\alpha_1, \cdots, m\alpha_n)$, $m \in \boldsymbol{Z}_+$, $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha \leqq \beta \Leftrightarrow \alpha_i \leqq \beta_i$, $1 \leqq i \leqq n$. For an integer $m$, $\alpha \equiv 0\ (m) \Leftrightarrow \alpha = m\beta$ for some $\beta$. When $\beta \leqq \alpha$ we put

$$\binom{\alpha}{\beta} \overset{\text{def}}{=} \frac{\alpha!}{\beta!\,(\alpha-\beta)!} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

For $a \in \boldsymbol{C}$, $n \in \boldsymbol{Z}_+$ we use Appell's notation $(a, n) = a(a+1) \cdots (a+n-1)$ for $n \geqq 1$ and $(a, 0) = 1$. For $a, b, c \in \boldsymbol{C}$, the hypergeometric series is defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}.$$

For a smooth function $f$ on an open set of $\boldsymbol{R}^n$, we put

$$\operatorname{grad} f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\right), \qquad \Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}.$$

We shall use the following formulas freely:

(0.4) $$(x_1 + \cdots + x_n)^\nu = \sum_{|\alpha|=\nu} \frac{\nu!}{\alpha!} x^\alpha, \quad \nu \in \boldsymbol{Z}_+, \quad x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n,$$

(0.5) when $|\alpha| = 2\nu$, we have $\dfrac{\Delta^\nu x^\alpha}{\nu!} = \dfrac{(2\beta)!}{\beta!}$ if $\alpha = 2\beta$, $=0$ if $\alpha \not\equiv 0\ (2)$.

## §1. Numbers $b_\nu(\ell;\lambda)$

Let $\nu \geqq 0$, $\ell \geqq 1$ be integers and let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \boldsymbol{R}^n$. We assume that $\ell\nu$ is even: $\ell\nu = 2k$, $k \in \boldsymbol{Z}_+$. We define a number $b_\nu(\ell;\lambda) \in \boldsymbol{Q}_+$ by

$$b_\nu(\ell;\lambda) = \frac{\nu!}{k!} \sum_{\substack{|\alpha|=k \\ 2\alpha\equiv 0(\ell)}} \frac{(2\alpha)!}{\alpha!\,(2\alpha/\ell)!} \lambda^{2\alpha/\ell} . \tag{1.1}$$

If, in particular, $\ell = 2$, then $k = \nu$ and

$$b_\nu(2;\lambda) = \sum_{|\alpha|=\nu} \binom{2\alpha}{\alpha} \lambda^\alpha . \tag{1.2}$$

If, $\lambda = 1_n$ in (1.2), we have

$$b_\nu(2;1_n) = \sum_{|\alpha|=\nu} \binom{2\alpha}{\alpha} . \tag{1.3}$$

Using the equality

$$4^k(\tfrac{1}{2}, k) = k!\binom{2k}{k}$$

which one verifies easily, we get the following equality as formal power series in $t$

$$\sum_{k=0}^{\infty} \binom{2k}{k} t^k = (1-4t)^{-1/2} \tag{1.4}$$

and, by (1.2), (1.4), we get

$$\sum_{\nu=0}^{\infty} b_\nu(2;\lambda)t^\nu = \prod_{i=1}^{n} (1-4\lambda_i t)^{-1/2} . \tag{1.5}$$

In particular, we have

$$\sum_{\nu=0}^{\infty} b_\nu(2;1_n)t^\nu = (1-4t)^{-n/2} \tag{1.6}$$

and hence, by (1.3), (1.6), we get

$$4^\nu\left(\frac{n}{2}, \nu\right) = \nu!\, b_\nu(2;1_n) . \tag{1.7}$$

## §2. Review of a mean value theorem in potential theory

Let $\varphi(x)$ be a complex valued smooth function defined on an open set $\Omega$ in $\boldsymbol{R}^n$ containing $S^{n-1}$. Assume that either (i) $\Delta^m\varphi = 0$ for some

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$m \geqq 1$ or (ii) $\Delta\varphi = \lambda\varphi$ for a constant $\lambda \in C$. In this situation, a mean value theorem in potential theory[3] tells us that

$$(2.1)\qquad \int_{S^{n-1}} \varphi(x) d\omega_{n-1} = \sum_{\nu=0}^{\infty} \frac{(\Delta^{\nu}\varphi)(0)}{4^{\nu}\nu!\,(n/2, \nu)} .$$

Needless to say, when $\varphi$ is harmonic, then $m = 1$ in (i) or $\lambda = 0$ in (ii) and (2.1) is the mean value theorem of Gauss:

$$\int_{S^{n-1}} \varphi(x) d\omega_{n-1} = \varphi(0) .$$

By (1.7), (2.2) can also be written as:

$$(2.2)\qquad \int_{S^{n-1}} \varphi(x) d\omega_{n-1} = \sum_{\nu=0}^{\infty} \frac{(\Delta^{\nu}\varphi)(0)}{(\nu!)^2 b_{\nu}(2; 1_n)} .$$

If $\varphi$ is a form of even degree $\ell = 2k$, then $\Delta^m\varphi = 0$ for $m > k$ and since $\Delta^m\varphi(0) = 0$ for $m < k$, we get from (2.2)

$$(2.3)\qquad \int_{S^{n-1}} \varphi(x) d\omega_{n-1} = \frac{\Delta^k\varphi}{(k!)^2 b_k(2; 1_n)} .$$

This shows also that

$$\int_{S^{n-1}} \varphi(x) d\omega_{n-1} = 0$$

if the degree of $\varphi$ is odd, a fact which can be proved directly. On the other hand, in case (ii), we have

$$(2.4)\qquad \begin{aligned} \int_{S^{n-1}} \varphi(x) d\omega_{n-1} &= \varphi(0) \sum_{\nu=0}^{\infty} \frac{\lambda^{\nu}}{(\nu!)^2 b_{\nu}(2; 1_n)} \\ &= \varphi(0) \frac{\Gamma(n/2)}{(\sqrt{-\lambda}/2)^{(n-2)/2}} J_{(n-2)/2}(\sqrt{-\lambda}) \end{aligned}$$

where $J_{\nu}(z)$ is the $\nu$-th Bessel function.

## § 3. Mean value of quadratic forms

Let

$$(3.1)\qquad \varphi(x) = \sum_{|\beta|=2k} c_{\beta} x^{\beta}$$

be a form of even degree $2k$. By (0.5), (2.3), we have

3) See Courant-Hilbert [3], pp. 258–261.

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(3.2) $$\int_{S^{n-1}} \varphi(x) d\omega_{n-1} = \frac{(1/k!) \sum_{|\alpha|=k} c_{2\alpha}((2\alpha)!/\alpha!)}{b_k(2; 1_n)}$$

Consider now a diagonal form $f(x) = \lambda_1 x_1^\ell + \cdots + \lambda_n x_n^\ell$ and put $\varphi(x) = f(x)^\nu$ with $\ell\nu = 2k$. Since we have

$$\varphi(x) = \sum_{|\sigma|=\nu} \frac{\nu!}{\sigma!} \lambda^\sigma x^{\ell\sigma} = \sum_{|\beta|=2k} c_\beta x^\beta$$

where

(3.3) $$c_\beta = \frac{\nu!}{\sigma!}\lambda^\sigma \qquad \text{if } \beta = \ell\sigma, \ =0 \text{ if } \beta \not\equiv 0 \ (\ell)\,,$$

by (3.2), (3.3), we have

(3.4) $$\int_{S^{n-1}} f(x)^\nu d\omega_{n-1} = \frac{(1/k!) \sum_{\substack{|\alpha|=k \\ 2\alpha=\ell\sigma}} (\nu!/\sigma!)\lambda^\sigma((2\alpha)!/\alpha!)}{b_k(2; 1_n)}$$

From (1.1), (3.4), it follows that

(3.5) $$\int_{S^{n-1}} (\lambda_1 x_1^\ell + \cdots + \lambda_n x_n^\ell)^\nu d\omega_{n-1} = \frac{b_\nu(\ell; \lambda)}{b_k(2; 1_n)}\,, \qquad \ell\nu = 2k\,.$$

If, in particular, $\ell = 2$, then $\nu = k$ and we have

(3.6) $$\int_{S^{n-1}} (\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2)^\nu d\omega_{n-1} = \frac{b_\nu(2; \lambda)}{b_\nu(2; 1_n)}$$

Consider a quadratic form $q(x)$ on $\boldsymbol{R}^n$ and the integral

(3.7) $$\int_{S^{n-1}} q(x)^\nu d\omega_{n-1}\,.$$

Since the change of variable $x \mapsto sx$, $s \in 0(\boldsymbol{R}^n)$, the orthogonal group, does not change the integral (3.7) and $q(x)$ can be brought to a diagonal form $\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$ by such a change of variable, (3.6) implies that

(3.8) $$\int_{S^{n-1}} q(x)^\nu d\omega_{n-1} = \frac{b_\nu(2; \lambda)}{b_\nu(2; 1_n)}\,, \qquad \nu \in \boldsymbol{Z}_+\,,$$

where $\lambda = (\lambda_1, \cdots, \lambda_n)$ denotes arbitrarily ordered eigenvalues of $q(x)$[4]. From (3.8), we get the following equality of formal power series

4) Note that $b_\nu(2; \lambda)$ is a symmetric function of $\lambda_i$'s.

$$\int_{S^{n-1}} \sum_{\nu=0}^{\infty} b_\nu(2; 1_n) q(x)^\nu t^\nu d\omega_{n-1} = \sum_{\nu=0}^{\infty} b_\nu(2; \lambda) t^\nu . \tag{3.9}$$

Replacing $t$ by $t/4$ in (3.9) and using (1.5), (1.6), we obtain an interesting equality

$$\int_{S^{n-1}} (1 - tq(x))^{-n/2} d\omega_{n-1} = \prod_{i=1}^{n} (1 - \lambda_i t)^{-1/2} \tag{3.10}$$

which makes sense if $|t|$ is sufficiently small.

## §4. $f_\nu(\xi)$ and $\sigma_\nu(f)$

As in Introduction, let $\Omega$ be an open set of $\boldsymbol{R}^n$ containing $S^{n-1}$ and let $f: \Omega \to \boldsymbol{R}^m$ be a smooth map. With each $\nu \geqq 0$, we associate a form $f_\nu(\xi)$ on $\boldsymbol{R}^m$ of degree $\nu$ by

$$f_\nu(\xi) = \int_{S^{n-1}} \langle \xi, f(x) \rangle^\nu d\omega_{n-1} . \tag{4.1}$$

We shall denote by $\sigma_\nu(f)$ the mean value of $f_\nu(\xi)$:

$$\sigma_\nu(f) = \int_{S^{m-1}} f_\nu(\xi) d\omega_{m-1} . \tag{4.2}$$

To study the numbers $\sigma_\nu(f)$ simultaneously for all $\nu \in \boldsymbol{Z}_+$, we introduce the generating function

$$\sigma(f; t) = \sum_{\nu=0}^{\infty} \sigma_\nu(f) \frac{t^\nu}{\nu!} . \tag{4.3}$$

As is easily seen, the series (4.3) converges for any $t \in \boldsymbol{C}$ and we have

$$\begin{aligned} \sigma(f; t) &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \int_{S^{m-1}} d\omega_{m-1} \int_{S^{n-1}} \langle \xi, f(x) \rangle^\nu d\omega_{n-1} \\ &= \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \int_{S^{n-1}} d\omega_{n-1} \int_{S^{m-1}} \langle \xi, f(x) \rangle^\nu d\omega_{m-1} \\ &= \int_{S^{n-1}} d\omega_{n-1} \int_{S^{m-1}} \sum_{\nu=0}^{\infty} \frac{1}{\nu!} (t\langle \xi, f(x) \rangle)^\nu d\omega_{m-1} \\ &= \int_{S^{n-1}} d\omega_{n-1} \int_{S^{m-1}} \exp (t\langle \xi, f(x) \rangle) d\omega_{m-1} . \end{aligned} \tag{4.4}$$

We are thus reduced to compute the integral

$$\int_{S^{m-1}} \varphi(\xi) d\omega_{m-1} \quad \text{with} \quad \varphi(\xi) = \exp (t\langle \xi, f(x) \rangle) . \tag{4.5}$$

A simple computation shows that

$$\Delta\varphi = \sum_{i=1}^{m} \frac{\partial^2\varphi}{\partial\xi_i^2} = t^2|f(x)|^2\varphi . \tag{4.6}$$

Since $\varphi(0) = 1$ and $\lambda = t^2|f(x)|^2$ is a constant for the variable $\xi$, we have, by (2.4),

$$\int_{S^{m-1}} \varphi(\xi)d\omega_{m-1} = \sum_{k=0}^{\infty} \frac{|f(x)|^{2k}}{(k!)^2 b_k(2; 1_m)} t^{2k}$$

and so

$$\sigma(f; t) = \sum_{k=0}^{\infty} \frac{N_k(f)}{(k!)^2 b_k(2; 1_m)} t^{2k} \tag{4.7}$$

with

$$N_k(f) = \int_{S^{n-1}} |f(x)|^{2k} d\omega_{n-1} . \tag{4.8}$$

Finally, by (4.3), (4.7), we have

$$\sigma_{2k}(f) = \frac{\binom{2k}{k}}{b_k(2; 1_m)} N_k(f) , \qquad k \in \boldsymbol{Z}_+ . \tag{4.9}$$

We can also write (4.9) as

$$\sigma_{2k}(f) = \frac{b_k(2; 1)}{b_k(2; 1_m)} N_k(f) , \qquad k \in \boldsymbol{Z}_+ . \tag{4.10}$$

We shall call a map $f: \Omega \to \boldsymbol{R}^m$ spherical if $f(S^{n-1}) \subset S^{m-1}$. Since $|f(x)| = 1$, $x \in S^{n-1}$, for a spherical map $f$, we have $N_k(f) = 1$ and so

$$\sigma_{2k}(f) = \frac{b_k(2; 1)}{b_k(2; 1_m)} \tag{4.11}$$

when $f$ is spherical.

## § 5. Examples

Example 1. Let $f_\rho$, $\rho \in \boldsymbol{R}$, be the map $\boldsymbol{R}^2 \to \boldsymbol{R}^2$ defined by $f_\rho(x) = (x_1^2 - x_2^2, 2\rho x_1 x_2)$. When $\rho = \pm 1$, $f_\rho$ sends $S^1$ onto $S^1$; when $\rho = 1$, $f_\rho$ is the map $z \to z^2$ of $\boldsymbol{C} = \boldsymbol{R}^2$ onto itself and when $\rho = -1$, $f_\rho$ is the map $z \to \bar{z}^2$. Now,

$$(f_\rho)_\nu(\xi) = \int_{S^1} q(x)^\nu d\omega_1$$

with $q(x) = \langle \xi, f_\rho(x) \rangle = {}^t x A_\rho x$ where

$$A_\rho = \begin{pmatrix} \xi_1 & \rho\xi_2 \\ \rho\xi_2 & -\xi_1 \end{pmatrix}$$

whose eigenvalues are $\lambda_1 = \sqrt{\xi_1^2 + \rho^2\xi_2^2}$, $\lambda_2 = -\sqrt{\xi_1^2 + \rho^2\xi_2^2}$. Therefore, by (1.7), (3.8), we have

$$\int_{S^1} q(x)^\nu d\omega_1 = \frac{b_\nu(2;\lambda)}{b_\nu(2;1_2)} = \frac{b_\nu(2;\lambda)}{4^\nu}.$$

Since we have

$$\prod_{i=1}^{2} (1 - 4\lambda_i t)^{-1/2} = (1 - 4\lambda_1 t)^{-1/2}(1 + 4\lambda_1 t)^{-1/2} = (1 - 16\lambda_1^2 t^2)^{-1/2}$$
$$= \sum_{k=0}^{\infty} (\tfrac{1}{2}, k) \frac{16^k \lambda_1^{2k}}{k!} t^{2k},$$

we get, by (1.5),

$$b_{2k}(2;\lambda) = \frac{(\frac{1}{2}, k)16^k}{k!}(\xi_1^2 + \rho^2\xi_2^2)^k.$$

Or we have

$$(f_\rho)_{2k}(\xi) = \int_{S^1} q(x)^{2k} d\omega_1 = \frac{(\frac{1}{2}, k)}{k!}(\xi_1^2 + \rho^2\xi_2^2)^k$$

and

$$\sigma_{2k}(f_\rho) = \frac{(\frac{1}{2}, k)}{k!} \int_{S^1} (\xi_1^2 + \rho^2\xi_2^2)^k d\omega_1.$$

Since $\xi_1^2 + \xi_2^2 = 1$ on $S^1$, we have

$$\sigma_{2k}(f_\rho) = \frac{(\frac{1}{2}, k)}{k!} \int_{S^1} (1 + (\rho^2 - 1)\xi_2^2)^k d\omega_1$$
$$= \frac{(\frac{1}{2}, k)}{k!} \sum_{m=0}^{k} \binom{k}{m} (\rho^2 - 1)^m \int_{S^1} \xi_2^{2m} d\omega_1.$$

As for the last integral, since the eigenvalues of the quadratic form $\xi_2^2$ are $\lambda = (0, 1)$, we have, by (1.7), (3.8),

$$\int_{S^1} \xi_2^{2m} d\omega_1 = \frac{b_m(2;(0,1))}{b_m(2;1_2)} = \frac{b_m(2;(0,1))}{4^m}.$$

Since we have

$$\prod_{i=1}^{2}(1-4\lambda_i t)^{-1/2}=(1-4t)^{-1/2}=\sum_{m=0}^{\infty}(\tfrac{1}{2},m)\frac{4^m t^m}{m!}\,,$$

we get, by (1.5),

$$b_m(2;(0,1))=(\tfrac{1}{2},m)\frac{4^m}{m!}$$

and

$$\sigma_{2k}(f_\rho)=\frac{(\frac{1}{2},k)}{k!}\sum_{m=0}^{k}\binom{k}{m}\frac{(\frac{1}{2},m)}{m!}(\rho^2-1)^m$$
$$=\frac{(\frac{1}{2},k)}{k!}F(-k,\tfrac{1}{2};1;1-\rho^2)$$

because

$$\binom{k}{m}=(-1)^m\frac{(-k,m)}{m!}\,.$$

Finally, from (4.10) we get

(5.1) $$N_k(f_\rho)=F(-k,\tfrac{1}{2};1;1-\rho^2)\,.$$

EXAMPLE 2. Let $f_\rho$, $\rho\in\boldsymbol{R}$, be the map $\boldsymbol{R}^4\to\boldsymbol{R}^3$ defined by $f_\rho(x)=(x_1^2+x_2^2-x_3^2-x_4^2,\ 2\rho(x_2x_3-x_1x_4),\ 2\rho(x_1x_3+x_2x_4))$. When $\rho=\pm1$, $f$ is the classical Hopf map sending $S^3$ onto $S^2$ (see Hopf [5]). Now,

$$(f_\rho)_\nu(\xi)=\int_{S^3}q(x)^\nu d\omega_3$$

with $q(x)=\langle\xi,f_\rho(x)\rangle={}^t xA_\rho x$ where

$$A_\rho=\begin{pmatrix}\xi_1 & 0 & \rho\xi_3 & -\rho\xi_2\\ 0 & \xi_1 & \rho\xi_2 & \rho\xi_3\\ \rho\xi_3 & \rho\xi_2 & -\xi_1 & 0\\ -\rho\xi_2 & \rho\xi_3 & 0 & -\xi_1\end{pmatrix}$$

whose eigenvalues are

$$\lambda_1=\lambda_2=\sqrt{\xi_1^2+\rho^2(\xi_2^2+\xi_3^2)}\,,\qquad \lambda_3=\lambda_4=-\sqrt{\xi_1^2+\rho^2(\xi_2^2+\xi_3^2)}\,.$$

Therefore, by (1.7), (3.8), we have

$$\int_{S^3}q(x)^\nu d\omega_3=\frac{b_\nu(2;\lambda)}{b_\nu(2;1_4)}=\frac{b_\nu(2;\lambda)}{4^\nu(\nu+1)}\,.$$

Since we have

$$\prod_{i=1}^{4}(1-4\lambda_i t)^{-1/2}=(1-4\lambda_1 t)^{-1}(1+4\lambda_1 t)^{-1}=(1-16\lambda_1^2t^2)^{-1}$$
$$=1+16\lambda_1^2t^2+16^2\lambda_1^4t^4+\cdots,$$

we get, by (1.5),

$$b_{2k}(2;\lambda)=16^k(\xi_1^2+\rho^2(\xi_2^2+\xi_3^2))^k .$$

Or we have

$$(f_\rho)_{2k}(\xi)=\int_{S^3}q(x)^{2k}d\omega_3=\frac{1}{2k+1}(\xi_1^2+\rho^2(\xi_2^2+\xi_3^2))^k$$

and

$$\sigma_{2k}(f_\rho)=\frac{1}{2k+1}\int_{S^2}(\xi_1^2+\rho^2(\xi_2^2+\xi_3^2))^k d\omega_2 .$$

Since $\xi_1^2+\xi_2^2+\xi_3^2=1$ on $S^2$, we have

$$\sigma_{2k}(f_\rho)=\frac{1}{2k+1}\int_{S^2}(1+(\rho^2-1)(\xi_2^2+\xi_3^2))^k d\omega_2$$
$$=\frac{1}{2k+1}\sum_{m=0}^{k}\binom{k}{m}(\rho^2-1)^m\int_{S^2}(\xi_2^2+\xi_3^2)^m d\omega_2 .$$

As for the last integral, since the eigenvalues of the quadratic form $\xi_2^2+\xi_3^2$ are $\lambda=(0,1,1)$, we have, by (1.7), (3.8)

$$\int_{S^2}(\xi_2^2+\xi_3^2)^m d\omega_2=\frac{b_m(2;(0,1,1))}{b_m(2;1_3)}=\frac{m!}{4^m(\frac{3}{2},m)}b_m(2;(0,1,1)) .$$

Since we have

$$\prod_{i=1}^{3}(1-4\lambda_i t)^{-1/2}=(1-4t)^{-1}=1+4t+4^2t^2+\cdots,$$

we get, by (1.5), $b_m(2;(0,1,1))=4^m$ and

$$\sigma_{2k}(f_\rho)=\frac{1}{2k+1}\sum_{m=0}^{k}\binom{k}{m}\frac{m!}{(\frac{3}{2},m)}$$
$$=\frac{1}{2k+1}F(-k,1;\tfrac{3}{2};(1-\rho^2)) .$$

Finally, from (4.10) we get

(5.2) $$N_k(f_\rho)=F(-k,1;\tfrac{3}{2};(1-\rho^2)) .$$

## Appendix. On Legendre polynomials

Let $f: \boldsymbol{R}^n \to \boldsymbol{R}^m$ be an affine map given by $f(x) = Ax + b$ where $A$ is an $(m \times n)$-matrix with real coefficients $a_{ij}$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, and $b$ is a real $m$-vector written vertically. Put $a_i = (a_{i1}, \cdots, a_{in})$, the $i$-th row of $A = (a_{ij})$, $M = (\langle a_i, a_j \rangle)$, an $(m \times m)$-symmetric real matrix, and $Q(\xi) = {}^t\xi M \xi$, the corresponding quadratic form. We can verify easily that $\lambda = t^2 Q(\xi)$ satisfies $\Delta\varphi = \lambda\varphi$ for $\varphi = \exp(t\langle \xi, f(x)\rangle)$. Hence, by (2.4), we have

$$\text{(a.1)} \qquad \begin{aligned} \sum_{\nu=0}^{\infty} f_\nu(\xi)\frac{t^\nu}{\nu!} &= \int_{S^{n-1}} \varphi(x)d\omega_{n-1} \\ &= \exp(t\langle \xi, b\rangle) \sum_{k=0}^{\infty} \frac{Q(\xi)^k t^{2k}}{4^k k!\,(n/2, k)} \end{aligned}$$

and so

$$\text{(a.2)} \qquad f_\nu(\xi) = \sum_{k=0}^{[\nu/2]} \frac{\nu!}{4^k k!\,(n/2, k)(\nu - 2k)!} Q(\xi)^k \langle \xi, b\rangle^{\nu - 2k} .$$

Denote by $H$ the algebraic set in $\boldsymbol{R}^m$ defined by

$$H = \{\xi \in \boldsymbol{R}^m;\ Q(\xi) = \langle \xi, b\rangle^2 - 1\}$$

and put $z = \langle \xi, b\rangle$. Then, for $\xi \in H$, we have

$$\text{(a.3)} \qquad \begin{aligned} f_\nu(\xi) &= \sum_{k=0}^{[\nu/2]} \frac{\nu!}{4^k k!\,(n/2, k)(\nu - 2k)!}(z^2 - 1)^k z^{\nu - 2k} \\ &= z^\nu \sum_{k=0}^{[\nu/2]} \frac{(-\nu/2, k)(-\nu/2 + \frac{1}{2}, k)}{(n/2, k)k!}\left(\frac{z^2 - 1}{z^2}\right)^k \\ &= z^\nu F\left(-\frac{\nu}{2}, -\frac{\nu}{2} + \frac{1}{2}; \frac{n}{2}; \frac{z^2 - 1}{z^2}\right) \\ &= F\left(-\nu, \nu + n - 1; \frac{n}{2}; \frac{1 - z}{2}\right) \\ &= P_{\nu, n+1}(z) , \end{aligned}$$

where the equality between two hypergeometric series follows from a formula of quadratic transformations[5] and the last equality is a well-known relation of the Legendre polynomial for $\boldsymbol{R}^{n-1}$ of order $\nu$ and the hypergeometric series[6]. On equating the first and the last terms of (a.3), we get

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5) See Magnus-Oberhettinger-Soni [6], p. 50, line 3 from the bottom.

6) See Hochstadt [4], p. 183, line 8.

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$$\text{(a.4)} \qquad P_{\nu, n+1}(\langle \xi, b \rangle) = \int_{S^{n-1}} \langle \xi, Ax + b \rangle^{\nu} d\omega_{n-1}\,, \qquad \xi \in H\,,$$

which is substantially the Laplace integral for the Legendre polynomials. If, in particular, $m = n + 1$,

$$A = \begin{bmatrix} 0 & 0 \cdots 0 \\ 1 & 0 \cdots 0 \\ \cdot & \\ & \cdot \\ & \quad \cdot \\ 0 & 0 \cdots 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $\xi_3 = \cdots = \xi_{n+1} = 0$, then $\xi_1^2 - \xi_2^2 = 1$ for $\xi \in H$ and we get

$$P_{\nu, n+1}(\xi_1) = \int_{S^{n-1}} (\xi_1 + \sqrt{\xi_1^2 - 1}\, x_1)^{\nu} d\omega_{n-1}\,,$$

the Laplace integral in its original form[7].

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7) See [4], p. 182, Theorem.