A NOTE ON THE ASYMPTOTIC EXPANSION OF A RATIO OF GAMMA FUNCTIONS

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MANY problems in mathematical analysis require a knowledge of the asymptotic behaviour of $\Gamma(z+\alpha)/\Gamma(z+\beta)$ for large values of $|z|$, where α and β are bounded quantities. Tricomi and Erdélyi in (**1**), gave the asymptotic expansion

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum_{j=0}^{\infty} \frac{(-1)^j\Gamma(\beta-\alpha+j)}{\Gamma(\beta-\alpha)j!} B_j^{(\alpha-\beta+1)}(\alpha)z^{\alpha-\beta-j},\ z\to\infty,$$

$$|\arg(z+\alpha)|<\pi,\ B_0^{(\alpha-\beta+1)}(\alpha)=1, \qquad (1)$$

where the $B_j^{(\alpha-\beta+1)}(\alpha)$ are the generalised Bernoulli polynomials, see (**2**), defined by

$$\left(\frac{t}{e^t-1}\right)^{\sigma} e^{xt} = \sum_{j=0}^{\infty} \frac{t^j}{j!} B_j^{(\sigma)}(x),\ |t|<2\pi. \qquad (2)$$

In this note, we show that if, instead of considering z to be the large variable, we consider a related large variable, (1) can be improved from a computational viewpoint. This result is included in the following

Theorem

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum_{j=0}^{\infty} \frac{\Gamma(\beta-\alpha+2j)}{\Gamma(\beta-\alpha)(2j)!} B_{2j}^{(2\rho)}(\rho)(z+\alpha-\rho)^{\alpha-\beta-2j},\ z\to\infty,$$

$$|\arg(z+\alpha)|<\pi,\ 2\rho=1+\alpha-\beta,\ B_0^{(2\rho)}(\rho)=1. \qquad (3)$$

Thus, the asymptotic series in (3) is essentially an even one. The proof of (3) follows directly from the loop integral representation used by Erdélyi in (**1**), to prove (1), i.e.,

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{\Gamma(1+\alpha-\beta)}{2\pi i}\int_{-\infty\cdot e^{i\delta}}^{(0+)} e^{(z+\alpha)t}(e^t-1)^{\beta-\alpha-1}dt,$$

$$\mathrm{Re}\,\{(z+\alpha)e^{i\delta}\}>0,\ |\delta|<\pi/2, \qquad (4)$$

and for small $|t|$,

$$\delta-\pi \leqq \arg(e^t-1) \leqq \delta+\pi. \qquad (5)$$

Thus (4) holds for all α and β with the trivial exception $\alpha-\beta \neq -1, -2, \ldots$, and for all z in the complex plane slit from $-\alpha$ to $-\alpha-\infty$. We rewrite (4) in the form

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = \frac{\Gamma(2\rho)}{2\pi i}\int_{-\infty\,.\,e^{i\delta}}^{(0+)} e^{(z+\alpha-\rho)t}t^{-2\rho}h(t)dt,$$

$$h(t) = e^{\rho t}t^{2\rho}(e^t-1)^{-2\rho} = \left[\frac{2\sinh(t/2)}{t}\right]^{-2\rho}, \tag{6}$$

and note that if $\arg(z+\alpha)$ is fixed, and $|z+\alpha|$ taken large enough,

$$\text{Re}\{(z+\alpha-\rho)e^{i\delta}\} = \text{Re}\{(z+\alpha)e^{i\delta}\}-\text{Re}\{\rho e^{i\delta}\}>0. \tag{7}$$

Also, it should be noted that if x is replaced by $\sigma-x$, (2) implies

$$B_j^{(\sigma)}(\sigma-x) = (-1)^j B_j^{(\sigma)}(x), \tag{8}$$

or

$$B_{2j+1}^{(\sigma)}(\sigma/2) \equiv 0. \tag{9}$$

Thus,

$$h(t) = \sum_{j=0}^{\infty}\frac{t^{2j}}{(2j)!}B_{2j}^{(2\rho)}(\rho),\ |t|<2\pi. \tag{10}$$

Then, since $h(t)$ is of bounded exponential growth along the path of integration, and

$$\frac{1}{2\pi i}\int_{-\infty\,.\,e^{i\delta}}^{(0+)} e^{zt}t^{\sigma-1}dt = \frac{z^{-\sigma}}{\Gamma(1-\sigma)},\ \text{Re}\{ze^{i\delta}\}>0,\ |\delta|<\pi/2, \tag{11}$$

Watson's lemma, see (3), is immediately applicable, and yields (3). For convenience, we note that

$$B_0^{(2\rho)}(\rho) = 1,\ B_2^{(2\rho)}(\rho) = -\rho/6,$$
$$B_4^{(2\rho)}(\rho) = \rho(5\rho+1)/(60),\ B_6^{(2\rho)}(\rho) = -\rho(35\rho^2+21\rho+4)/(504), \tag{12}$$

and that the $B_{2j}^{(2\rho)}(\rho)$ can be defined recursively by

$$B_{2j+2}^{(2\rho)}(\rho) = (-2\rho)\sum_{k=0}^{j}\frac{1}{(2k+2)}\binom{2j+1}{2k+1}B_{2k+2}^{(1)}(0)B_{2j-2k}^{(2\rho)}(\rho), \tag{13}$$

where the $B_{2k+2}^{(1)}(0)$ are the ordinary Bernoulli numbers defined by (2). If α and β are real, and $\arg z = 0$, then (3) is related to a result of Frame, see (4).

From the theorem, two corollaries are readily deduced.

Corollary 1. If $E(v)$ is a function analytic in a neighbourhood of $v = 0$, $E(0) = 1$, and w is defined implicitly by

$$z+\alpha-\rho = z+\tfrac{1}{2}(\alpha+\beta-1) = wE(w^{-2}), \tag{14}$$

then there exist numbers c_j such that

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim \sum_{j=0}^{\infty} c_j w^{\alpha-\beta-2j},\ z\to\infty,\ |\arg(z+\alpha)|<\pi,\ c_0 = 1. \tag{15}$$

One case of Corollary 1 is particularly interesting. If

$$z+\alpha-\rho = w\sqrt{1+(\alpha-\rho)^2w^{-2}}, \tag{16}$$

then

$$w = \sqrt{z(z+\alpha+\beta-1)}. \tag{17}$$

Finally, we have for the hypergeometric polynomials ${}_2F_1\left(\begin{matrix}-n,\ n+\lambda\\ \beta\end{matrix}\middle|\, z\right)$, see (5),

Corollary 2. If n is a positive integer and β is not a non-positive integer, then there exist numbers $c_j(\mu)$, $\mu = 1, 2$, such that

$${}_2F_1\left(\begin{matrix}-n,\ n+\lambda\\ \beta\end{matrix}\middle|\, 1\right) = (-1)^n \frac{\Gamma(\beta)\Gamma(n+1+\lambda-\beta)}{\Gamma(1+\lambda-\beta)\Gamma(n+\beta)},$$

$$\sim (-1)^n \frac{\Gamma(\beta)}{\Gamma(1+\lambda-\beta)} \sum_{j=0}^{\infty} c_j(\mu)(N_\mu)^{1+\lambda-2\beta-2j}, \quad n\to\infty, \tag{18}$$

where

$$N_1 = n+\lambda/2,$$
$$N_2 = \sqrt{n(n+\lambda)}, \tag{19}$$

and

$$c_0(\mu) = 1, \ \mu = 1 \text{ or } 2. \tag{20}$$

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