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Q_p SPACES ON RIEMANN SURFACES

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ABSTRACT. We study the function spaces $Q_p(R)$ defined on a Riemann surface R, which were earlier introduced in the unit disk of the complex plane. The nesting property $Q_p(R) \subseteq Q_q(R)$ for $0 < p < q < \infty$ is shown in case of arbitrary hyperbolic Riemann surfaces. Further, it is proved that the classical Dirichlet space $\mathrm{AD}(R) \subseteq Q_p(R)$ for any p, $0 < p < \infty$, thus sharpening T. Metzger's well-known result $\mathrm{AD}(R) \subseteq \mathrm{BMOA}(R)$. Also the first author's result $\mathrm{AD}(R) \subseteq \mathrm{VMOA}(R)$ for a regular Riemann surface R is sharpened by showing that, in fact, $\mathrm{AD}(R) \subseteq Q_{p,0}(R)$ for all p, $0 < p < \infty$. The relationships between $Q_p(R)$ and various generalizations of the Bloch space on R are considered. Finally we show that $Q_p(R)$ is a Banach space for $0 < p < \infty$.

1. **Introduction.** Let R be an open Riemann surface having a Green's function, *i.e.*, $R \notin O_G$. Denote the Green's function on R with singularity at α by $g_R(w,\alpha)$. Let $A(R)$ denote the collection of all functions analytic on R. For $0 < p < \infty$, we define

$$Q_p(R) = \Big\{ F \in A(R) : \|F\|^2_{Q_p(R)} = \sup_{\alpha \in R} \int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} < \infty \Big\}$$

and

$$Q_{p,0}(R) = \Big\{ F \in A(R) : \lim_{\alpha \to \partial R} \int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} = 0 \Big\},$$

where ∂R is the ideal boundary of R and $dw\, d\bar{w} = 2\, du\, dv$ for a local parameter $w = u + iv$. For the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, $Q_p(\Delta)$ and $Q_{p,0}(\Delta)$ have been defined and studied in [4] and [6]. It is proved in [4] that $Q_p(\Delta) = B(\Delta)$ and $Q_{p,0}(\Delta) = B_0(\Delta)$ for $1 < p < \infty$. Earlier, in [13] and [14], it was proved that $Q_2(\Delta) = B(\Delta)$ and $Q_{2,0}(\Delta) = B_0(\Delta)$, respectively. Recall that the Bloch space $B(\Delta)$ and the little Bloch space $B_0(\Delta)$ are defined as follows:

$$B(\Delta) = \big\{ f \in A(\Delta) : \|f\|_B = \sup_{z \in \Delta} |f'(z)|(1 - |z|^2) < \infty \big\}$$

and

$$B_0(\Delta) = \big\{ f \in A(\Delta) : \lim_{|z| \to 1} |f'(z)|(1 - |z|^2) = 0 \big\}.$$

It is proved in [6] that, for $0 < p_1 < p_2 \le 1$, $Q_{p_1}(\Delta) \underset{\neq}{\subset} Q_{p_2}(\Delta)$.

For $p = 1$ and $R = \Delta$, it is known that $Q_1(R) = \mathrm{BMOA}(R)$ and $Q_{1,0}(R) = \mathrm{VMOA}(R)$ and so this has been taken as the definition of BMOA and VMOA on a Riemann surface R (*cf.* [9, 10, 1]). BMO-spaces of harmonic functions on Riemann surfaces have been

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considered by Y. Gotoh in [7]. In [5], the relationships between $Q_2(R)$, $Q_{2,0}(R)$ and various generalizations of the Bloch space on Riemann surfaces have been studied. Before introducing these results, we first look at some basic facts on hyperbolic geometry.

Let R be a Riemann surface such that $R \notin O_G$. It is well known that the universal covering surface of R is the unit disc Δ. Let $\lambda_\Delta(z) = 1/(1-|z|^2)$ be the density of the hyperbolic distance in Δ. Then the hyperbolic distance between two points z and a in Δ is given by

$$d_\Delta(z,a) = \inf\Big\{\int_\gamma \lambda_\Delta(\zeta)|d\zeta| : \gamma \text{ is a curve in } \Delta \text{ from } a \text{ to } z\Big\}.$$

Now let $\pi\colon \Delta \longrightarrow R$ denote the universal covering mapping, and let $w, \alpha \in R$. We define the hyperbolic distance between w and α on R by

$$d_R(w,\alpha) = \inf\{d_\Delta(z,a) : \pi(z) = w \text{ and } \pi(a) = \alpha\}.$$

Thus the density of d_R at the point α is given by

$$\lambda_R(\alpha) = \inf\{\lambda_\Delta(a) : \pi(a) = \alpha\}.$$

We can generalize the Bloch space and the little Bloch space onto R as follows:

$$\mathcal{B}(R) = \left\{F \in A(R) : \|F\|_{\mathcal{B}(R)} = \sup_{\alpha\in R} \frac{|F'(\alpha)|}{\lambda_R(\alpha)} < \infty\right\}$$

and

$$\mathcal{B}_0(R) = \left\{F \in A(R) : \lim_{\alpha\longrightarrow\partial R} \frac{|F'(\alpha)|}{\lambda_R(\alpha)} = 0\right\}.$$

To introduce another kind of generalization of the Bloch space on R, we note that if R is a Riemann surface with Green's function $g_R(w,\alpha)$, then, by using local coordinates in a neighborhood of α, we can define the Robin's constant $\gamma_R(\alpha)$ by

$$\gamma_R(\alpha) = \lim_{w\longrightarrow\alpha}\left(g_R(w,\alpha) - \log\frac{1}{|w-\alpha|}\right).$$

Let $c_R(\alpha) = \exp\big(-\gamma_R(\alpha)\big)$ be the capacity density of R at α. It is known that if $F \in A(R)$, then $|F'(\alpha)|/c_R(\alpha)$ is a conformal invariant (*cf.*, for example, [12]). Thus we can define the spaces $C\mathcal{B}(R)$ and $C\mathcal{B}_0(R)$ by

$$C\mathcal{B}(R) = \left\{F \in A(R) : \|F\|_{C\mathcal{B}(R)} = \sup_{\alpha\in R} \frac{|F'(\alpha)|}{c_R(\alpha)} < \infty\right\}$$

and

$$C\mathcal{B}_0(R) = \left\{F \in A(R) : \lim_{\alpha\longrightarrow\partial R} \frac{|F'(\alpha)|}{c_R(\alpha)} = 0\right\}.$$

It is easy to check that, for $R = \Delta$, both $\mathcal{B}(R)$ ($\mathcal{B}_0(R)$) and $C\mathcal{B}(R)$ ($C\mathcal{B}_0(R)$) coincide with the Bloch space $\mathcal{B}(\Delta)$ (the little Bloch space $\mathcal{B}_0(\Delta)$).

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The following inclusions are given in [5],

$$\mathrm{BMOA}(R) \subseteq Q_2(R) \subseteq \mathcal{C}\mathcal{B}(R) \subseteq \mathcal{B}(R) \tag{1.1}$$

and

$$\mathrm{VMOA}(R) \subseteq Q_{2,0}(R) \subseteq \mathcal{C}\mathcal{B}_0(R) \subseteq \mathcal{B}_0(R). \tag{1.2}$$

(Note that in [5], $Q_2(R)$ and $Q_{2,0}(R)$ were denoted by $\mathrm{BMOA}(R, m)$ and $\mathrm{VMOA}(R, m)$, respectively.) It turns out that, on general Riemann surfaces R, $Q_2(R)$ ($Q_{2,0}(R)$) and $\mathcal{C}\mathcal{B}(R)$ ($\mathcal{C}\mathcal{B}_0(R)$) do not always coincide with $\mathcal{B}(R)$ ($\mathcal{B}_0(R)$). There is a Riemann surface $R \notin O_G$ such that $\mathcal{C}\mathcal{B}(R) \neq \mathcal{B}(R)$ and $Q_2(R) \neq \mathcal{B}(R)$ ([5, Theorem 4.2 and Theorem 7.2]). There is also another Riemann surface R such that $\mathcal{C}\mathcal{B}_0(R) \neq \mathcal{B}_0(R)$ and $Q_{2,0}(R) \neq \mathcal{B}_0(R)$ ([5, Theorem 7.3]).

In this paper we study the relations between $Q_p(R)$ and various generalizations of the Bloch spaces on Riemann surfaces as well as $\mathrm{BMOA}(R)$. One of our main results is to generalize the inclusion relations (1.1) and (1.2) to $Q_p(R)$, $Q_q(R)$ and $Q_{p,0}(R)$, $Q_{q,0}(R)$, by showing the nesting properties

$$Q_p(R) \subseteq Q_q(R), \quad Q_{p,0}(R) \subseteq Q_{q,0}(R) \tag{1.3}$$

and the inclusions

$$Q_p(R) \subseteq \mathcal{C}\mathcal{B}(R), \quad Q_{p,0}(R) \subseteq \mathcal{C}\mathcal{B}_0(R) \tag{1.4}$$

for $0 < p < q < \infty$. By (1.1) and (1.2) we have also proved

$$Q_p(R) \subseteq \mathcal{B}(R), \quad Q_{p,0}(R) \subseteq \mathcal{B}_0(R) \tag{1.5}$$

for $0 < p < \infty$. These will be proved in Section 2 and Section 4, respectively. The main result in Section 3 sharpens T. Metzger's result

$$\mathrm{AD}(R) \subseteq \mathrm{BMOA}(R)$$

(*cf.* [9, Theorem 1]) showing that, in fact,

$$\mathrm{AD}(R) \subseteq Q_p(R) \tag{1.6}$$

for all p, $0 < p < \infty$. Further, the first author's result $\mathrm{AD}(R) \subseteq \mathrm{VMOA}(R)$ for regular Riemann surfaces R (*cf.* [1, Theorem 1(a)]) is sharpened by showing

$$\mathrm{AD}(R) \subseteq Q_{p,0}(R) \tag{1.7}$$

for all p, $0 < p < \infty$, in case of regular Riemann surfaces R. In Section 5, we will prove that for $0 < p < \infty$, $Q_p(R)$ is a Banach space and $Q_{p,0}(R)$ is a closed subspace of $Q_p(R)$. We will also give a criterion for $Q_p(R)$ by regular exhaustions of R.

Finally we note that in [2] all these inclusions (1.3)–(1.7) have been proved by using a different technique.

2. $Q_p(R) \subseteq Q_q(R)$. In this section, we show the nesting properties of the spaces $Q_p(R)$ and $Q_{p,0}(R)$ as a function of parameter values p. In [2, Theorem 4] different proofs for these nesting properties are given. For proving the inclusions we need several lemmas which are derived in the following.

First we show that $1 - e^{-t} \leq \frac{1}{p}t^p$ for $t > 0$ and $0 < p \leq 1$. If $t \geq 1$, then $1 - e^{-t} \leq 1 \leq \frac{1}{p}t^p$. Let $0 < t < 1$ and $f(t) = \frac{1}{p}t^p - (1 - e^{-t})$. Then $f'(t) = t^{p-1} - e^{-t} \geq 1 - e^{-t} > 0$, and thus $f(t)$ is increasing when $0 < t < 1$. Since $f(0) = 0$ we get $f(t) \geq 0$, and so $1 - e^{-t} \leq \frac{1}{p}t^p$ for $0 < t < 1$. By using this we get the first lemma

LEMMA 2.1. *Let R be a Riemann surface, let $R \notin O_G$ and let $0 < p \leq 1$. Then, for $F \in A(R)$,*

$$\int_R |F'(w)|^2 g_R(w,\alpha)\,dw\,d\bar{w} \leq \frac{2^p}{p} \int_R |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\bar{w}.$$

PROOF. By [8, Lemma 2] we have

$$\int_R |F'(w)|^2 g_R(w,\alpha)\,dw\,d\bar{w} \leq \int_R |F'(w)|^2 (1 - e^{-2g_R(w,\alpha)})\,dw\,d\bar{w}$$

and using the above consideration

$$\int_R |F'(w)|^2 g_R(w,\alpha)\,dw\,d\bar{w} \leq \frac{2^p}{p} \int_R |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\bar{w}. \qquad \blacksquare$$

This gives as a corollary

COROLLARY 2.2. $Q_p(R) \subseteq \mathrm{BMOA}(R)$ *for all p, $0 < p \leq 1$.*

By the inequality $1 - e^{-t} \leq t$ for $t > 0$ and [8, Lemma 2] we get

PROPOSITION 2.3. $\int_R |F'(w)|^2 g_R(w,\alpha)\,dw\,d\bar{w} \sim \int_R |F'(w)|^2 (1 - e^{-2kg_R(w,\alpha)})\,dw\,d\bar{w}$ *for any positive integer k.*

In the above, we use the notation $a \sim b$ to denote comparability of the quantities, *i.e.*, there are absolute positive constants c_1, c_2 satisfying $c_1 b \leq a \leq c_2 b$. For proving the nesting properties of the spaces $Q_p(R)$, $Q_q(R)$ and $Q_{p,0}(R)$, $Q_{q,0}(R)$ we first derive area integral estimates for parameter values p and q. By using a different method these inequalities with different constant factors have been shown in [2, Theorem 2].

LEMMA 2.4. *Let R be a Riemann surface, let $R \notin O_G$ and let $0 < p < q < \infty$. Then, for $F \in A(R)$,*

$$\int_R |F'(w)|^2 g_R^q(w,\alpha)\,dw\,d\bar{w} \leq c_{p,q} \int_R |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\bar{w},$$

where $c_{p,q} = 2^{1+p-q}\frac{\Gamma(q+1)}{p}e^2$ for $1 < q < \infty$ and $c_{p,q} = 2^p\frac{q}{p}$ for $0 < q \leq 1$.

PROOF. We will prove the result for the case where R is a compact bordered Riemann surface. For the general case, the conclusion follows by taking a regular exhaustion of R.

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Let $F \in A(R)$ and let $R_{1,\alpha} = \{w \in R : g_R(w,\alpha) > 1\}$. Then

$$\int_{R\setminus R_{1,\alpha}} |F'(w)|^2 g_R^q(w,\alpha)\, dw\, d\overline{w} \le \int_{R\setminus R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w}. \tag{2.1}$$

Let $B_\varepsilon(\alpha)$ be a disk in $R_{1,\alpha}$ with center at α and radius ε, and let $R_{1,\alpha,\varepsilon} = R_{1,\alpha} \setminus B_\varepsilon(\alpha)$. By using Green's formula we get

$$\begin{aligned}\int_{R_{1,\alpha,\varepsilon}} &\Big[g_R^q(w,\alpha)\Delta\big(|F(w)-F(\alpha)|^2\big) - |F(w)-F(\alpha)|^2 \Delta g_R^q(w,\alpha)\Big]\, dw\, d\overline{w} \\ &= 2\int_{\partial R_{1,\alpha,\varepsilon}} \left[|F(w)-F(\alpha)|^2 \frac{\partial g_R^q(w,\alpha)}{\partial n} - g_R^q(w,\alpha)\frac{\partial |F(w)-F(\alpha)|^2}{\partial n}\right] ds,\end{aligned} \tag{2.2}$$

where Δ denotes the Laplacian, $\frac{\partial}{\partial n}$ differentiation in the inner normal direction and ds arc length measure on $\partial R_{1,\alpha,\varepsilon}$. By computing we get

$$\Delta|F(w)-F(\alpha)|^2 = 4|F'(w)|^2$$

and

$$\Delta g_R^q(w,\alpha) = q(q-1)g_R^{q-2}(w,\alpha)|\nabla g_R(w,\alpha)|^2,$$

where ∇ denotes the gradient operator. Further,

$$\frac{\partial g_R^q(w,\alpha)}{\partial n} = q g_R^{q-1}(w,\alpha)\frac{\partial g_R(w,\alpha)}{\partial n} = q\frac{\partial g_R(w,\alpha)}{\partial n}$$

for $w \in \partial R_{1,\alpha}$.

Let $H_{1,\alpha}(w)$ be the least harmonic majorant of $|F(w)-F(\alpha)|^2$ on $R_{1,\alpha}$. Let $g_R^*(w,\alpha)$ be the conjugate of $g_R(w,\alpha)$. Then

$$\exp h_R(w,\alpha) = \exp[g_R(w,\alpha) + i g_R^*(w,\alpha)]$$

is a meromorphic function with a simple pole at α. Since

$$\phi_{1,\alpha}(w) = \Big|\big(F(w)-F(\alpha)\big)\exp h_R(w,\alpha)\Big|^2 = |F(w)-F(\alpha)|^2 e^{2g_R(w,\alpha)}$$

is a subharmonic function on $R_{1,\alpha}$ and

$$\phi_{1,\alpha}(w) = e^2|F(w)-F(\alpha)|^2$$

for $w \in \partial R_{1,\alpha}$, we get by the maximum principle

$$|F(w)-F(\alpha)|^2 \le e^2 H_{1,\alpha}(w) e^{-2g_R(w,\alpha)} \tag{2.3}$$

for $w \in R_{1,\alpha}$.

Let $g_{R_{1,\alpha}}(w,\alpha)$ be a Green's function of $R_{1,\alpha}$ with logarithmic singularity at α. Now $\Delta g_{R_{1,\alpha}}(w,\alpha) = 0$ in $R_{1,\alpha} \setminus \{\alpha\}$ and $g_{R_{1,\alpha}}(w,\alpha) = 0$ for $w \in \partial R_{1,\alpha}$ and similar to the proof in [5, Lemma 2.1] we get

$$\frac{1}{\pi}\int_{R_{1,\alpha}} |F'(w)|^2 g_{R_{1,\alpha}}(w,\alpha)\, dw\, d\overline{w} = \frac{1}{2\pi}\int_{\partial R_{1,\alpha}} |F(w)-F(\alpha)|^2 \frac{\partial g_{R_{1,\alpha}}(w,\alpha)}{\partial n}\, ds = H_{1,\alpha}(\alpha). \tag{2.4}$$

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For $t > 0$, let $S_{t,\alpha} = \{w \in R : g_R(w,\alpha) = t\}$. Since $g_R(w,\alpha) = t$ on $S_{t,\alpha}$ we have $dt = \frac{\partial g_R}{\partial n} dn$. Further, in the conclusion below we use $|\nabla g_R(w,\alpha)|^2 = (\partial g_R(w,\alpha)/\partial n)^2$ for $w \in S_{t,\alpha}$. Taking the limit as ε tends to zero, (2.2) becomes

(2.5)

$$\begin{aligned}
I_{1,q}(\alpha) &= 4\int_{R_{1,\alpha}} |F'(w)|^2 g_R^q(w,\alpha)\,dw\,d\bar{w}\\
&= \int_{R_{1,\alpha}} |F(w)-F(\alpha)|^2 \Delta g_R^q(w,\alpha)\,dw\,d\bar{w}\\
&\quad + 2\int_{\partial R_{1,\alpha}} \left[|F(w)-F(\alpha)|^2 \frac{\partial g_R^q(w,\alpha)}{\partial n} - g_R^q(w,\alpha)\frac{\partial |F(w)-F(\alpha)|^2}{\partial n}\right] ds\\
&= q(q-1)\int_{R_{1,\alpha}} |F(w)-F(\alpha)|^2 g_R^{q-2}(w,\alpha)|\nabla g_R(w,\alpha)|^2\,dw\,d\bar{w}\\
&\quad + 2q\int_{\partial R_{1,\alpha}} |F(w)-F(\alpha)|^2 \frac{\partial g_R(w,\alpha)}{\partial n}\,ds - 2\int_{\partial R_{1,\alpha}} \frac{\partial |F(w)-F(\alpha)|^2}{\partial n}\,ds\\
&= q(q-1)\int_{R_{1,\alpha}} |F(w)-F(\alpha)|^2 g_R^{q-2}(w,\alpha)|\nabla g_R(w,\alpha)|^2\,dw\,d\bar{w}\\
&\quad + 2q\int_{\partial R_{1,\alpha}} |F(w)-F(\alpha)|^2 \frac{\partial g_R(w,\alpha)}{\partial n}\,ds + 4\int_{R_{1,\alpha}} |F'(w)|^2\,dw\,d\bar{w},
\end{aligned}$$

where we have used the equality

$$2\int_{R_{1,\alpha}} |F'(w)|^2\,dw\,d\bar{w} = -\int_{\partial R_{1,\alpha}} \frac{\partial |F(w)-F(\alpha)|^2}{\partial n}\,ds$$

obtained by Green's formula.

We first suppose that $1 < q < \infty$. Then, by Lemma 2.1, (2.3), (2.4) and the inequality $g_{R_{1,\alpha}}(w,\alpha) \le g_R(w,\alpha)$,

(2.6)

$$\begin{aligned}
I_{1,q}(\alpha) &\le q(q-1)e^2\int_{R_{1,\alpha}} H_{1,\alpha}(w) g_R^{q-2}(w,\alpha)|\nabla g_R(w,\alpha)|^2 e^{-2g_R(w,\alpha)}\,dw\,d\bar{w}\\
&\quad + 4q\pi H_{1,\alpha}(\alpha) + 4\int_{R_{1,\alpha}} |F'(w)|^2\,dw\,d\bar{w}\\
&\le 2q(q-1)e^2\int_1^\infty \left(\int_{S_{t,\alpha}} H_{1,\alpha}(w)\frac{\partial g_R(w,\alpha)}{\partial n}\,ds\right) g_R^{q-2}(w,\alpha) e^{-2g_R(w,\alpha)}\,dt\\
&\quad + 4q\int_{R_{1,\alpha}} |F'(w)|^2 g_{R_{1,\alpha}}(w,\alpha)\,dw\,d\bar{w} + 4\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\bar{w}\\
&\le 4q(q-1)e^2\pi H_{1,\alpha}(\alpha)\int_1^\infty t^{q-2}e^{-2t}\,dt + 4q\frac{2^p}{p}\int_{R_{1,\alpha}} |F'(w)|^2 g_{R_{1,\alpha}}^p(w,\alpha)\,dw\,d\bar{w}\\
&\quad + 4\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\bar{w}\\
&\le 2^{3-q}\Gamma(q+1)e^2\int_{R_{1,\alpha}} |F'(w)|^2 g_{R_{1,\alpha}}(w,\alpha)\,dw\,d\bar{w}\\
&\quad + 4q\frac{2^p}{p}\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\bar{w} + 4\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\bar{w}\\
&\le 2^{3+p-q}\frac{\Gamma(q+1)}{p}e^2\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\bar{w},
\end{aligned}$$

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since $2^{3+p-q}\frac{\Gamma(q+1)}{p} > 2^{2+p}\frac{q}{p} > 4$. For $0 < q \le 1$ we have, by Lemma 2.1, (2.4) and the inequality $g_{R_{1,\alpha}}(w,\alpha) \le g_R(w,\alpha)$, the estimate

$$
\begin{aligned}
I_{1,q}(\alpha) &\le 4q\pi H_{1,\alpha}(\alpha) + 4\int_{R_{1,\alpha}} |F'(w)|^2\, dw\, d\bar{w} \\
&\le 4q\int_{R_{1,\alpha}} |F'(w)|^2 g_{R_{1,\alpha}}(w,\alpha)\, dw\, d\bar{w} + 4\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} \\
&\le 4q\frac{2^p}{p}\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} + 4\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} \\
&\le 2^{2+p}\frac{q}{p}\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w},
\end{aligned}
\tag{2.7}
$$

since $q - 1 \le 0$.

Combining (2.1) and (2.6) we get for $1 < q < \infty$,

$$
\begin{aligned}
\int_R |F'(w)|^2 g_R^q(w,\alpha)\, dw\, d\bar{w} &= \int_{R\setminus R_{1,\alpha}} |F'(w)|^2 g_R^q(w,\alpha)\, dw\, d\bar{w} \\
&\quad + \int_{R_{1,\alpha}} |F'(w)|^2 g_R^q(w,\alpha)\, dw\, d\bar{w} \\
&\le 2^{1+p-q}\frac{\Gamma(q+1)}{p} e^2 \int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w}
\end{aligned}
\tag{2.8}
$$

and similarly combining (2.1) and (2.7), for $0 < q \le 1$,

$$\int_R |F'(w)|^2 g_R^q(w,\alpha)\, dw\, d\bar{w} \le 2^p\frac{q}{p}\int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w}.$$

This proves the lemma. ∎

Thus the nesting property of the $Q_p(R)$ spaces is a direct consequence of Lemma 2.4.

THEOREM 2.5. *Let R be a Riemann surface, $R \notin O_G$, and let $0 < p < q < \infty$. Then*
(i) $Q_p(R) \subseteq Q_q(R)$,
(ii) $Q_{p,0}(R) \subseteq Q_{q,0}(R)$.

We note that a different proof of this result is shown in [2, Theorem 4].

3. $\mathrm{AD}(R) \subseteq Q_p(R)$. In this section we will sharpen T. Metzger's result that the classical Dirichlet space $\mathrm{AD}(R) = \{F \in A(R) : \int_R |F'(w)|^2\, dw\, d\bar{w} < \infty\}$ is included in BMOA(R) (*cf.* [9, Theorem 1]) by proving

$$\mathrm{AD}(R) \subseteq Q_p(R)$$

for any p, $0 < p < \infty$. The first author proved in [1, Theorem 1(a)] that $\mathrm{AD}(R) \subseteq$ VMOA(R) for a regular Riemann surface R. Also this result is strengthened by using the $Q_{p,0}(R)$ spaces.

We are now ready to prove

THEOREM 3.1. *$AD(R) \subseteq Q_p(R)$ for any p, $0 < p < \infty$.*

PROOF. Applying Theorem 2.5 for $1 = p < q < \infty$ we get $BMOA(R) \subseteq Q_q(R)$. By T. Metzger's result $AD(R) \subseteq BMOA(R)$ [9, Theorem 1] we have

$$AD(R) \subseteq Q_q(R) \tag{3.1}$$

for $1 \le q < \infty$.

So we can concentrate on the case $0 < p < 1$. By (2.5) we get in case of $R_{1,\alpha} = \{w \in R : g_R(w,\alpha) > 1\}$,

$$\begin{aligned} 4\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} &\le p(p-1)\int_{R_{1,\alpha}} |F(w)-F(\alpha)|^2 g_R^{p-2}(w,\alpha)|\nabla g_R(w,\alpha)|^2\, dw\, d\bar{w} \\ &\quad + 4p\pi H_{1,\alpha}(\alpha) + 4\int_{R_{1,\alpha}} |F'(w)|^2\, dw\, d\bar{w} \\ &\le 4p\pi H_{1,\alpha}(\alpha) + 4\int_R |F'(w)|^2\, dw\, d\bar{w}. \end{aligned} \tag{3.2}$$

The latter inequality follows because $p - 1 < 0$. If now $F \in AD(R)$, then $\int_R |F'(w)|^2\, dw\, d\bar{w} = M < \infty$. On the other hand, by T. Metzger's result $F \in BMOA(R)$ and (2.4),

$$\begin{aligned} H_{1,\alpha}(\alpha) &= \frac{1}{\pi}\int_{R_{1,\alpha}} |F'(w)|^2 g_{R_{1,\alpha}}(w,\alpha)\, dw\, d\bar{w} \\ &\le \frac{1}{\pi}\int_R |F'(w)|^2 g_R(w,\alpha)\, dw\, d\bar{w} \le K < \infty \end{aligned} \tag{3.3}$$

for all $\alpha \in R$. By (3.2) and (3.3),

$$\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} \le p\pi K + M \tag{3.4}$$

for all $\alpha \in R$.

Further, trivially

$$\begin{aligned} \int_{R\setminus R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} &\le \int_{R\setminus R_{1,\alpha}} |F'(w)|^2\, dw\, d\bar{w} \\ &\le \int_R |F'(w)|^2\, dw\, d\bar{w} = M. \end{aligned} \tag{3.5}$$

Thus, by (3.4) and (3.5),

$$\sup_{\alpha\in R}\int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w} \le p\pi K + 2M,$$

and hence $F \in Q_p(R)$. Combining this result with (3.1) we have

$$AD(R) \subseteq Q_p(R)$$

for all p, $0 < p < \infty$. The theorem is proved. ∎

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REMARK. Theorem 3.1 sharpens T. Metzger's result $\mathrm{AD}(R) \subseteq \mathrm{BMOA}(R)$, since even in the case of the unit disk Δ, $Q_p(\Delta) \underset{\neq}{\subset} \mathrm{BMOA}(\Delta)$, for $0 < p < 1$ (*cf.* [6, Theorem 2 and Corollary 3]).

We recall that R is a regular Riemann surface if for each $w \in R$,

$$\lim_{\alpha \longrightarrow \partial R} g_R(w, \alpha) = 0.$$

Otherwise, we say that R is a non-regular Riemann surface. The first author proved that $\mathrm{AD}(R) \subseteq \mathrm{VMOA}(R)$ for regular Riemann surfaces. He also showed that $\mathrm{VMOA}(R)$ contains only constant functions for non-regular Riemann surfaces. This result is generalized to the space $Q_{2,0}(R)$ in [5, Theorem 2.5]. It is also true for $Q_{p,0}(R)$ for $0 < p < \infty$ as the next theorem shows. Since even for the unit disk Δ, $Q_{p,0}(\Delta) \underset{\neq}{\subset} \mathrm{VMOA}(\Delta)$ as $0 < p < 1$, the case (i) of the below theorem sharpens the first author's result [1, Theorem 1(a)], and by Theorem 2.5(ii) the case (ii) generalizes [1, Theorem 1(b)]. Finally we note that Theorem 3.2(i) has been proved in [2, Theorem 7] by using a different technique.

THEOREM 3.2. *Let $0 < p < \infty$. Then*

(i) if R is a regular Riemann surface, $\mathrm{AD}(R) \subseteq Q_{p,0}(R)$,

(ii) if R is a non-regular Riemann surface, $Q_{p,0}(R)$ contains only constant functions.

PROOF. (i) For $1 \leq p < \infty$ this is a direct consequence of Theorem 2.5(ii) and [1, Theorem 1(a)], since $Q_{1,0}(R) = \mathrm{VMOA}(R)$. Therefore let $0 < p < 1$ and let ε, $0 < \varepsilon < 1$, be arbitrary but fixed during the consideration. If $F \in \mathrm{AD}(R)$, then, by (3.2) and (3.3),

(3.6)
$$4 \int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w} \leq 4p \int_R |F'(w)|^2 g_R(w,\alpha)\, dw\, d\overline{w} + 4 \int_{R_{1,\alpha}} |F'(w)|^2\, dw\, d\overline{w},$$

where $R_{1,\alpha} = \{w \in R : g_R(w,\alpha) > 1\}$. By [1, Theorem 1(a)] we know that the integral $\int_R |F'(w)|^2 g_R(w,\alpha)\, dw\, d\overline{w}$ tends to 0 as α tends to ∂R. Since R is a regular Riemann surface, $R_{1,\alpha}$ as a compact set tends to ∂R when α tends to ∂R. Hence $\int_{R_{1,\alpha}} |F'(w)|^2\, dw\, d\overline{w} \longrightarrow 0$ for $\alpha \longrightarrow \partial R$. Thus, by (3.6),

$$\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w} < \varepsilon \tag{3.7}$$

as $\alpha \in R \setminus K_1$, where K_1 is a compact subset of R. Let $R_\varepsilon = \{w \in R \mid g_R(w,\alpha) > (\varepsilon/M)^{1/p}\}$, where $\int_R |F'(w)|^2\, dw\, d\overline{w} = M$. We can suppose that $\epsilon/M < 1$. Then

$$\begin{aligned} \int_{R \setminus R_\varepsilon} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w} &\leq \frac{\varepsilon}{M} \int_{R \setminus R_\varepsilon} |F'(w)|^2\, dw\, d\overline{w} \\ &\leq \frac{\varepsilon}{M} \int_R |F'(w)|^2\, dw\, d\overline{w} = \frac{\varepsilon}{M} \cdot M = \varepsilon. \end{aligned} \tag{3.8}$$

Now $R_\varepsilon \setminus R_{1,\alpha}$ is a compact set and $R_\varepsilon \setminus R_{1,\alpha}$ tends to ∂R as α tends to ∂R. Since $F \in \mathrm{AD}(R)$, there exists a compact set A such that $\int_{R \setminus A} |F'(w)|^2\, dw\, d\overline{w} < \varepsilon$. On the other hand, there is a compact set K_2 such that when $\alpha \in R \setminus K_2$, then $R_\varepsilon \setminus R_{1,\alpha} \subseteq R \setminus A$. Thus, for $\alpha \in R \setminus K_2$,

$$\text{(3.9)} \qquad \begin{aligned} \int_{R_\varepsilon \setminus R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w} &\leq \int_{R_\varepsilon \setminus R_{1,\alpha}} |F'(w)|^2\, dw\, d\overline{w} \\ &\leq \int_{R \setminus A} |F'(w)|^2\, dw\, d\overline{w} < \varepsilon. \end{aligned}$$

Hence, for $\alpha \in R \setminus K_1 \cup K_2$, by combining (3.7), (3.8) and (3.9) we get

$$\begin{aligned} \int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w} = &\int_{R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w} \\ &+ \int_{R_\varepsilon \setminus R_{1,\alpha}} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w} \\ &+ \int_{R \setminus R_\varepsilon} |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w} < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Thus $F \in Q_{p,0}(R)$ for $0 < p < 1$.

(ii) Because of the nesting property in Theorem 2.5(ii) it is enough to prove the assertion for $1 < p < \infty$, and then we can follow the proof of Theorem 1(b) in [1] by noticing that, by Hölder's inequality,

$$\left(\int_{V_{r_1 r_2}} g_R(w,\alpha)\, dw\, d\overline{w}\right)^p \leq \left(\pi(r_2^2 - r_1^2)\right)^{p-1} \int_{V_{r_1 r_2}} g_R^p(w,\alpha)\, dw\, d\overline{w},$$

where $V_{r_1 r_2} = \{w : r_1 < |w - \alpha| < r_2\}$ is a part of the parameter disk. We omit the details here. ∎

DEFINITION 3.3. Let $E(\zeta) = \sum_{n=1}^\infty a_n \zeta^n$ be an entire function with $a_n \geq 0$. We define

$$Q_E(R) = \left\{F \in A(R) : \sup_{\alpha \in R} \int_R |F'(w)|^2 E\big(g_R(w,\alpha)\big)\, dw\, d\overline{w} < \infty\right\}$$

and

$$Q_{E,0}(R) = \left\{F \in A(R) : \lim_{\alpha \to \partial R} \int_R |F'(w)|^2 E\big(g_R(w,\alpha)\big)\, dw\, d\overline{w} = 0\right\}.$$

THEOREM 3.4. *Let $E(\zeta) = \sum_{n=1}^\infty a_n \zeta^n$ be an entire function with $a_n \geq 0$ and $a_1 > 0$. If its growth order ρ and type σ satisfy one of the following conditions:*

(i) $\rho = 1$, $\sigma < 2$, or

(ii) $\rho > 1$, σ arbitrary, then $\mathrm{BMOA}(R) = Q_E(R)$ *and* $\mathrm{VMOA}(R) = Q_{E,0}(R)$.

PROOF. Since $a_1 > 0$ and $a_n \geq 0$, it is obvious that $Q_E(R) \subseteq \mathrm{BMOA}(R)$ and $Q_{E,0}(R) \subseteq \mathrm{VMOA}(R)$. For the converse, we use (2.8) for $p = 1$ and q a positive integer n, and get

$$\begin{aligned} I_E(\alpha) &= \int_R |F'(w)|^2 E\big(g_R(w,\alpha)\big)\, dw\, d\overline{w} \\ &= \sum_{n=1}^\infty a_n \int_R |F'(w)|^2 g_R^n(w,\alpha)\, dw\, d\overline{w} \\ &\leq 4e^2 \int_R |F'(w)|^2 g_R(w,\alpha)\, dw\, d\overline{w} \sum_{n=1}^\infty A_n, \end{aligned}$$

where $A_n = a_n\Gamma(n+1)/2^n$. Similar to the proof of [3, Theorem 1.1] it is not hard to show that $\sum_{n=1}^{\infty} A_n$ is convergent under the condition (i) or (ii). Therefore we have

$$\int_R |F'(w)|^2 E\big(g_R(w,\alpha)\big)\,dw\,d\overline{w} \le M\int_R |F'(w)|^2 g_R(w,\alpha)\,dw\,d\overline{w},$$

where $M > 0$ is a constant. Thus, by definition, we get $\mathrm{BMOA}(R) \subseteq Q_E(R)$ and $\mathrm{VMOA}(R) \subseteq Q_{E,0}(R)$, and the proof is completed. ■

COROLLARY 3.5. *Let $0<\beta<2$ and let $F \in A(R)$. Then $F \in \mathrm{BMOA}(R)$ if and only if for every $\alpha \in R$ and every $t > 0$, there is a constant $K > 0$ such that*

$$\int_{R_{t,\alpha}} |F'(w)|^2 g_R(w,\alpha)\,dw\,d\overline{w} \le Ke^{-\beta t}, \tag{3.10}$$

where $R_{t,\alpha} = \{w \in R : g_R(w,\alpha) > t\}$.

PROOF. Assume that $F \in \mathrm{BMOA}(R)$. Let $E_\beta(\zeta) = \zeta e^{\beta\zeta} = \sum_{n=1}^{\infty} \beta^{n-1}\zeta^n/(n-1)!$. Then it is easy to check that the growth order ρ and type σ of the entire function $E_\beta(\zeta)$ satisfy $p = 1$ and $\sigma = \beta < 2$. Thus, by Theorem 3.4, for every $\alpha \in R$ and every $t > 0$,

$$\begin{aligned} e^{\beta t}\int_{R_{t,\alpha}} |F'(w)|^2 g_R(w,\alpha)\,dw\,d\overline{w} &\le \int_{R_{t,\alpha}} |F'(w)|^2 g_R(w,\alpha) e^{\beta g_R(w,\alpha)}\,dw\,d\overline{w} \\ &\le \int_R |F'(w)|^2 E_\beta\big(g_R(w,\alpha)\big)\,dw\,d\overline{w} \le K < \infty. \end{aligned}$$

Hence

$$\int_{R_{t,\alpha}} |F'(w)|^2 g_R(w,\alpha)\,dw\,d\overline{w} \le Ke^{-\beta t}.$$

On the contrary, if F satisfies (3.10), we let t tend to 0 and get

$$\sup_{\alpha\in R}\int_R |F'(w)|^2 g_R(w,\alpha)\,dw\,d\overline{w} \le \lim_{t\to 0} Ke^{-\beta t} = K < \infty.$$

Thus $F \in \mathrm{BMOA}(R)$ and the proof is completed. ■

4. The Bloch space and $Q_p(R)$.

In this section we study the relationship between the spaces $B(R)$, $CB(R)$ and $Q_p(R)$ for $0 < p < \infty$. Since in [5] the theorems below of this section have been proved in a special case for parameter value $p = 2$, we will not give the proofs in a detailed way. We first draft the proof of the following result.

THEOREM 4.1. *Let $0 < p < \infty$. Then*
(i) $Q_p(R) \subseteq CB(R)$,
(ii) $Q_{p,0}(R) \subseteq CB_0(R)$.

PROOF. Because of the nesting property for the spaces $Q_p(R)$ in Theorem 2.5 and by Theorem 7.7 in [5] we need only consider parameter values $1 < p < \infty$. But in this case our proof differs from the proof of Theorem 7.10 in [5] for a special case $p = 2$ only in a few points which we now show. First replacing $R_{1,\alpha,\varepsilon}$ by $R_{\alpha,\varepsilon} = R \setminus B_\varepsilon(\alpha)$ and letting ε tend to 0 we get

$$\int_R |F'(w)|^2 g_R^p(w,\alpha)\,dw\,d\overline{w} = \frac{p(p-1)}{2}\int_R |F(w)-F(\alpha)|^2 g_R^{p-2}(w,\alpha)|\nabla g_R(w,\alpha)|^2\,dw\,d\overline{w}. \tag{4.1}$$

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Thus we need replace $|\nabla g_R(w,\alpha)|^2$ by $g_R^{p-2}(w,\alpha)|\nabla g_R(w,\alpha)|^2$ and then consider the integral $\int_0^\infty H_{t,\alpha}(\alpha)t^{p-2}\,dt$ instead of $\int_0^\infty H_{t,\alpha}(\alpha)\,dt$. By these changes using the same inequality

$$\left(\frac{|F'(\alpha)|}{c_t(\alpha)}\right)^2 \le H_{t,\alpha}(\alpha)$$

for the capacity density $c_t(\alpha)$ of $R_{t,\alpha}$ at α as in the proof of [5, Theorem 7.10] we get the inequality

$$\int_R |F(w)-F(\alpha)|^2 g_R^{p-2}(w,\alpha)|\nabla g_R(w,\alpha)|^2\,dw\,d\bar{w} \ge 2^{2-p}\Gamma(p-1)\pi\left(\frac{|F'(\alpha)|}{c_R(\alpha)}\right)^2 \tag{4.2}$$

which proves the theorem. ∎

COROLLARY 4.2. *Let $0<p<\infty$. Then*
(i) $Q_p(R)\subseteq B(R)$,
(ii) $Q_{p,0}(R)\subseteq B_0(R)$.

This is obvious since $CB(R)\subseteq B(R)$ and $CB_0(R)\subseteq B_0(R)$ ([5, Theorem 7.1]).

THEOREM 4.3. *There exist Riemann surfaces R_1 and R_2 for which $Q_p(R_1)\neq B(R_1)$ and $Q_{p,0}(R_2)\neq B_0(R_2)$ for any p, $0<p<\infty$.*

PROOF. By obvious changes the proofs are the same as in [5, Theorem 4.2] and [5, Theorem 5.4], respectively. ∎

Next we give a sufficient condition for which $Q_p(R)=B(R)$ for $1<p<\infty$. To this end, we define

$$C(R)=\sup\left\{\frac{d_R(w,\alpha)}{l_R(w,\alpha)} : w,\alpha\in R\right\},$$

where $d_R(w,\alpha)$ is the hyperbolic distance between w and α and

$$l_R(w,\alpha)=\frac{1}{2}\log\left(\frac{\exp\bigl(g_R(w,\alpha)\bigr)+1}{\exp\bigl(g_R(w,\alpha)\bigr)-1}\right).$$

We note that $C(R)\ge 1$ and the equality holds if and only if R is simply connected.

THEOREM 4.4. *If $C(R)<\infty$, then $Q_p(R)=B(R)$ for all $1<p<\infty$.*

PROOF. If $g_k(w,\alpha)$ is a Green's function of R_k in a regular exhaustion $\{R_k\}$ of R, we denote $h_k(w,\alpha)=g_k(w,\alpha)+ig_k^*(w,\alpha)$. The similar notation is introduced for $l_k(w,\alpha)$. Following the proof of Theorem 4.4 in [5] we get

$$\begin{aligned}
&\int_{R_k}\bigl(l_k(w,\alpha)\bigr)^2 g_k^{p-2}(w,\alpha)|h_k'(w,\alpha)|^2\,dw\,d\bar{w}\\
&\qquad=\frac{1}{2}\int_0^\infty\left(\int_{S_{t,\alpha,k}}\frac{\partial g_k(w,\alpha)}{\partial n}\,ds\right)\left(\log\frac{e^t+1}{e^t-1}\right)^2 t^{p-2}\,dt\\
&\qquad=\pi\int_0^\infty\left(\log\frac{e^t+1}{e^t-1}\right)^2 t^{p-2}\,dt\\
&\qquad=4\pi\int_0^\infty t^{p-2}e^{-2t}\bigl(1+O(e^{-2t})\bigr)\,dt=K<\infty,
\end{aligned} \tag{4.3}$$

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where $S_{t,\alpha,k} = \{w \in R_k : g_k(w,\alpha) = t\}$. From (4.3) and the proof of [5, Theorem 4.4] we conclude that $\mathcal{B}(R) \subseteq Q_p(R)$ for all $1 < p < \infty$. In Corollary 4.2 we have shown $Q_p(R) \subseteq \mathcal{B}(R)$, and thus the theorem is proved. ∎

Next we consider the relations between $Q_E(R)$ and $\mathcal{B}(R)$ ($C\mathcal{B}(R)$). Note that in the following we do not restrict $a_1 > 0$.

THEOREM 4.5. *Let $E(\zeta) = \sum_{n=1}^{\infty} a_n \zeta^n$ be an entire function with $a_n \geq 0$. If its growth order ρ and type σ satisfy one of the following conditions:*

(i) $\rho = 1$, $\sigma < 2$, or

(ii) $\rho < 1$, σ arbitrary, then $Q_E(R) \subseteq C\mathcal{B}(R)$ and $Q_{E,0}(R) \subseteq C\mathcal{B}_0(R)$.

PROOF. Using (4.1) and (4.2), we get

$$\int_R |F'(w)|^2 E\big(g_R(w,\alpha)\big)\, dw\, d\overline{w} = \sum_{n=1}^{\infty} a_n \int_R |F'(w)|^2 g_R^n(w,\alpha)\, dw\, d\overline{w}$$
$$\geq 2\pi \sum_{n=1}^{\infty} A_n \left(\frac{|F'(\alpha)|}{c_R(\alpha)}\right)^2,$$

where $A_n = a_n \Gamma(n+1)/2^n$. As before, if (i) or (ii) is satisfied, then $\sum_{n=1}^{\infty} A_n = M < \infty$. Thus

$$\int_R |F'(w)|^2 E\big(g_R(w,\alpha)\big)\, dw\, d\overline{w} \geq 2\pi M \left(\frac{|F'(\alpha)|}{c_R(\alpha)}\right)^2.$$

Both inclusions $Q_E(R) \subseteq C\mathcal{B}(R)$ and $Q_{E,0}(R) \subseteq C\mathcal{B}_0(R)$ follow from this inequality. ∎

COROLLARY 4.6. *Under the same conditions as in Theorem 4.5, we have $Q_E(R) \subseteq \mathcal{B}(R)$ and $Q_{E,0}(R) \subseteq \mathcal{B}_0(R)$.*

5. $Q_p(R)$ as a Banach space.

The main result of this section is the following.

THEOREM 5.1. *Let R be a Riemann surface, $R \notin O_G$, and let $0 < p < \infty$. Then $Q_p(R)$ is a Banach space with the norm*

$$\|F\| = |F(\alpha_0)| + \left(\sup_{\alpha \in R} \int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w}\right)^{1/2}, \quad \alpha_0 \in R,$$

and the point evaluation is a continuous functional on $Q_p(R)$.

PROOF. Suppose $0 < p < \infty$. It is easy to check that $\|\cdot\|$ is a norm. For $F \in Q_p(R)$ let

$$I_p(\alpha) = \int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\overline{w}.$$

From (4.1), (4.2) and the fact $c_R(\alpha) \leq \lambda_R(\alpha)$ for every $\alpha \in R$ (*cf.* [11, Theorem 2]), we have

$$\left(\frac{|F'(\alpha)|}{\lambda_R(\alpha)}\right)^2 \leq \left(\frac{|F'(\alpha)|}{c_R(\alpha)}\right)^2 \leq C I_p(\alpha),$$

where $C > 0$ is a constant independent of F. Let $\pi\colon \Delta \longrightarrow R$ be the universal covering mapping such that $\pi(0) = \alpha_0$, and let $f = F \circ \pi$. Then it is well known that $f \in B(\Delta)$ and for every $z \in \Delta$,

$$|f(z)| \leq |f(0)| + M(z)\|f\|_{B(\Delta)},$$

where $M(z)$ is a constant depending on z. Thus, by $f = F \circ \pi$ and $\|F\|_{B(R)} = \|f\|_{B(\Delta)}$, we get

$$|F(w)| \leq |F(\alpha_0)| + M(w)\|F\|_{B(R)} \leq |F(\alpha_0)| + C^{1/2}M(w)\Big(\sup_{\alpha\in R} I_p(\alpha)\Big)^{1/2} \leq \tilde{M}(w)\|F\|,$$

where $M(w)$ and $\tilde{M}(w)$ are constants depending on w. Thus the point evaluation is a continuous functional with respect to $\|\cdot\|$. By a standard argument, we can prove that $Q_p(R)$ is a Banach space under the norm $\|\cdot\|$ (*cf.*, for example, the proof of Theorem 2.10 in [15]). ∎

THEOREM 5.2. *Let R be a Riemann surface, let $R \notin O_G$ and let $0 < p < \infty$. Then $Q_{p,0}(R)$ is a closed subspace of $Q_p(R)$.*

PROOF. By the same method as in the proof of Theorem 3.1 in [5], we can prove that $Q_{p,0}(R) \subseteq Q_p(R)$ for $0 < p < \infty$. Since the point evaluation is a continuous linear functional on $Q_p(R)$, we can prove by a standard argument that $Q_{p,0}(R)$ is a closed subspace of $Q_p(R)$ (*cf.*, for example, [15, Theorem 2.15]). We omit the details here. ∎

To close this section, we give a characterization of $Q_p(R)$ by regular exhaustions.

THEOREM 5.3. *Let R be a Riemann surface, let $R \notin O_G$, let $\{R_k\}$ be a regular exhaustion of R, and let $F \in A(R)$. If we denote*

$$\|F\|_k^2 = \sup_{\alpha\in R_k} \int_{R_k} |F'(w)|^2 g_k^p(w,\alpha)\, dw\, d\bar{w},$$

where $g_k(w,\alpha)$ is the Green's function on R_k, then for $0 < p < \infty$,

$$\|F\|_{Q_p(R)}^2 = \lim_{k\to\infty} \|F\|_k^2.$$

PROOF. It is easy to see that $\{\|F\|_k^2\}$ is increasing with respect to k and $\|F\|_k^2 \leq \|F\|_{Q_p(R)}^2$. Thus

$$\|F\|_{Q_p(R)}^2 \geq \lim_{k\to\infty} \|F\|_k^2.$$

On the contrary, since

$$\|F\|_{Q_p(R)}^2 = \sup_{\alpha\in R} \int_R |F'(w)|^2 g_R^p(w,\alpha)\, dw\, d\bar{w},$$

we know that there is a sequence of points $\{\alpha_n\}$ in R such that

$$\|F\|_{Q_p(R)}^2 = \lim_{n\to\infty} \int_R |F'(w)|^2 g_R^p(w,\alpha_n)\, dw\, d\bar{w}.$$

For every $\alpha \in R$ and every $k > 1$, let

$$\tilde{g}_k(w,\alpha) = \begin{cases} g_k(w,\alpha), & w \in R_k, \\ 0, & w \in R \setminus R_k. \end{cases}$$

Then, from $\lim_{k\to\infty} g_k(w,\alpha) = g_R(w,\alpha)$, we know

$$\lim_{k\to\infty} \tilde{g}_k(w,\alpha) = g_R(w,\alpha).$$

Let n be an arbitrary positive integer. Because $\{R_k\}$ is a regular exhaustion of R, there is a k_n such that $\alpha_n \in R_{k_n}$. Thus for every $k \geq k_n$, $\alpha_n \in R_k \subseteq R$, and so

$$\int_R |F'(w)|^2 \tilde{g}_k^p(w,\alpha_n)\, dw\, d\bar{w} \leq \sup_{\alpha\in R_k} \int_R |F'(w)|^2 \tilde{g}_k^p(w,\alpha)\, dw\, d\bar{w}.$$

Then, by Fatou's Lemma,

$$\begin{aligned} \int_R |F'(w)|^2 g_R^p(w,\alpha_n)\, dw\, d\bar{w} &= \int_R |F'(w)|^2 \lim_{k\to\infty} \tilde{g}_k^p(w,\alpha_n)\, dw\, d\bar{w} \\ &\leq \lim_{k\to\infty} \int_R |F'(w)|^2 \tilde{g}_k^p(w,\alpha_n)\, dw\, d\bar{w} \\ &\leq \lim_{k\to\infty} \sup_{\alpha\in R_k} \int_R |F'(w)|^2 \tilde{g}_k^p(w,\alpha)\, dw\, d\bar{w} \\ &= \lim_{k\to\infty} \sup_{\alpha\in R_k} \int_{R_k} |F'(w)|^2 \tilde{g}_k^p(w,\alpha)\, dw\, d\bar{w} \\ &= \lim_{k\to\infty} \|F\|_k^2. \end{aligned}$$

Since the right hand side is independent of n, we have

$$\|F\|_{Q_p(R)}^2 = \lim_{n\to\infty} \int_R |F'(w)|^2 g_R^p(w,\alpha_n)\, dw\, d\bar{w} \leq \lim_{k\to\infty} \|F\|_k^2.$$

The proof is complete. ■

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