



On Intrinsic Quadrics

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Abstract. An intrinsic quadric is a normal projective variety with a Cox ring defined by a single quadratic relation. We provide explicit descriptions of these varieties in the smooth case for small Picard numbers. As applications, we figure out in this setting the Fano examples and (affirmatively) test Fujita's freeness conjecture.

1 Introduction

Intrinsic quadrics were introduced in [5] as an example class of normal, projective, algebraic varieties that are accessible by elementary combinatorial methods similar to toric varieties. Recall that the normal projective toric varieties X are characterized by the property that their divisor class group $\text{Cl}(X)$ is finitely generated and their Cox ring

$$\mathcal{R}(X) = \bigoplus_{\text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D))$$

is a polynomial ring. An *intrinsic quadric* is by definition a normal projective variety X with finitely generated divisor class group $\text{Cl}(X)$ and a finitely generated Cox ring admitting homogeneous generators such that the associated ideal of relations is generated by a single, purely quadratic polynomial. In that sense, studying intrinsic quadrics is a quite moderate step beyond toric geometry. Some well known non-toric examples are the usual smooth quadrics $X \subseteq \mathbb{P}_n$ for $n \geq 4$ and several cubic surfaces in \mathbb{P}_3 . We refer the reader to [7] for a sample use of intrinsic quadrics as a testing class.

In this article, we take a closer look at smooth intrinsic quadrics of small Picard number, but arbitrarily high dimension. For toric varieties, the analogous idea has been pursued by Kleinschmidt [19] in Picard number two and by Batyrev [4] in Picard number three. Moreover, in [13], we described all smooth, rational varieties of Picard number two that come with a torus action of complexity one. Similarly to the toric setting, where the restriction of being smooth of Picard number one allows just the projective spaces, the situation turns out to be simple for intrinsic quadrics: in Picard number one, we only find the classical smooth quadrics $X \subseteq \mathbb{P}_n$; see Proposition 3.1. In Picard number two, we obtain a considerably larger class. The first main result of the paper provides a full description of these varieties X in terms of their $\text{Cl}(X)$ -graded Cox ring and the semiample cone $\tau_X \subseteq \text{Cl}_{\mathbb{Q}}(X)$. This collection of data indeed fixes X ; see Section 2 for a brief reminder and [3] for more background.

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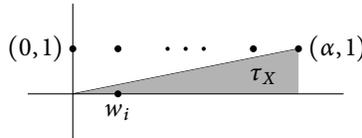
Theorem 1.1 *Let X be a smooth intrinsic quadric of Picard number $\rho(X) = 2$. Then X has divisor class group $\text{Cl}(X) \cong \mathbb{Z}^2$ and, with suitable integers $n, m \in \mathbb{Z}_{\geq 0}$, the Cox ring of X is given by*

$$\mathcal{R}(X) \cong \mathbb{K}[T_1, \dots, T_n, S_1, \dots, S_m] / \langle g \rangle,$$

$$g = \begin{cases} T_1 T_2 + \dots + T_{n-1} T_n, & n \text{ even,} \\ T_1 T_2 + \dots + T_{n-2} T_{n-1} + T_n^2, & n \text{ odd.} \end{cases}$$

The possible constellations for the $\text{Cl}(X)$ -grading of $\mathcal{R}(X)$ and the semiample cone $\tau_X \subseteq \text{Cl}_{\mathbb{Q}}(X)$ are listed below; we distinguish four types and write $w_i := \text{deg}(T_i)$ and $u_j := \text{deg}(S_j)$ for the $\text{Cl}(X)$ -degrees.

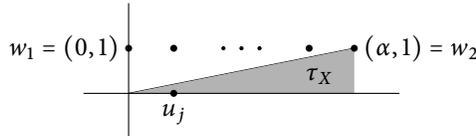
Type 1: Fix $\alpha \in \mathbb{Z}_{\geq 0}$. We have $n \geq 5$ and $m \geq 2$. Moreover, $w_1 = \dots = w_n = (1, 0)$ and $u_j = (a_j, 1)$ with $0 = a_1 \leq a_2 \leq \dots \leq a_m = \alpha$ holds.



Here, X is the projectivization $\mathbb{P}(\mathcal{O}_Y(a_1) \oplus \mathcal{O}_Y(a_2) \oplus \dots \oplus \mathcal{O}_Y(a_m))$ of the split vector bundle defined by a_1, \dots, a_m over the smooth quadric $Y = V(g) \subseteq \mathbb{P}_{n-1}$.

Type 2: Fix $\alpha \in \mathbb{Z}_{\geq 0}$. We have $n \geq 5$ and $m \geq 2$. Moreover, $u_1 = \dots = u_m = (1, 0)$ holds and we have $w_i = (a_i, 1)$ with $0 \leq a_i \leq \alpha$ for $i = 1, \dots, n$ such that

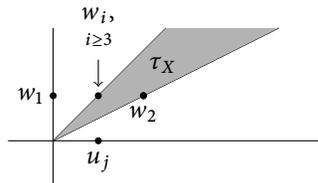
- (a) $w_1 = (0, 1)$ and $w_2 = (\alpha, 1)$,
- (b) $w_i + w_{i+1} = (\alpha, 2)$ for all odd $i < n$ and $2w_n = (\alpha, 2)$, if n is odd.



Here, X admits a locally trivial fibration $X \rightarrow \mathbb{P}_{m-1}$ with fibers isomorphic to the smooth quadric $V(g) \subseteq \mathbb{P}_{n-1}$.

Type 3: We have $n \geq 5$ and $m \geq 1$. Moreover, $u_1 = \dots = u_m = (1, 0)$ holds and the w_i satisfy

- (a) $w_1 = (0, 1)$ and $w_2 = (2, 1)$,
- (b) $w_3 = \dots = w_n = (1, 1)$.



Here, X is the blowing-up of the projective space \mathbb{P}_{n+m-3} centered at the smooth quadric $V(g - T_1 T_2, S_1, \dots, S_m) \subseteq \mathbb{P}_{n+m-3}$.

Note that in Theorem 1.1, the variety X is of dimension $n + m - 3$. Thus, the above table provides us in particular for every dimension with the numbers of (almost) Fano smooth intrinsic quadrics of Picard number two. The overlap with the classification of smooth Fano threefolds by Mori and Mukai consists of the threefold Type 3 with $n = 5, m = 1$, and the threefold of Type 4 with $n = 6, m = 0$, which occur as No. 2.30 and No. 2.32 in [20], respectively.

For a Fano, not necessarily smooth, full intrinsic quadric X , we see in Proposition 5.1 that its Picard number is bounded by $\rho(X) \leq 3$. Moreover, if X is smooth, then we can further show $\rho(X) \leq 2$ and arrive at the following theorem.

Theorem 1.3 *Let X be a Fano smooth full intrinsic quadric. Then X is of Picard number $\rho(X) \leq 2$ and*

- (i) *if $\rho(X) = 1$, X is isomorphic to the smooth projective quadric $V(T_0^2 + \dots + T_n^2) \subseteq \mathbb{P}_n$, where $n \geq 4$;*
- (ii) *if $\rho(X) = 2$, X is isomorphic to $V(T_0S_0 + \dots + T_nS_n) \subseteq \mathbb{P}_n \times \mathbb{P}_n$, the flag variety of type $(1, n - 1, 1)$, where $n \geq 2$.*

We use our results to test Fujita’s freeness conjecture, which says that for any smooth projective variety X with canonical divisor C_X , the divisor $C_X + sD$ is base point free provided that D is ample and $s \geq \dim(X) + 1$ holds; see [15]. This statement is known to hold for varieties with torus action of complexity at most one [1, 14] and in general up to dimension five [11, 18, 22, 24]. Corollary 4.4 verifies Fujita’s freeness conjecture for smooth intrinsic quadrics of Picard number at most two.

We turn to Picard number three. Recall that smooth toric varieties of Picard number three have been described by Batyrev in [4] in terms of primitive collections. In the setting of intrinsic quadrics, we obtain a complete picture in the full case.

Theorem 1.4 *Let X be a full smooth intrinsic quadric of Picard number three. Then X has divisor class group $\text{Cl}(X) = \mathbb{Z}^3$, and, with a suitable even integer $n \geq 8$, the Cox ring of X is given by*

$$\mathcal{R}(X) \cong \mathbb{K}[T_1, \dots, T_n]/\langle g \rangle, \quad g := T_1T_2 + \dots + T_{n-1}T_n.$$

The possible constellations for the $\text{Cl}(X)$ -gradings and the ample cone of X are the following. There is an integer $a \geq 0$ such that the degree of the relation is $\mu := \deg(g) = (1, a, 1)$, and we have

$$\begin{aligned} w_1 = w_3 = (0, 1, 0), \quad w_2 = w_4 = (1, a - 1, 1), \\ w_5 = (0, 0, 1), \quad w_6 = (1, a, 0), \quad w_7 = (1, 0, 0), \quad w_8 = (0, a, 1). \end{aligned}$$

For all odd $9 \leq i < n$, the generator degrees w_i, w_{i+1} coincide either with w_1, w_2 or are located on the line segments $\text{conv}(w_5, w_8)$ and $\text{conv}(w_6, w_7)$.

Moreover, as indicated in Figure 1, the semiample cone $\tau_X \subseteq \text{Cl}(X)_{\mathbb{Q}}$ of X is given as the intersection of two cones:

$$\tau_X = \text{cone}(w_1, w_2, w_6) \cap \text{cone}(w_1, w_6, w_8).$$

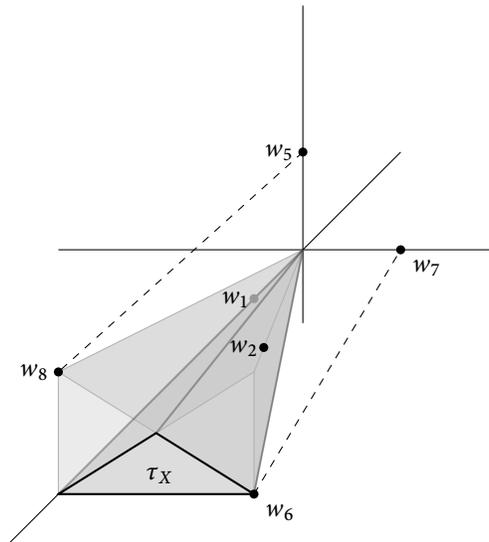


Figure 1

Conversely, each of the above constellations defines a full smooth intrinsic quadric X of Picard number 3. Moreover, each such X admits a locally trivial fibration with fibers a projective space onto a smooth projective toric variety of Picard number two.

Again, we use this description to verify Fujita’s freeness conjecture for full smooth intrinsic quadrics of Picard number three; see Corollary 6.1. As soon as we leave the full case, the situation in Picard number three becomes much more ample; complete descriptions in the dimensions three and four have been elaborated in [12].

2 Basics on Intrinsic Quadrics

We first discuss purely quadratic polynomials in general and present a graded normal form in Proposition 2.1. Then we provide a quick guide to the general combinatorial theory of [3, Chap. 3] adapted to the sample class of intrinsic quadrics. This allows us in particular to encode and read off the necessary geometric properties.

Throughout the whole article, we work over an algebraically closed field \mathbb{K} of characteristic zero. A grading of a \mathbb{K} -algebra R by a finitely generated abelian group K is a direct sum decomposition

$$R = \bigoplus_{w \in K} R_w$$

into vector subspaces $R_w \subseteq R$ being compatible with multiplication in the sense that $R_w R_{w'} \subseteq R_{w+w'}$ holds for all $w, w' \in K$. A homomorphism of graded algebras $R = \bigoplus_K R_w$ and $S = \bigoplus_L S_u$ is a pair (ψ, F) consisting of an algebra homomorphism $\psi: R \rightarrow S$ and a group homomorphism $F: K \rightarrow L$ such that one always has $\psi(R_w) \subseteq S_{F(w)}$.

In this situation, we speak of a graded homomorphism if $K = L$ holds and F is the identity map.

Proposition 2.1 *Let K be a finitely generated abelian group, consider a K -grading on the polynomial ring $\mathbb{K}[T_1, \dots, T_s]$ such that the variables T_1, \dots, T_s and the following quadratic polynomial are K -homogeneous:*

$$g = \sum_{1 \leq i < j \leq s} a_{ij} T_i T_j \in \mathbb{K}[T_1, \dots, T_s].$$

Then there are a linear automorphism $\psi: \text{lin}(T_1, \dots, T_s) \rightarrow \text{lin}(T_1, \dots, T_s)$ inducing a graded automorphism $\Psi: \mathbb{K}[T_1, \dots, T_s] \rightarrow \mathbb{K}[T_1, \dots, T_s]$ and non-negative integers q, t with $q + t \leq s$, such that

$$\Psi(g) = g_{q,t} := T_1 T_2 + \dots + T_{q-1} T_q + T_{q+1}^2 + \dots + T_{q+t}^2$$

and $\deg(T_{q+k}) \neq \deg(T_{q+l})$ holds for all $0 < k < l < t$. In this setting, $s - q - t$ is the dimension of the singular locus of $V(g_{q,t}) \subseteq \mathbb{K}^s$ and t counts the $u \in K$ with $2u = \deg(g_{q,t})$ such that the number of T_i of degree u showing up in $g_{q,t}$ is odd.

Proof Suitably renumbering the variables, we can assume that T_1, \dots, T_r are precisely the variables that show up in g . Let $w_1, \dots, w_n \in K$ be the degrees of T_1, \dots, T_r ; we impose $w_k \neq w_l$ for $k \neq l$ here. Moreover, set $\mu := \deg(g) \in K$. Further suitable renumbering of variables yields

$$w_1 + w_2 = \dots = w_m + w_{m+1} = \mu, \quad 2w_{m+2} = \dots = 2w_n = \mu$$

with a unique odd number $-1 \leq m < n$. Some of the variables T_1, \dots, T_s may share the same degree, and we have

$$V := \text{lin}(T_1, \dots, T_s) = V_1 \oplus \dots \oplus V_n \oplus V_0,$$

where V_k is the linear subspace generated by all $T_i, 1 \leq i \leq r$, of degree w_k , and V_0 is the linear subspace generated by the variables T_{r+1}, \dots, T_s . Suitably renumbering the T_i again, we achieve

$$T_1, \dots, T_{d_1} \in V_1, \quad \dots \quad T_{d_{n-1}+1}, \dots, T_{d_n} \in V_n, \quad T_{d_{n+1}}, \dots, T_s \in V_0.$$

The idea is to build up ψ stepwise from appropriate endomorphisms $V \rightarrow V$. First, consider variables $T_i \in V_1$ and $T_j \in V_2$ with $a_{ij} \neq 0$. Define a linear automorphism

$$\psi_{ij}: V \longrightarrow V, \quad T_j \longmapsto a_{ij}^{-1} T_j - a_{ij}^{-1} \sum_{k \neq j} a_{ik} T_k, \quad T_l \longmapsto T_l \text{ for } l \neq j.$$

Then ψ_{ij} respects the direct sum decomposition of V and restricts to the identity on all components different from V_2 . Moreover, ψ_{ij} extends to an automorphism Ψ_{ij} of the K -graded algebra $\mathbb{K}[T_1, \dots, T_s]$, and we have

$$\Psi_{ij}(g) = \left(T_i + \sum_{k \neq i} a_{ij}^{-1} a_{kj} T_k \right) T_j + \sum_{k \neq i, l \neq j} \tilde{a}_{kl} T_k T_l$$

with some $\tilde{a}_{kl} \in \mathbb{K}$. Now define a linear automorphism

$$\psi_{ji}: V \longrightarrow V, \quad T_i \longmapsto T_i - a_{ij}^{-1} \sum_{k \neq i} a_{kj} T_k, \quad T_l \longmapsto T_l \text{ for } l \neq i.$$

Similarly as before, ψ_{ji} respects the direct sum decomposition of V and restricts to the identity on all components different from V_1 . Again, ψ_{ji} extends to an automorphism Ψ_{ji} of the K -graded algebra $\mathbb{K}[T_1, \dots, T_s]$. This time we have

$$\Psi_{ji}(\Psi_{ij}(g)) = T_i T_j + \sum_{k \neq i, l \neq j} \tilde{a}_{kl} T_k T_l.$$

Thus, a suitable composition of the automorphisms $\Psi_{ji} \circ \Psi_{ij}$ turns g into the desired form with respect to the variables from V_1 and V_2 . Proceeding similarly, we can settle all other pairs V_l and V_{l+1} for $l = 3, 5, \dots, m$.

On each subspace V_k for $k > m + 1$, the variables all have the same K -degree and, if a variable of a given monomial of g belongs to V_k , then all variables of this monomial belong to V_k . Thus, we can treat the part q_k of q built from variables of V_k separately. The usual diagonalization procedure for the Gram matrix of q_k leads to a presentation of q_k as a sum of squares. If the number c_k of these squares is even, then we turn the whole q_k into a sum of terms $T_i T_j$ with $i \neq j$. Otherwise, we turn q_k into a sum of $T_i T_j$ with $i \neq j$ plus one single square. ■

We call $g_{q,t} \in \mathbb{K}[T_1, \dots, T_s]$ as in Proposition 2.1 a *standard K -homogeneous quadratic polynomial*. As the supplement of the proposition shows, a given standard K -homogeneous quadratic polynomial $g_{q,t}$ can be transformed via an automorphism of graded algebras into another one, say $g_{q',t'}$, if and only if $q = q'$ and $t = t'$ hold. If for some $g_{q,t}$ the sum $q + t$ is odd, then we must have $t \geq 1$. Let us briefly discuss what happens if $t > 1$ holds.

Remark 2.2 Let $g_{q,t} \in \mathbb{K}[T_1, \dots, T_s]$ be a standard K -homogeneous quadratic polynomial with $t > 1$. Then, for any two $1 \leq i < j \leq t$, twice the degree of T_{q+i} as well as twice the degree of T_{q+j} equal the degree of $g_{q,t}$ and thus we have

$$2(\deg(T_{q+i}) - \deg(T_{q+j})) = 0 \in K.$$

In particular, the number t is bounded by the order of the subgroup $K_2 \subseteq K$ consisting of all elements annihilated by multiplication with 2. Here is a concrete example: Take

$$K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \quad g = T_1^2 + T_2^2 \in \mathbb{K}[T_1, T_2].$$

Define a K -grading on $\mathbb{K}[T_1, T_2]$ by setting $\deg(T_1) := (\bar{1}, \bar{0})$ and $\deg(T_2) := (\bar{0}, \bar{1})$. Then $g = g_{0,2}$ is a standard K -homogeneous quadratic polynomial in $\mathbb{K}[T_1, T_2]$.

We turn to the construction of intrinsic quadrics. Recall that every Mori dream space, that means, every irreducible, normal, projective variety X with finitely generated divisor class group $\text{Cl}(X)$ and finitely generated Cox ring

$$\mathcal{R}(X) = \bigoplus_{\text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D))$$

can be retrieved from $\mathcal{R}(X)$ as follows. The above grading defines an action of the quasitorus $H = \text{Spec } \mathbb{K}[\text{Cl}(X)]$ on the total coordinate space $\bar{X} = \text{Spec } \mathcal{R}(X)$. If $u \in \text{Cl}(X)$ is any ample class of X , then the associated set of semistable points is

$$\bar{X}^{ss}(u) = \{x \in \bar{X}; f(x) \neq 0 \text{ for some } f \in \mathcal{R}(X)_{nu}, \text{ where } n > 0\} \subseteq \bar{X}.$$

This is an open H -invariant set and the variety X is obtained as the associated geometric invariant theory quotient $X = \overline{X}^{ss}(u) // H$. We refer the reader to [3] for more background.

Reversing the picture just drawn, we can produce all Mori dream spaces from suitable finitely generated, normal, integral, K -graded \mathbb{K} -algebras

$$R = \bigoplus_{w \in K} R_w,$$

where “suitable” characterizes the Cox rings among these algebras. Let us briefly recall from [3] what that means. First, R has to be K -factorial in the sense that we have unique factorization in the multiplicative monoid $R_\times \subseteq R$ of non-zero homogeneous elements of R ; for instance, R can be a unique factorization domain in the classical sense. For the further conditions, fix any system f_1, \dots, f_s of pairwise non-associated K -prime, i.e., prime in R_\times , generators of R and consider the (convex, polyhedral) cones

$$\kappa_0 := \text{cone}(\deg(f_1), \dots, \deg(f_s)), \quad \kappa_1 := \bigcap_{i=1}^s \text{cone}(\deg(f_j); j \neq i)$$

in the rational vector space $K_{\mathbb{Q}} = K \otimes_{\mathbb{Z}} \mathbb{Q}$ associated with K . Then we ask the K -grading to be pointed in the sense that $R_0 = \mathbb{K}$ holds and the weight cone κ_0 contains no lines. Moreover, the K -grading must be almost free in the sense that any $s - 1$ of the $\deg(f_i)$ generate K as a group. Finally, the moving cone κ_1 has to be of full dimension in $K_{\mathbb{Q}}$.

Example 2.3 Let a finitely generated abelian group K and a pointed, almost free K -grading of the polynomial ring $\mathbb{K}[T_1, \dots, T_s]$ be given such that all variables T_i are K -homogeneous and the moving cone is of full dimension in $K_{\mathbb{Q}}$. Moreover, let $g_{q,t} \in \mathbb{K}[T_1, \dots, T_s]$ be a standard K -homogeneous quadratic polynomial and consider the factor algebra $R := \mathbb{K}[T_1, \dots, T_s] / \langle g_{q,t} \rangle$ with its induced K -grading.

(i) If $q + t \geq 5$ holds, then R is a unique factorization domain and the K -grading of R is factorial.

(ii) For $q + t < 5$, the ring is normal, integral with factorial K -grading if and only if $K = \mathbb{Z}^s / M$ and $\deg(T_i) = e_i + M$ hold, where M is the row space of an $r \times s$ matrix with $r < s$ of the following shape

$$\begin{aligned} q = 0, \quad t = 4: & \begin{bmatrix} -2 & 2 & 0 & 0 & 0 & \dots & 0 \\ -2 & 0 & 2 & 0 & 0 & \dots & 0 \\ -2 & 0 & 0 & 2 & 0 & \dots & 0 \\ d_1 & d_2 & d_3 & d_4 & d'_1 & \dots & d'_m \end{bmatrix}, \\ q = 2, \quad t = 2: & \begin{bmatrix} -1 & -1 & 2 & 0 & 0 & \dots & 0 \\ -1 & -1 & 0 & 2 & 0 & \dots & 0 \\ d_1 & d_2 & d_3 & d_4 & d'_1 & \dots & d'_m \end{bmatrix}, \\ q = 0, \quad t = 3: & \begin{bmatrix} -2 & 2 & 0 & 0 & \dots & 0 \\ -2 & 0 & 2 & 0 & \dots & 0 \\ d_1 & d_2 & d_3 & d'_1 & \dots & d'_m \end{bmatrix}. \end{aligned}$$

In Case (ii), the conditions “almost free”, “pointed” and “full-dimensional moving cone” on the K -grading mean that the columns of the listed matrices are pairwise

different primitive lattice points in \mathbb{Z}^r generating \mathbb{Q}^r as a cone. For the last two cases, the statement on K -factoriality follows from the results of [16] and for the first one, a proof in a more general framework will be presented elsewhere.

We are ready for explicit construction of intrinsic quadrics. The notation introduced in the subsequent two constructions will be used throughout the whole article.

Construction 2.4 (Standard intrinsic quadrics) Consider a pointed K -grading of the polynomial ring $\mathbb{K}[T_1, \dots, T_n, S_1, \dots, S_m]$, where K denotes a finitely generated abelian group and where all variables T_i and S_j are K -homogeneous, any $n + m - 1$ of their degrees generate K as a group and the moving cone is of full dimension in $K_{\mathbb{Q}}$. Moreover, let

$$g_{q,t} \in \mathbb{K}[T_1, \dots, T_n, S_1, \dots, S_m]$$

be a standard K -homogeneous quadratic polynomial with $3 \leq q + t = n$; thus, by choice of notation, $g_{q,t}$ depends precisely on the variables T_1, \dots, T_n . Assume that the K -grading is factorial; that means that Condition 2.3(i) or (ii) is satisfied. Take any $u \in K$ from the relative interior of the moving cone. Then we obtain a commutative diagram

$$\begin{array}{ccccc} V(g_{q,t}) & = & \overline{X} & \subseteq & \overline{Z} & = & \mathbb{K}^{n+m} \\ & & \cup & & \cup & & \\ & & \overline{X}^{ss}(u) & \subseteq & \overline{Z}^{ss}(u) & & \\ \parallel H \downarrow & & \downarrow & & \downarrow & \parallel H & \\ & & X & \longrightarrow & Z, & & \end{array}$$

where $H = \text{Spec } \mathbb{K}[K]$ is the quasitorus corresponding to K , the downwards arrows are the GIT-quotients defined by u and the bottom horizontal arrow is a closed embedding. Moreover, X and Z are normal projective varieties and we have

$$\dim(X) + 1 = \dim(Z) = n + m - \dim(K_{\mathbb{Q}}), \quad \text{Cl}(X) = \text{Cl}(Z) = K$$

for the respective dimensions and divisor class groups. Moreover, Z is a toric variety and we call $X = X(q, t, m, u)$ a *standard intrinsic quadric*. The Cox ring of X is given as K -graded factor algebra

$$\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_n, S_1, \dots, S_m] / \langle g_{q,t} \rangle.$$

By a *full intrinsic quadric*, we mean an intrinsic quadric with a defining quadratic polynomial g such that the normal form of g is $g_{q,t} \in \mathbb{K}[T_1, \dots, T_n]$ with $n = q + t$, that means that there are no free variables S_j .

By definition, intrinsic quadrics are normal projective varieties X admitting a presentation of the Cox ring by $\text{Cl}(X)$ -homogeneous generators such that the ideal of relations is generated by single, purely quadratic, $\text{Cl}(X)$ -homogeneous polynomial. As an immediate consequence of Proposition 2.1, we obtain the following proposition.

Proposition 2.5 *Every intrinsic quadric is isomorphic to a standard intrinsic quadric.*

Note that in Construction 2.4 the set of semistable points $\overline{Z}^{ss}(u)$ is an open toric subvariety of $\overline{Z} = \mathbb{K}^{n+m}$ and the quotient map $\pi: \overline{Z}^{ss}(u) \rightarrow Z$ for the action of H is a toric morphism; in fact this is the usual quotient presentation of the toric variety Z from [9]. Cutting down the orbit decomposition from the ambient toric variety Z to X yields a decomposition of X into locally closed subvarieties which we call the *pieces* of X . We need to identify these pieces explicitly.

Construction 2.6 Notation as in Construction 2.4. The degree homomorphism $Q: \mathbb{Z}^{n+m} \rightarrow K$ sending the i -th canonical basis vector $e_i \in \mathbb{Z}^{n+m}$ to the weight $\deg(T_i) \in K$ gives rise to a pair of mutually dual exact sequences of abelian groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \longrightarrow & \mathbb{Z}^{n+m} & \xrightarrow{P} & \mathbb{Z}^r \\ 0 & \longleftarrow & K & \xleftarrow{Q} & \mathbb{Z}^{n+m} & \xleftarrow{P^*} & \mathbb{Z}^r & \longleftarrow & 0. \end{array}$$

For every face $\gamma_0 \leq \gamma$ of the positive orthant $\gamma = \mathbb{Q}_{\geq 0}^{n+m}$, denote by $\overline{Z}(\gamma_0) \subseteq \overline{Z}$ the set of all points $z \in \overline{Z}$ having coordinates $z_i \neq 0$ if $e_i \in \gamma_0$ and $z_i = 0$ otherwise. This sets up a bijection

$$\{\text{faces of } \gamma\} \longrightarrow \{\text{toric orbits of } \overline{Z}\}, \quad \gamma_0 \longmapsto \overline{Z}(\gamma_0).$$

A face $\gamma_0 \leq \gamma$ is called *Z-relevant*, if the cone $Q(\gamma_0) \subseteq K_{\mathbb{Q}}$ contains u in its relative interior. The set of semistable points $\overline{Z}^{ss}(u)$ is the union of all toric orbits $\overline{Z}(\gamma_1)$, where $\gamma_0 \leq \gamma_1$ with a *Z-relevant* $\gamma_0 \leq \gamma$. Via the quotient map $\pi: \overline{Z}^{ss}(u) \rightarrow Z$, we obtain a bijection

$$\{\text{Z-relevant faces of } \gamma\} \longrightarrow \{\text{toric orbits of } Z\}, \quad \gamma_0 \longmapsto Z(\gamma_0) := \pi(\overline{Z}(\gamma_0)).$$

We say that $\gamma_0 \leq \gamma$ is an *X-face* if $\overline{X}(\gamma_0) := \overline{X} \cap \overline{Z}(\gamma_0)$ is non-empty and we call it *X-relevant* if in addition γ_0 is *Z-relevant*. The *X-relevant* faces of γ correspond to the toric orbits of Z intersecting X non-trivially. This leads to a bijection

$$\{\text{X-relevant faces of } \gamma\} \longrightarrow \{\text{pieces of } X\}, \quad \gamma_0 \longmapsto X(\gamma_0) := X \cap Z(\gamma_0).$$

The *covering collection* of X is the set $\text{cov}(X)$ of all minimal *X-relevant* faces of γ . The union over all affine toric charts $Z_{\gamma_0} \subseteq Z$, where γ_0 stems from the covering collection, is the *minimal toric ambient variety* of X ; it is the minimal open toric subvariety of Z containing X as a closed subvariety.

Remark 2.7 Due to the specific form of $g_{q,t} \in \mathbb{K}[T_1, \dots, T_n, S_1, \dots, S_m]$, we can explicitly describe the faces $\gamma_0 \leq \gamma$ defining a non-empty set $\overline{X}(\gamma_0) = \overline{X} \cap \overline{Z}(\gamma_0)$. For any sequence $1 \leq i_1 < \dots < i_k \leq n + m$, we denote

$$\gamma_{i_1, \dots, i_k} := \text{cone}(e_{i_1}, \dots, e_{i_k}) \leq \gamma.$$

This gives us all the faces of the orthant $\gamma = \mathbb{Q}_{\geq 0}^{n+m}$. We consider the following four basic types of faces:

- (i) $\gamma_{i, i+1, j, j+1}$ with $1 \leq i < j < q$ odd,
- (ii) $\gamma_{i, i+1, j}$ with $1 \leq i < q$ odd and $q + 1 \leq j \leq q + t$,
- (iii) $\gamma_{i, j}$ with $q + 1 \leq i < j \leq q + t$,

- (iv) $\gamma_{i_1, \dots, i_k, q+t+1, \dots, q+t+m}$, where $i_1 \in \{1, 2\}$, $i_2 \in \{3, 4\}, \dots, i_k \in \{q-1, q\}$ with $k = q/2$.

Then $\gamma_0 \leq \gamma$ is an \bar{X} -face, i.e., the set $\bar{X}(\gamma_0)$ is non-empty, if and only if one of the following holds

- $\tau \leq \gamma_0$ with a face $\tau \leq \gamma$ of type (i), type (ii), or type (iii).
- $\gamma_0 \leq \tau$ with a face $\tau \leq \gamma$ of type (iv).

A point $x \in X$ of a variety is *factorial* if the local ring $\mathcal{O}_{X,x}$ is a unique factorization domain. A variety X is called *locally factorial* if all its points are factorial; this is equivalent to the property that every Weil divisor of X is Cartier. We say that a standard intrinsic quadric X arising from Construction 2.1 is *quasismooth* if $\bar{X}^{ss}(u)$ is smooth; this implies that X has at most abelian quotient singularities.

Proposition 2.8 *Let $X = X(q, t, m, u)$ be a standard intrinsic quadric arising from Construction 2.4.*

(i) *Let $\gamma_m := \text{cone}(e_{q+t+1}, \dots, e_{q+t+m}) \leq \gamma$. Then the singular locus of the total coordinate space $\bar{X} = V(g_{q,t})$ is given by*

$$\bar{X}^{\text{sing}} = V(T_1, \dots, T_{q+t}) = \bigcup_{\gamma_0 \leq \gamma_m} \bar{X}(\gamma_0) \subseteq \bar{X}.$$

(ii) *The variety X is quasismooth if and only if every X -relevant face $\gamma_0 \leq \gamma$ contains some e_i with $1 \leq i \leq q+t$.*

(iii) *The piece $X(\gamma_0)$ associated with an X -relevant $\gamma_0 \leq \gamma$ consists of locally factorial points of X if and only if $Q(\text{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m})$ generates K as a group.*

(iv) *The variety X is locally factorial if and only if for every $\gamma_0 \in \text{cov}(X)$, the image $Q(\text{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m})$ generates K as a group.*

(v) *The piece $X(\gamma_0)$ associated with an X -relevant $\gamma_0 \leq \gamma$ consists of smooth points of X if and only if the following two statements hold:*

- (a) $Q(\text{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m})$ generates K as a group;
- (b) $e_i \in \gamma_0$ holds for some $1 \leq i \leq q+t$.

(vi) *The variety X is smooth if and only if it is quasismooth and for every $\gamma_0 \in \text{cov}(X)$, the image $Q(\text{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m})$ generates K as a group.*

Proof The first statement is obvious and the remaining ones are the adapted versions of [3, Cor. 3.3.1.8 and Prop. 3.3.1.10]. ■

The following three statements are proved in more generality in [3, Sec. 3.3]. Below, we denote for a convex, polyhedral cone σ in a rational vector space V , its relative interior by σ° .

Proposition 2.9 *Let $X = X(q, t, m, u)$ be a standard intrinsic quadric arising from Construction 2.4. Then the Picard group of X is given as*

$$\text{Pic}(X) = \bigcap_{\gamma_0 \in \text{cov}(X)} Q(\text{lin}_{\mathbb{Z}}(\gamma_0) \cap E) \subseteq K = \text{Cl}(X).$$

Proposition 2.10 *Let $X = X(q, t, m, u)$ be a standard intrinsic quadric arising from Construction 2.4. Then the cones of effective, movable, semiample, and ample divisor classes of X in $\text{Cl}_{\mathbb{Q}}(X) = K_{\mathbb{Q}}$ are given as*

$$\begin{aligned} \text{Eff}(X) &= Q(\gamma), & \text{Mov}(X) &= \bigcap_{\gamma_0 \leq \gamma \text{ facet}} Q(\gamma_0), \\ \text{SAmple}(X) &= \bigcap_{\gamma_0 \in \text{cov}(X)} Q(\gamma_0), & \text{Ample}(X) &= \bigcap_{\gamma_0 \in \text{cov}(X)} Q(\gamma_0)^\circ. \end{aligned}$$

Moreover, for every $u' \in \text{Ample}(X)$ we have $\overline{X}^{\text{ss}}(u) = \overline{X}^{\text{ss}}(u')$ for the sets of semistable points and thus $X = X(q, t, m, u')$.

Proposition 2.11 *Let $X = X(q, t, m, u)$ be a standard intrinsic quadric arising from Construction 2.4. Then the following statements are equivalent.*

- (i) X is \mathbb{Q} -factorial.
- (ii) For every X -relevant $\gamma_0 \leq \gamma$ the image $Q(\gamma_0)$ is of full dimension in $K_{\mathbb{Q}}$.
- (iii) The semiample cone $\text{SAmple}(X)$ is of full dimension in $K_{\mathbb{Q}}$.

3 Picard Numbers One and Two: Classification

First, we describe all locally factorial intrinsic quadrics of Picard number one. Then we show that locally factorial intrinsic quadrics of Picard number two have torsion free divisor class group, see Proposition 3.3. Finally, as the first part of the proof of Theorem 1.1, we establish the normal forms for the smooth intrinsic quadrics given there.

Proposition 3.1 *Let X be a locally factorial intrinsic quadric of Picard number one. Then X has divisor class group $\text{Cl}(X) \cong \mathbb{Z}$ and, with suitable integers $n \geq 5$ and $m \geq 0$, the Cox ring of X is given by*

$$\begin{aligned} \mathcal{R}(X) &\cong \mathbb{K}[T_1, \dots, T_n, S_1, \dots, S_m] / \langle g \rangle, \\ g &= \begin{cases} T_1 T_2 + \dots + T_{n-1} T_n, & n \text{ even,} \\ T_1 T_2 + \dots + T_{n-2} T_{n-1} + T_n^2, & n \text{ odd.} \end{cases} \end{aligned}$$

The $\text{Cl}(X)$ -grading of $\mathcal{R}(X)$ is given by $\deg(T_i) = \deg(S_j) = 1$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$. Thus, X is isomorphic to the classical quadric $V(g) \subseteq \mathbb{P}^{n+m-1}$ with singular locus $V(T_1, \dots, T_n)$. In particular, X is smooth if and only if $m = 0$ holds.

Proof Since X is locally factorial, we have $\text{Pic}(X) = \text{Cl}(X)$. In particular, $\text{Cl}_{\mathbb{Q}}(X)$ is of dimension one. We can assume that X arises from Construction 2.4 with a standard $\text{Cl}(X)$ -homogeneous quadratic polynomial $g_{q,t}$ and that the ample cone $\text{Ample}(X)$ is the positive ray in $\text{Cl}_{\mathbb{Q}}(X) = \mathbb{Q}$. Consider the faces

$$\gamma_0 := \text{cone}(e_i), \quad i = 1, \dots, q \quad \text{or} \quad i = n + 1, \dots, n + m.$$

Each of these faces is X -relevant. Since X is locally factorial, $Q(e_i)$ generates $\text{Cl}(X)$ as a group; see Proposition 2.8. In particular, if $q \geq 2$ holds, then we can conclude

$\text{Cl}(X) = \mathbb{Z}$ and

$$\deg(T_i) = 1, \quad i = 1, \dots, n, \quad \deg(S_j) = 1, \quad j = 1, \dots, m.$$

This implies that $t \leq 1$. The cases $n = 3, 4$ are impossible: then the Cox ring $\mathcal{R}(X)$ wouldn't admit unique factorization, but it has to do so, because of the torsion free divisor class group $\text{Cl}(X)$, see [3, Prop. 1.4.1.5]. Thus, we also have $n \geq 5$, if $q \geq 2$ holds.

We exclude the case $q = 0$. Here, $t \geq 3$ must hold. Thus, we have the X -relevant face $\gamma_0 = \text{cone}(e_1, e_2)$. Thus, $\text{Cl}(X)$ is generated by $\deg(T_1)$ and $\deg(T_2)$, which implies $\text{Cl}(X) = \mathbb{Z} \oplus \Gamma$ with a cyclic group $\Gamma = \mathbb{Z}/k\mathbb{Z}$ and, after applying a suitable automorphism of $\text{Cl}(X)$, we can assume

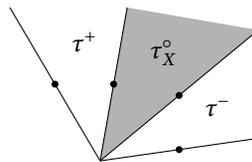
$$\deg(T_1) = (1, \bar{0}), \quad \deg(T_2) = (1, \bar{1}).$$

Since $2 \deg(T_1) = 2 \deg(T_2)$ holds, we obtain $k = 2$. But then there is no way to assign to T_3 a degree in $\text{Cl}(X)$ differing from the degrees of T_1 and T_2 , a contradiction. ■

Remark 3.2 Let X be a \mathbb{Q} -factorial standard intrinsic quadric with $\text{Cl}_{\mathbb{Q}}(X)$ of dimension two arising from Construction 2.4. Then the effective cone $\text{Eff}(X)$ is uniquely decomposed into three convex sets

$$\text{Eff}(X) = \tau^+ \cup \tau_X^\circ \cup \tau^-$$

such that τ^+ and τ^- do not intersect $\tau_X^\circ = \text{Ample}(X)$ and $\tau^+ \cap \tau^-$ consists of the origin. Because of $\tau_X^\circ \subseteq \text{Mov}(X)$, each of τ^+ and τ^- contains at least two (not necessarily different) degrees of the Cox ring generators $T_1, \dots, T_n, S_1, \dots, S_m$.



Note that τ_X° is an open cone of dimension two, whereas τ^- as well as τ^+ might be one-dimensional. The closure $\tau_X = \text{SAmple}(X)$ of τ_X° is the intersection of two X -relevant faces, see Proposition 2.10, and thus we find degrees of variables on its boundary. Moreover, apart from $\deg(T_n)$ when $t = 1$, no degree of a T_i or a S_j can lie in τ_X° , use again Proposition 2.10.

Proposition 3.3 Let X be an intrinsic quadric of Picard number two. If X is locally factorial, then $\text{Cl}(X) = \text{Pic}(X) = \mathbb{Z}^2$ holds.

Proof Since X is locally factorial, every Weil divisor is principal, and thus we have $\text{Cl}(X) = \text{Pic}(X)$. The remaining task is to show that $\text{Pic}(X)$ is torsion free. For this, we can assume that X arises from Construction 2.4. We claim that it suffices to find a two-dimensional X -relevant face $\gamma_0 \leq \gamma$. Indeed, Proposition 2.9 tells us that $\text{Pic}(X)$ is a subgroup of $Q(\text{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m}) \subseteq \text{Cl}(X)$. In particular, $Q(\text{lin}_{\mathbb{Q}}(\gamma_0))$ is of dimension two. Consequently, being generated by two elements, $Q(\text{lin}_{\mathbb{Q}}(\gamma_0) \cap \mathbb{Z}^{n+m})$ is torsion free. Then also $\text{Pic}(X)$ must be torsion free.

Now, the ample cone τ_X° is two-dimensional and, according to Remark 3.2, we find two degrees $v_1^-, v_2^- \in \tau^-$ and two degrees $v_1^+, v_2^+ \in \tau^+$ stemming from, in total, four of the generators T_i and S_j such that v_1^- and v_1^+ generate the effective cone $\text{Eff}(X)$. After suitably renumbering the T_i and the T_j , we are in one of the following cases:

Case 1. We have $\text{deg}(S_1) \in \tau^-$ and $\text{deg}(S_2) \in \tau^+$. Then $\gamma_0 = \text{cone}(e_{n+1}, e_{n+2})$ is the desired X -relevant face.

Case 2. We have $\text{deg}(S_1) \in \tau^-$ and τ^+ contains no degrees of variables S_j . If $\text{deg}(T_i) \in \tau^+$ holds for some $1 \leq i \leq q$, then $\gamma_0 = \text{cone}(e_i, e_{n+1})$ is the desired X -relevant face.

Suppose that there is no $1 \leq i \leq q$ with $\text{deg}(T_i) \in \tau^+$. Then one and hence all $\text{deg}(T_{q+i})$ lie on the ray through v_1^+ . This implies $q = 0$. Now, $\text{cone}(e_1, e_2, e_{t+1})$ is an X -relevant face. Since X is locally factorial, the corresponding degrees generate $\text{Cl}(X)$. We conclude $\text{Cl}(X) = \mathbb{Z}^2 \oplus \Gamma$ with a cyclic torsion part $\Gamma = \mathbb{Z}/k\mathbb{Z}$ and, applying a suitable automorphism of Γ , we achieve

$$\text{deg}(T_1) = (1, 0, \bar{0}), \quad \text{deg}(T_2) = (1, 0, \bar{1}), \quad \text{deg}(S_1) = (0, 1, \bar{0}).$$

Because of $2 \text{deg}(T_1) = 2 \text{deg}(T_2)$, we obtain $k = 2$. Since the T_i must have different degrees in $\text{Cl}(X)$, we obtain that there are no T_i for $i \geq 3$. Thus, no $1 \leq i \leq q$ with $\text{deg}(T_i) \in \tau^+$ is impossible.

Case 3. There are no $\text{deg}(S_j)$ in $\tau^- \cup \tau^+$. Then we can assume $v_1^- = \text{deg}(T_1)$ and obtain the desired X -relevant face $\gamma_0 = \text{cone}(e_1, e_i)$ by choosing $i \neq 2$ such that $\text{deg}(T_i)$ is one of v_1^+, v_2^+ . ■

Proof of Theorem 1.1, Part I We show that all smooth intrinsic quadrics of Picard number $\rho(X) = 2$ are isomorphic to one of the varieties described in Theorem 1.1. Proposition 3.3 yields $\text{Cl}(X) = \mathbb{Z}^2$. Moreover, according to Proposition 2.5, we can assume that X is a standard intrinsic quadric. Then the Cox ring of X is given as

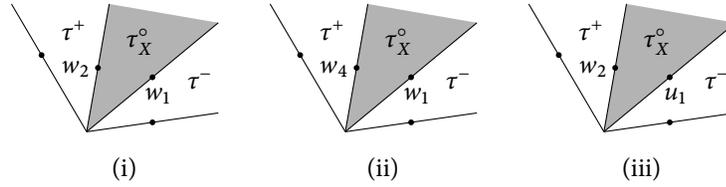
$$\mathcal{R}(X) = \mathbb{K}[T_1, \dots, T_n, S_1, \dots, S_m] / \langle g \rangle,$$

$$g = \begin{cases} T_1 T_2 + \dots + T_{n-1} T_n & \text{if } n \text{ is even,} \\ T_1 T_2 + \dots + T_{n-2} T_{n-1} + T_n^2 & \text{if } n \text{ is odd.} \end{cases}$$

Note that we have $n \geq 5$, because $\text{Cl}(X)$ is torsion free, and thus $\mathcal{R}(X)$ must be a unique factorization domain. As outlined in Remark 3.2, the effective cone of X is the disjoint union of three convex sets,

$$\text{Eff}(X) = \tau^- \cup \tau_X^\circ \cup \tau^+,$$

where $\tau_X^\circ \subseteq \text{Cl}_\mathbb{Q}(X)$ is the ample cone. Since X is smooth, there are no X -relevant faces of the form $\text{cone}(e_i, e_j)$ with $n + 1 \leq i < j \leq n + m$; see Proposition 2.8. Consequently, the $\text{deg}(S_j)$ either all lie in τ^- or all in τ^+ . After suitably renumbering the variables T_i and S_j , we are left with the following cases:



Here, we set $w_i := \deg(T_i)$ and $u_j := \deg(S_j)$. Observe that in Case (iii), we can indeed assume $u_1 \in \tau^-$, because $\gamma_{2,n+1}$ is an X -relevant face and thus Proposition 2.8 allows us to interchange τ^- and τ^+ via a linear coordinate change if necessary. We now go through the cases, using the notation of Remark 2.7 for X -relevant faces and writing $\mu = (\mu_1, \mu_2) \in \mathbb{Z}^2$ for the degree of g .

Case (i): We have $\tau_X = \text{cone}(w_1, w_2)$ with $w_1 \in \tau^-$ and $w_2 \in \tau^+$. Then $\mu \in \tau_X$ holds. Thus, we can assume $w_3 \in \tau^-$ and $w_4 \in \tau^+$. Applying Proposition 2.8 to $\gamma_{1,4}$, we see that w_1, w_4 form a \mathbb{Z} -basis for \mathbb{Z}^2 . By a suitable coordinate change, we achieve $w_1 = (1, 0)$ and $w_4 = (0, 1)$. Then $w_1 + w_2 = w_3 + w_4 = \mu$ implies $w_2 = (\mu_1 - 1, \mu_2)$ and $w_3 = (\mu_1, \mu_2 - 1)$. Like w_1 and w_4 , w_3 and w_2 also form a \mathbb{Z} -basis for $\text{Cl}(X)$, being positively oriented, because $\text{Eff}(X)$ is pointed and we have $w_2 \in \tau^+$ and $w_3 \in \tau^-$. This implies

$$1 = \det(w_3, w_2) = \mu_1 + \mu_2 - 1.$$

From $\mu \in \tau_X \subseteq \text{cone}(w_1, w_4)$ we infer $\mu_1, \mu_2 > 0$ and conclude that $\mu_1 = \mu_2 = 1$. In particular, we have $w_2 = (0, 1)$, $w_3 = (1, 0)$ and $\tau_X = \mathbb{Q}_{\geq 0}^2$. Moreover, $\mu = (1, 1)$ implies that n is even. Suitably renumbering the T_i with $i \geq 5$, we achieve $w_i \in \tau^-$ and $w_{i+1} \in \tau^+$ for $i = 5, 7, \dots, n - 1$. Then, for every odd i , Proposition 2.8 and the homogeneity of g provide us with the conditions

$$\det(w_i, w_2) = 1, \quad w_i + w_{i+1} = \mu = (1, 1), \quad \det(w_1, w_{i+1}) = 1.$$

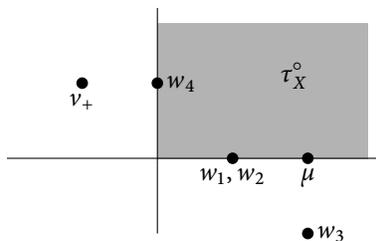
We conclude that $w_i = (1, 0)$ and $w_{i+1} = (0, 1)$ for all $i = 5, 7, \dots, n - 1$. The weights $u_j = \deg(S_j)$ are contained either all in τ^- or all in τ^+ . We can assume all in τ^+ . Applying Proposition 2.8 to $\gamma_{1,j}$, where $j = 1, \dots, m$ yields $u_j = (a_j, 1)$ with some $a_j \in \mathbb{Z}_{\leq 0}$. A suitable linear coordinate change in \mathbb{Z}^2 leads to Type 4.

Case (ii): We have $\tau_X = \text{cone}(w_1, w_4)$ with $w_1 \in \tau^-$ and $w_4 \in \tau^+$. If $w_2 \in \tau^+$ holds, then we are in Case (i) just settled. We treat the case $w_2 \in \tau^-$. Since $\gamma_{1,4}$ is an X -relevant face, Proposition 2.8 says that w_1, w_4 form a \mathbb{Z} -basis of \mathbb{Z}^2 . Thus, a suitable linear coordinate change in \mathbb{Z}^2 yields $w_1 = (1, 0)$ and $w_4 = (0, 1)$.

As $\gamma_{2,4}$ is an X -relevant face, we can apply Proposition 2.8 again and see that w_2, w_4 is a positively oriented \mathbb{Z} -basis of \mathbb{Z}^2 . Thus, $\det(w_2, w_4) = 1$ holds and we conclude that $w_2 = (1, x)$ with some $x \in \mathbb{Z}_{< 0}$. By the same arguments, if $v^+ \in \tau^+$ is a degree of any of the $T_i \neq T_4$ or the S_j , we see that w_1, v^+ is a positively oriented \mathbb{Z} -basis and conclude that $v^+ = (y, 1)$ with some $y \in \mathbb{Z}_{\leq 0}$. Moreover, arguing further along this line gives

$$1 = \det(w_2, v^+) = 1 - xy, \quad 1 = \det(w_3, v^+) = 2 - xy + y,$$

where we use $w_3 = \mu - w_4 = w_1 + w_2 - w_4$ for the last equality. This implies $x = 0$ and $y = -1$, and thus $w_2 = (1, 0)$ and $v^+ = (-1, 1)$. We conclude that $\mu = w_1 + w_2 = (2, 0)$ and $w_3 = \mu - w_4 = (2, -1)$. So far, the situation looks as follows:



In particular, if n is odd, we must have $w_n = (1, 0)$. Because of $\mu \in \tau^-$, we can renumber the other T_i with $i \geq 5$ such that $w_i \in \tau^-$ holds for all odd i . Now consider any odd i with $5 \leq i < n$. Because of $\mu = (2, 0)$ we either have $w_i = w_{i+1} = (1, 0)$ or $w_{i+1} = v^+ = (-1, 1)$ and $w_i = (3, -1)$. The second case is excluded, because $\gamma_{4,i}$ is then an X -relevant face, contradicting the smoothness condition of Proposition 2.8. Thus, $w_i = (1, 0)$ holds for all $i \geq 5$. Consequently, there must be at least one u_j and all u_j equal $v^+ = (-1, 1)$. A suitable linear coordinate change in \mathbb{Z}^2 and renumbering the variables leads to Type 3.

Case (iii): We have $\tau_X = \text{cone}(u_1, w_2)$ with $u_1 \in \tau^-$ and $w_2 \in \tau^+$. Then $\gamma_{2,n+1}$ is an X -relevant face. By Proposition 2.8, we achieve $u_1 = (1, 0)$ and $w_2 = (0, 1)$ via a suitable linear coordinate change. We distinguish the following two subcases.

First assume that $w_1 \in \tau^+$. Then $\mu \in \tau^+$ holds. Thus, we may assume that all w_i with i odd are contained in τ^+ . Proposition 2.8 shows that u_1, w_i is a \mathbb{Z} -basis for the odd $i < n$, and thus $w_i = (x_i, 1)$ holds in these cases, where $x_i \in \mathbb{Z}_{\leq 0}$ due to $w_i \in \tau^+$. In particular, we have

$$\mu = w_1 + w_2 = (x_1, 2), \quad w_{i+1} = \mu - w_i = (x_1 - x_i, 1),$$

where $i < n$ is odd. Thus, we obtain $w_1, w_2, \dots, w_n \in \tau^+$. Consequently, $m \geq 2$ holds. Because of $u_1 \in \tau^-$, we have $u_j \in \tau^-$ for all $j = 1, \dots, m$. Moreover, u_j, w_2 is a \mathbb{Z} -basis due to Proposition 2.8, and thus $u_j = (1, y_j)$ holds, where $y_j \in \mathbb{Z}_{\leq 0}$. Repeating the same argument with all pairings u_j, w_i yields $w_i = (0, 1)$ for all i or $u_j = (1, 0)$ for all j . Applying a suitable linear coordinate change, we arrive at Type 1 or Type 2, respectively.

Now assume $w_1 \in \tau^-$. Then $\mu \in \tau_X \cup \tau^-$ holds. Suitably renumbering the T_i , we achieve $w_i \in \tau^-$ for all odd $i < n$. Moreover, as all u_j lie in τ^- and there must be the degree of a second variable in τ^+ , we can assume $w_4 \in \tau^+$. Proposition 2.8 applied to the X -relevant faces $\gamma_{2,i}$ for the $i \geq 3$ with $w_i \in \tau^-$ and $\gamma_{i,n+1}$ for the i with $w_i \in \tau^+$ shows

$$w_i = \begin{cases} (1, y_i) \text{ with } y_i \in \mathbb{Z}_{\leq 0} & \text{if } w_i \in \tau^-, \\ (x_i, 1) \text{ with } x_i \in \mathbb{Z}_{\leq 0} & \text{if } w_i \in \tau^+, \end{cases}$$

unless $i = 1$ or $i = n$ with n odd. Now consider any even i with $4 \leq i \leq n$. Then the degree of g is $\mu = w_3 + w_4 = (1 + x_4, y_3 + 1)$. Because of $\mu \in \tau_X \cup \tau^-$, we conclude $x_4 = 0$ and obtain $\mu = (1, y_3 + 1)$. In particular, we see that n is even and $w_i \in \tau^+$ holds for all even i . Moreover, $w_1 = \mu - w_2 = w_3$ holds and for every even i we have

$$(1, y_3 + 1) = \mu = w_{i-1} + w_i = (1 + x_i, y_{i-1} + 1).$$

Consequently, $x_i = 0$ and thus $w_i = (0, 1)$ holds for all even i . Thus, for the odd i , we obtain $w_i = (1, y_3)$. Finally u_j, w_2 is a \mathbb{Z} -basis for all $j = 1, \dots, m$, and thus we have $u_j = (1, a_j)$ with $a_j \in \mathbb{Z}_{\leq 0}$. So, a suitable linear coordinate change leads to Type 4. ■

4 Picard Number Two: Geometry

We discuss geometric aspects of the intrinsic quadrics listed in Theorem 1.1. First, we enter their Mori theory and prove the still open geometric statements made in Theorem 1.1. Then we figure out the Fano examples from Theorem 1.1 and thus prove Corollary 1.2. Moreover, we obtain base point freeness for numerically effective divisors (see Corollary 4.3), and thus can verify Fujita’s freeness conjecture for all smooth intrinsic quadrics of Picard number at most two (see Corollary 4.4). Finally, we discuss Mukai’s conjecture in Example 4.5.

The morphisms providing the geometric descriptions of Theorem 1.1 are examples of so called elementary contractions [10]. We obtain them by looking at the Mori chamber decomposition, which in our case is easy to compute. Before entering the details, let us briefly recall some general background. Every effective divisor D on a normal projective variety X defines a rational map

$$\varphi_D: X \dashrightarrow X(D), \quad X(D) := \text{Proj} \left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(X, \mathcal{O}_X(nD)) \right).$$

Two divisors are called Mori equivalent if they define the same map. The Mori chamber decomposition is the subdivision of the effective cone into the classes arising from Mori equivalence. In the case of a Mori dream space X , there is a fundamental connection to geometric invariant theory, as observed by Hu and Keel [17]. Namely, we have the action of the quasitorus $H = \text{Spec } \mathbb{K}[\text{Cl}(X)]$ on $\overline{X} := \text{Spec } \mathcal{R}(X)$ and thus the GIT-fan Λ_X describing the variation of GIT-quotients in the sense that two classes $w_1, w_2 \in \text{Cl}(X)$ define the same sets of semistable points $\overline{X}^{ss}(w_i)$ if and only if they lie in the relative interior of a common cone $\lambda \in \Lambda_X$. Now, the crucial observation is that, inside the moving cone, the Mori chambers of X are the precisely the relative interiors of the cones of the GIT-fan.

For a Mori dream space $X = \overline{X}^{ss}(u) // H$, the cone $\lambda(u) \in \Lambda_X$ containing u in its relative interior, is the semiample cone of X . A divisor D defines a morphism $\varphi_D: X \rightarrow X(D)$ if and only if for the class w of D , the associated cone $\lambda(w) \in \Lambda_X$ is a face of $\lambda(u)$. In this case, $\varphi_D: X \rightarrow X(D)$ is called a contraction, and, in the GIT picture, φ_D is the induced map of GIT-quotients making the following diagram commutative:

$$\begin{array}{ccc} \overline{X}^{ss}(u) & \subseteq & \overline{X}^{ss}(w) \\ \downarrow //H & & \downarrow //H \\ X & \xrightarrow{\varphi_D} & X(D). \end{array}$$

A contraction $\varphi: X \rightarrow X(D)$ is called elementary if $X(D)$ is of Picard number one less than X . There are three possibilities for such an elementary contraction, according to the possible positions of the class of D in the effective cone:

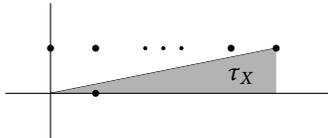
- The class of D lies on the boundary of $\text{Eff}(X)$. Then φ_D is of *fiber type*, i.e., the dimension of $X(D)$ is strictly less than that of X .
- The class of D lies on a boundary of $\text{Mov}(X)$, but not on the boundary of $\text{Eff}(X)$. Then φ_D is a *birational divisorial contraction*, i.e., it is birational and contracts precisely a prime divisor of X .
- The class of D lies in the interior $\text{Mov}(X)$. Then φ_D is a *birational small contraction*, i.e., it is birational and contracts only a subvariety of codimension at least two.

Remark 4.1 Construction 2.4 produces an intrinsic quadric X in an ambient toric variety Z by passing to a quotient of the action of H on \bar{X} and $\bar{Z} = \mathbb{K}^{n+m}$. The cones of the (finite) GIT-fans Λ_X and Λ_Z in $K_{\mathbb{Q}} = \text{Cl}_{\mathbb{Q}}(X) = \text{Cl}_{\mathbb{Q}}(Z)$ are

$$\lambda_X(w) = \bigcap_{\substack{w \in Q(\gamma_0), \\ \bar{X}(\gamma_0) \neq \emptyset}} Q(\gamma_0), \quad \lambda_Z(w) = \bigcap_{w \in Q(\gamma_0)} Q(\gamma_0),$$

respectively, where w runs through $K_{\mathbb{Q}}$ and Remark 2.7 tells which are the faces $\gamma_0 \leq \gamma = \mathbb{Q}^{n+m}$ such that $\bar{X}(\gamma_0)$ is non-empty. In particular, the fan Λ_Z refines the fan Λ_X , which in turn connects the Mori theory of X with that of Z .

Proof of Theorem 1.1, Part II We first discuss the varieties X of Types 1, 2, and 4. In these cases, the configurations of weights and the semiample cone are of the following shape:



We work with the toric embedding $X \subseteq Z$ provided by Construction 2.4. From Remark 2.7, we infer $\tau_X = \tau_Z$ for the semiample cones. Thus, for the divisor class $w = (1, 0)$, a representing toric divisor E on Z and its restriction D on X , we obtain a commutative diagram

$$\begin{array}{ccccc} X & \subseteq & Z & \cong & \mathbb{P}(\mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_k)) \\ \varphi_D \downarrow & & \downarrow \varphi_E & & \downarrow \\ X(D) & \subseteq & Z(E) & \cong & \mathbb{P}_{l-1}, \end{array}$$

where the inclusions are closed embeddings, l is the number of coordinates of $\bar{Z} = \mathbb{K}^{n+m}$ of degree $(1, 0)$, call them f_1, \dots, f_l , and k is the number of remaining coordinates, call them h_1, \dots, h_k . So, we have $n + m = l + k$. In terms of homogeneous coordinates on \bar{Z} and on $\bar{Z}(E) = \mathbb{K}^l$, local trivializations of the bundle projection

$\varphi_E: Z \rightarrow Z(E)$ are given by

$$\begin{array}{ccc}
 \mathbb{K}_{f_i}^{n+m} \setminus V(h_1, \dots, h_k) & \xrightarrow{\left(f_1, \dots, f_l, \frac{h_1}{f_i^{a_1}}, \dots, \frac{h_k}{f_i^{a_k}} \right)} & \mathbb{K}_{f_i}^l \times (\mathbb{K}^k \setminus \{0\}) \\
 & \searrow (f_1, \dots, f_l) & \swarrow \text{pr}_{\mathbb{K}^l} \\
 & & \mathbb{K}_{f_i}^l
 \end{array}$$

If X is of Type 1, then $l = n$ and $k = m$ hold, the f_i are the variables T_i and the h_j are the variables S_j . We see directly that X maps onto $V(g) \subseteq \mathbb{P}_{n-1}$ and that $X = \varphi_E^{-1}(V(g))$ holds. Thus, $\varphi_D: X \rightarrow X(D)$ is a bundle projection as wanted.

If X is of Type 2, then $l = m$ and $k = n$ hold, the f_i are the variables S_i , and the h_j are the variables T_j . Using the fact that the relation g is K -homogeneous, we see that the above local trivializations respect g . We conclude that $X(D) = Z(E) = \mathbb{P}_{m-1}$ holds and that locally with respect to the base $X(D)$, the variety X is a product of the smooth quadric $V(g) \subseteq \mathbb{P}_{n-1}$ and \mathbb{P}_{m-1} .

If X is of Type 4, then $l = n/2$ and $k = m + n/2$ hold, the f_i are the variables T_i with i odd and the h_j subsume the variables T_i with i even as well as the variables S_j . Using the above local trivializations, we see that X projects onto the base, which means that $X(D) = Z(E) = \mathbb{P}_{n/2-1}$ holds. Moreover, on each fiber $\varphi_E^{-1}([z])$, the relation g becomes a linear form in the coordinates T_i with i even and thus cuts out a hyperplane of $\varphi_E^{-1}([z]) \cong \mathbb{P}_{n/2+m-1}$. Consequently, $\varphi_D: X \rightarrow X(D)$ is as claimed.

Finally, let X be of Type 3. Observe that the ambient toric variety Z is not smooth in this case. We take the divisors E on Z and D on X corresponding to the generator T_3 of the respective Cox rings. Then $\varphi_E: Z \rightarrow Z(E)$ and $X \rightarrow X(E)$ contract the respective divisors defined by T_1 . We obtain that $Z(E)$ is the weighted projective space $\mathbb{P}_{(1, \dots, 1, 2)}$ of dimension $n + m - 2$ and $X(D) \subseteq Z(E)$ is defined by the equation $g - T_1 T_2 + T_2 = 0$. This gives $X(D) = \mathbb{P}_{n+m-3}$. Moreover, the fan of the ambient toric variety Z is obtained from the fan of $Z(E) = \mathbb{P}_{(1, \dots, 1, 2)}$ by barycentric subdivision of the cone over the rays corresponding to u_1, \dots, u_m, w_2 . We conclude that the center of the modification $X \rightarrow X(D)$ is the smooth quadric $V(g - T_1 T_2, S_1, \dots, S_m) \subseteq \mathbb{P}_{n+m-3}$. The fact that $X \rightarrow X(D)$ is indeed the blowing-up is checked directly in the affine charts of $X(D) = \mathbb{P}_{n+m-3}$. ■

We turn to the (almost) Fano varieties among the smooth intrinsic quadrics of Picard number two. Adapting [3, Prop. 3.3.3.2] leads to the following explicit description of the anticanonical class.

Proposition 4.2 *Let $X = X(q, t, m, u)$ be a standard intrinsic quadric arising from Construction 2.1. Then the anticanonical class of X is given by*

$$-\mathcal{K}_X = \frac{q-2}{2} \deg(g_{q,t}) + \sum_{i=1}^t \deg(T_{q+i}) + \sum_{k=1}^m \deg(S_k) \in K = \text{Cl}(X).$$

Proof of Corollary 1.2 In the situation of Theorem 1.1, the formula of Proposition 4.2 simplifies to

$$-\mathcal{K}_X = \frac{n-2}{2} \deg(g) + \deg(u_1) + \dots + \deg(u_m).$$

The variety X is Fano if and only if $-\mathcal{K}_X$ lies in the interior of the semiample cone τ_X specified in the theorem, and X is truly almost Fano if $-\mathcal{K}_X$ lies on the boundary of τ_X and in the interior of $\text{Eff}(X)$. One immediately computes:

Type	$-\mathcal{K}_X$	τ_X
1	$\binom{n-2}{0} + \binom{a_1+\dots+a_m}{m}$	$\text{cone} \left(\binom{1}{0}, \binom{\alpha}{1} \right)$
2	$\frac{n-2}{2} \binom{\alpha}{2} + \binom{m}{0}$	$\text{cone} \left(\binom{1}{0}, \binom{\alpha}{1} \right)$
3	$(n-2) \binom{1}{1} + \binom{m}{0}$	$\text{cone} \left(\binom{1}{1}, \binom{2}{1} \right)$
4	$\frac{n-2}{2} \binom{a+1}{1} + \binom{a_1+\dots+a_m}{m}$	$\text{cone} \left(\binom{1}{0}, \binom{\alpha}{1} \right).$

From this, we directly derive the Fano and truly almost Fano conditions. Note that for Type 4, we must have $w_2 = (\alpha, 1)$ in order to obtain a (truly almost) Fano variety and $w_2 = (0, 1)$ produces further truly almost Fano varieties. ■

Corollary 4.3 *Let X be a smooth intrinsic quadric of Picard number at most two. Then every numerically effective divisor on X is base point free.*

Proof We can assume that X arises from Construction 2.1. Consider the monoid $\text{BPF}(X) \subseteq \text{Cl}(X)$ of divisor classes admitting a base point free representative. Using [3, Prop. 3.3.2.8], we obtain

$$\text{BPF}(X) = \bigcap_{\gamma_0 \in \text{cov}(X)} Q(\gamma_0 \cap \mathbb{Z}^{n+m}) \subseteq \text{Cl}(X).$$

By Proposition 2.10, the cone in $\text{Cl}_{\mathbb{Q}}(X)$ generated by $\text{BPF}(X)$ equals the cone $\text{SAmple}(X)$ of semiample divisor classes. As for any Mori dream space, $\text{SAmple}(X)$ coincides with the cone of numerically effective divisor classes.

Our task is to show that $\text{BPF}(X)$ is saturated in $\text{Cl}(X)$ in the sense that given $w \in \text{Cl}(X)$ and $n \in \mathbb{Z}_{>0}$ with $nw \in \text{BPF}(X)$, one has $w \in \text{BPF}(X)$. Since the intersection of saturated submonoids is saturated, it suffices to show that every monoid $Q(\gamma_0 \cap \mathbb{Z}^{n+m})$, where $\gamma_0 \in \text{cov}(X)$, is saturated. If X is of Picard number one, then Proposition 3.1 tells us $\text{Cl}(X) = \mathbb{Z}$ and $Q(\gamma_0 \cap \mathbb{Z}^{n+m}) = \mathbb{Z}_{\geq 0}$ for all faces $\gamma_0 \in \text{cov}(X)$, proving that $\text{BPF}(X)$ is saturated.

Assume that X is of Picard number two. Then $\text{Cl}(X) = \mathbb{Z}^2$ holds according to Theorem 1.1. Moreover, for any two-dimensional face $\gamma_{i,j} = \text{cone}(e_i, e_j)$ of $\text{cov}(X)$, Proposition 2.8(v) says that $Q(e_i)$ and $Q(e_j)$ form a \mathbb{Z} -basis for $\text{Cl}(X)$. We conclude that $Q(\gamma_{i,j} \cap \mathbb{Z}^{n+m})$ is saturated for all two-dimensional faces $\gamma_{i,j} = \text{cone}(e_i, e_j)$ of $\text{cov}(X)$. Theorem 1.1 specifies the semiample cone for each of the Types 1 to 4. Combining this with Remark 2.7 allows us to determine the set $\text{cov}(X)$ of minimal X -relevant faces explicitly. If X is of Types 1, 2, or 4, then we see that in fact all

$\gamma_0 \in \text{cov}(X)$ are two-dimensional and thus $\text{BPF}(X)$ is saturated. We are left with discussing X of Type 3. If n is even, then again all $\gamma_0 \in \text{cov}(X)$ are two-dimensional. If n is odd, then, besides the two-dimensional ones, we find one more face in $\text{cov}(X)$, namely $\gamma_{1,2,n} = \text{cone}(e_1, e_2, e_n)$. The corresponding images under Q are $(0, 1)$, $(2, 1)$ and $(1, 1)$, generating the saturated monoid $\text{cone}(Q(\gamma_{1,2,n})) \cap \mathbb{Z}^2$. ■

Corollary 4.4 *Every smooth intrinsic quadric X of Picard number at most two fulfills Fujita’s freeness conjecture. That means that $C_X + sD$ is base point free for any canonical divisor C_X , all $s \geq \dim(X) + 1$ and all ample divisors D on X .*

Proof Fujita proved that $C_X + sD$ is numerically effective under the above assumptions [15, Thm. 1] on s and D . Thus, Corollary 4.3 gives the assertion. ■

Mukai’s conjecture [21] predicts that $\rho(X)(q(X) - 1) \leq \dim(X)$ with equality if and only if X is the $\rho(X)$ -fold product of the projective space $\mathbb{P}_{q(X)-1}$ for every smooth Fano variety X of Picard number $\rho(X)$ and Fano index $q(X)$. The conjecture is proved for toric X and in general for $\rho(X) \leq 2$ as well as for $\dim(X) \leq 5$; see [2, 6, 8, 23]. Let us revisit the case $\rho(X) \leq 2$.

Example 4.5 We show how to obtain Mukai’s conjecture for smooth Fano intrinsic quadrics X of Picard number $\rho(X) \leq 2$ from our results. In the case $\rho(X) = 1$, Proposition 3.1 tells us that X is a smooth quadric in a projective space and thus satisfies Mukai’s conjecture. So, assume $\rho(X) = 2$. We can assume that X arises from Construction 2.1 with input data given by Theorem 1.1. Note that we have

$$\dim(X) = n + m - 3.$$

Corollary 1.2 provides us with the Fano condition. Moreover, the anticanonical class $-\mathcal{K}_X \in \text{Cl}(X) = \mathbb{Z}^2$ is specified in the table shown in the proof of Corollary 1.2, and the Fano index $q(X)$ equals the greatest common divisor of the two entries of the vector $-\mathcal{K}_X$. We now go through the four different Types of Theorem 1.1.

Let X be of Type 1. If $\alpha = 0$ holds, then we have $-\mathcal{K}_X = (n - 2, m)$ and thus $q(X) = \text{gcd}(n - 2, m)$. We conclude

$$2(q(X) - 1) \leq 2 \min(n - 2, m) - 2 \leq (n - 2 + m) - 2 < \dim(X).$$

Now let $\alpha > 0$. With $k := n - 2 + a_1 + \dots + a_m$, we have $-\mathcal{K}_X = (k, m)$ and $q(X)$ divides both entries. This implies $q(X) \leq m$. If $q(X) < m$ holds, then, because of $n \geq 5$, we obtain

$$2(q(X) - 1) \leq 2\left(\frac{m}{2} - 1\right) = m - 2 \leq n + m - 7 < \dim(X).$$

If we have $q(X) = m$, then m divides k . Thus, the Fano condition $\alpha m < k$ implies $(\alpha + 1)m \leq k$. Moreover, $\alpha > 0$ implies $a_1 + \dots + a_m < \alpha m$. Together, we obtain

$$2(q(X) - 1) = 2m - 2 \leq k - (\alpha + 1 - 2)m - 2 < n - 2 + m - 2 < \dim(X).$$

Let X be of Type 2. Then we have $-\mathcal{K}_X = ((n/2 - 1)\alpha + m, n - 2)$, and $q(X)$ divides both entries. If $q(X) < n - 2$ holds, then using $m \geq 2$, we obtain

$$2(q(X) - 1) \leq 2\left(\frac{n - 2}{2} - 1\right) = n - 4 \leq n + m - 6 < \dim(X).$$

We are left with discussing the case $q(X) = n - 2$. If $\alpha = 0$ holds, then we obtain $-\mathcal{K}_X = (m, n - 2)$ and thus $n - 2 \leq m$. We conclude

$$2(q(X) - 1) = 2((n - 2) - 1) = (n - 2) + (n - 4) \leq n + m - 4 < \dim(X).$$

Next, let $\alpha = 1$. Then $-\mathcal{K}_X = ((n - 2)/2 + m, n - 2)$ holds and $q(X) = n - 2$ divides the first entry. Thus, with a suitable $k \in \mathbb{Z}$, we have

$$m = \frac{2k + 1}{2}(n - 2).$$

In the case $\alpha = 1$, the Fano condition reads as $m > (n - 2)/2$, and thus $k \geq 1$ holds. Moreover, because of $n > 4$, we have $n/2 - 1 < n - 3$ and thus obtain

$$2(q(X) - 1) = 2(n - 2 - 1) < \frac{3}{2}(n - 2) + \frac{n - 2}{2} \leq m + \frac{n}{2} - 1 < \dim(X).$$

Finally, let $\alpha \geq 2$. Then the Fano condition says $n - 2 < 2m/\alpha$. Consequently, we obtain

$$2(q(X) - 1) = (n - 2) + (n - 4) < \frac{2m}{\alpha} + (n - 4) \leq n + m - 4 < \dim(X).$$

Let X be of Type 3. Then $-\mathcal{K}_X = (n - 2 + m, n - 2)$ holds and the Fano condition yields $m < n - 2$. As $q(X)$ divides both entries of $-\mathcal{K}_X$, we see $q(X) \neq n - 2$ and thus $q(X) \leq (n - 2)/2$. We conclude

$$2(q(X) - 1) \leq 2\left(\frac{n - 2}{2} - 1\right) = n - 4 < \dim(X).$$

Let X be of Type 4. Then $-\mathcal{K}_X = ((n/2 - 1)(\alpha + 1) + a_1 + \cdots + a_m, n/2 - 1 + m)$ holds. In the case $\alpha = 0$, all the a_i vanish as well, we obtain $q(X) \leq n/2 - 1$ and thus

$$2(q(X) - 1) \leq n - 4 \leq n + m - 4 < \dim(X).$$

Let $\alpha > 0$. If $q(X) < n/2 - 1 + m$ holds, then even $q(X) \leq (n/2 - 1 + m)/2$ must hold and, because of $n < 0$, we obtain

$$2(q(X) - 1) \leq \frac{n}{2} - 1 + m - 2 < \dim(X).$$

We discuss the case $q(X) = n/2 - 1 + m$. The first component of $-\mathcal{K}_X$ equals $\beta q(X)$ with some positive integer β . Plugging the Fano condition

$$\alpha m - (n/2 - 1) < a_1 + \cdots + a_m$$

into this equality leads to the estimate $\alpha + 1 \leq \beta$. Comparing $(\alpha + 1)q(X)$ with the first component $\beta q(X)$ of $-\mathcal{K}_X$ gives

$$(\alpha + 1)q(X) \leq \left(\frac{n}{2} - 1\right)(\alpha + 1) + a_1 + \cdots + a_m < \left(\frac{n}{2} - 1\right)(\alpha + 1) + \alpha m,$$

where the last inequality is due to the fact that $\alpha > 0$ forces vanishing of at least one of the a_j . This allows us to conclude the discussion by

$$\begin{aligned} 2(q(X) - 1) &= ((\alpha + 1) - (\alpha + 1 - 2))q(X) - 2 \\ &< \left(\left(\frac{n}{2} - 1\right)(\alpha + 1) + \alpha m\right) - (\alpha - 1)q(X) - 2 \\ &= n + m - 4 < \dim(X). \end{aligned}$$

5 Proof of Theorem 1.3

A first step is the general bound for the Picard number of (possibly singular) Fano full intrinsic quadrics provided in Proposition 5.1. Then we prepare the proof of Theorem 1.3, which is given at the end of the section. We will mostly work in the setting of standard intrinsic quadrics $X = X(q, t, m, u)$ arising from Construction 2.4. We write $g = g_{q,t}$ for the relation, and the degrees of the variables in $\text{Cl}(X) = K$ will be denoted as

$$\begin{aligned} w_i &= \deg(T_i) = Q(e_i) \text{ for } i = 1, \dots, n, \\ w_{n+j} &= \deg(S_j) = Q(e_{n+j}) \text{ for } j = 1, \dots, m. \end{aligned}$$

Proposition 5.1 *Let X be a Fano full standard intrinsic quadric arising from Construction 2.4.*

- (i) *If $t > 1$ holds, i.e., g has at least two squares, then we have $\rho(X) = 1$.*
- (ii) *If $t = 1$ holds, i.e., g has one square, then we have $\rho(X) \leq 2$.*
- (iii) *If $t = 0$ holds, i.e., g has no squares, then we have $\rho(X) \leq 3$.*
- (iv) *If $\rho(X) = 3$ holds, then we have $t = 0$ and X is \mathbb{Q} -factorial.*

Proof We have $X = X(q, t, m, u)$ with $m = 0$, and, according to Proposition 2.10, we may assume that u is the anticanonical class. Proposition 4.2 tells us that in $K_{\mathbb{Q}}$, we have

$$u = \frac{q + t - 2}{2} \deg(g).$$

A face $\gamma_0 \leq \gamma$ is X -relevant if and only if it satisfies the conditions of Remark 2.7 and one has $u \in \text{relin}(Q(\gamma_0))$. For Assertions (i), (ii), and (iii), we consider the following X -relevant faces of γ :

$$(i) \ \gamma' = \gamma_{q+1, q+2}, \quad (ii) \ \gamma' = \gamma_{1, 2, q+1}, \quad (iii) \ \gamma' = \gamma_{1, 2, 3, 4}.$$

We have $\dim(Q(\gamma')) \leq 1, 2, 3$ according to the cases, because homogeneity of g yields the following linear relations in the respective images $Q(\text{lin}_{\mathbb{Q}}(\gamma_0))$:

$$2w_{q+1} = 2w_{q+2}, \quad w_1 + w_2 = w_{q+1}, \quad w_1 + we_2 = we_3 + w_4.$$

The first three assertions thus follow from the description of the Picard group provided by Proposition 2.9: in each of the three cases, we have

$$\text{Pic}(X) = \bigcap_{\gamma_0 \in \text{cov}(X)} Q(\text{lin}_{\mathbb{Z}}(\gamma_0)) \subseteq Q(\gamma').$$

In order to prove (iv), assume $\rho(X) = 3$. Assertions (i) and (ii) yield $t = 0$. To obtain \mathbb{Q} -factoriality of X , we have to show that $K_{\mathbb{Q}} = \text{Cl}(X)_{\mathbb{Q}}$ is of dimension three. The assumption $\rho(X) = 3$ together with Proposition 2.9 yields $\dim(Q(\gamma_0)) \geq 3$ for all X -relevant faces $\gamma_0 \leq \gamma$. Consider the faces

$$\gamma(i, j) := \gamma_{i, i+1, j, j+1} \leq \gamma,$$

where i, j are odd with $1 \leq i < j \leq q - 1$. These are all X -relevant and $Q(\gamma(i, j))$ is of dimension three. Using $\rho(X) = 3$ and Proposition 2.9 again, we conclude that

the $Q(\gamma(i, j))$ generate all the same 3-dimensional vector subspace $V \subseteq K_{\mathbb{Q}}$. Thus $\dim(K_{\mathbb{Q}}) = 3$ follows from

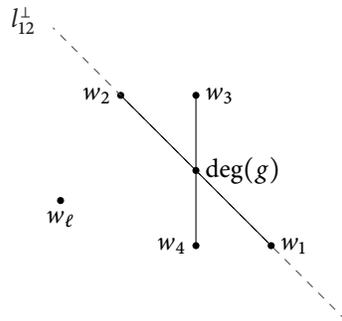
$$K_{\mathbb{Q}} = Q(\mathbb{Q}^{n+m}) = Q(\text{lin}_{\mathbb{Q}}(\gamma(1, 3)) + \cdots + \text{lin}_{\mathbb{Q}}(\gamma(q - 3, q))) = V. \quad \blacksquare$$

Corollary 5.2 *Let X be a Fano full intrinsic quadric. Then $\rho(X) \leq 3$ holds, and if we have $\rho(X) = 3$, then X is \mathbb{Q} -factorial.*

We start our preparations for the proof of Theorem 1.3. When performing a renumberation of variables, we always keep $g = g_{q,t}$ a standard K -homogeneous quadratic polynomial, which means that our renumberings respect monomials and take place only inside the q -, t - and m -blocks. Moreover, when visualizing the situation, we draw (parts of) the intersection of $\text{Eff}(X) = Q(\gamma)$ with an affine hyperplane passing orthogonally through an inner vector of $\text{Eff}(X)$ and we will indicate the ray through, for instance, w_i by a dot with label w_i .

Lemma 5.3 *Let $X = X(q, t, m, u)$ be a \mathbb{Q} -factorial standard intrinsic quadric of Picard number three with $q \geq 4$. If there is an ℓ with $5 \leq \ell \leq q$ or $n + 1 \leq \ell \leq n + m$ such that $\gamma_{1,2,\ell} \leq \gamma$ is Z -relevant, then there are $1 \leq i \leq 2$ and $3 \leq j \leq 4$ such that $\gamma_{i,j,\ell} \leq \gamma$ is X -relevant.*

Proof Let l_{12} be a linear form on $K_{\mathbb{Q}} = \text{Cl}_{\mathbb{Q}}(X)$ with $l_{12}(w_1) = l_{12}(w_2) = 0$ and $l_{12}(w_{\ell}) \geq 0$. Then $l_{12}(\text{deg}(g)) = 0$ holds, and we can assume $l_{12}(w_3) \leq 0$.



Moreover, $Q(\gamma_{1,2,\ell})$ is contained in $Q(\gamma_{1,3,\ell}) \cup Q(\gamma_{2,3,\ell})$. Thus, u lies in the relative interior of a face τ of one of the latter two cones. We have $\tau = Q(\gamma_0)$ with a face γ_0 of $\gamma_{1,3,\ell}$ or $\gamma_{2,3,\ell}$. Since γ_0 is an \bar{X} -face, Proposition 2.11 yields that $Q(\gamma_0)$ must be of dimension three. Thus, γ_0 equals $\gamma_{1,3,\ell}$ or $\gamma_{2,3,\ell}$. \blacksquare

Lemma 5.4 *Let $X = X(q, t, m, u)$ be a \mathbb{Q} -factorial standard intrinsic quadric of Picard number three with $q \geq 6$. If $\gamma_{1,2} \leq \gamma$ is Z -relevant, then $\gamma_{i,3,j}$ is X -relevant for some $1 \leq i \leq 2$ and $5 \leq j \leq 6$.*

Proof Let l_{12} be a linear form on $K_{\mathbb{Q}}$ with $l_{12}(w_1) = l_{12}(w_2) = 0$ and $l_{12}(w_3) \leq 0$. We can assume that $l_{12}(w_5) \geq 0$, and then we have $Q(\gamma_{1,2}) \subseteq Q(\gamma_{1,3,5}) \cup Q(\gamma_{2,3,5})$. As in the previous proof, we conclude that u lies in the relative interior of $Q(\gamma_{1,3,5})$ or that of $Q(\gamma_{2,3,5})$. \blacksquare

Lemma 5.5 *Let $X = X(q, t, m, u)$ be a \mathbb{Q} -factorial full standard intrinsic quadric of Picard number three. Then we have $q \geq 6$ and, after suitable renumbering of variables, $\gamma_{1,3,5}$ is X -relevant.*

Proof As the moving cone of X is of dimension three, we must have $q \geq 6$; use Proposition 2.10. The effective cone of X is generated by w_1, \dots, w_q . Thus, Carathéodory’s theorem yields a Z -relevant face $\tau \leq \gamma$ generated by at most three of e_1, \dots, e_q . Suitably renumbering the variables, we achieve $\tau \leq \gamma_{1,3,5}$ or $\tau \leq \gamma_{1,2,3}$. Since all rays of τ are X -relevant, Proposition 2.11 shows that τ is at least of dimension two. If $\dim(\tau) = 2$ holds, then Proposition 2.11 yields that τ is not an \bar{X} -face, which means that $\tau = \gamma_{1,2}$. In this case, Lemma 5.4 gives the assertion. If τ is three-dimensional, then Lemma 5.3 completes the proof. ■

Lemma 5.6 *Let $X = X(q, t, m, u)$ be a standard intrinsic quadric of Picard number three. If there are pairwise different odd integers $1 \leq a, b, c \leq q - 1$ such that $\tau_0 := \gamma_{a,b,c}$ and $\tau_1 := \gamma_{a,b,c+1}$ are X -relevant, then X is not locally factorial.*

Proof Assume that X is locally factorial. Applying Proposition 2.8(iii) to τ_0 gives $\text{Cl}(X) \cong \mathbb{Z}^3$. Using K -homogeneity of g and suitable coordinates on $K = \text{Cl}(X)$, we achieve

$$[w_a, w_{a+1}, w_b, w_{b+1}, w_c, w_{c+1}] = \left[\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & d_2 & 1 & d_2 - 1 & 0 & d_2 \\ 0 & d_3 & 0 & d_3 & 1 & d_3 - 1 \end{array} \right],$$

where $(d_1, d_2, d_3) = \text{deg}(g)$. Applying Proposition 2.8(iii) to τ_1 gives $d_3 \in \{0, 2\}$. Let $\tau_2 := \gamma_{a,a+1,b,b+1}$. Since $Q(\tau_0)^\circ \cap Q(\tau_1)^\circ$ is three-dimensional and contained in $Q(\tau_2)$, we conclude that the \bar{X} -face τ_2 is X -relevant. Proposition 2.8 applied to τ_2 yields $d_3 \in \{-1, 1\}$, a contradiction. ■

Lemma 5.7 *Let $X = X(q, t, m, u)$ be a locally \mathbb{Q} -factorial standard intrinsic quadric of Picard number three with $q \geq 6$. Then, after suitably renumbering the variables, $\gamma_{1,3,5}$ is X -relevant and $\text{deg}(g) \in Q(\gamma_{1,3,5})$ holds.*

Proof According to Lemma 5.5, we can assume that $\gamma_{1,3,5}$ is X -relevant. If $\text{deg}(g)$ is contained in $Q(\gamma_{1,3,5})$, then we are done. Otherwise, suitably renumbering the variables once more, we arrive at one of the situations in Figure 2.

In the right-hand setting, exchanging T_1 and T_2 yields the assertion. So, consider the left-hand setting. Applying Lemma 5.6 to the X -relevant face $\gamma_{1,3,5}$ yields that neither $\gamma_{1,3,6}$ nor $\gamma_{2,3,5}$ is X -relevant. Note that we have

$$u \in Q(\gamma_{1,3,5})^\circ \subseteq Q(\gamma_{1,3,6}) \cup Q(\gamma_{2,3,6}) \cup Q(\gamma_{2,3,5})$$

and that all faces of $\gamma_{1,3,6}, \gamma_{2,3,6}$ and $\gamma_{2,3,5}$ are \bar{X} -faces. Proposition 2.11 shows that $\gamma_{2,3,6}$ is X -relevant. After exchanging T_1 and T_2 as well as T_5 and T_6 , the new $\gamma_{1,3,5}$ is X -relevant, and we have $\text{deg}(g) \in Q(\gamma_{1,3,5})$. ■

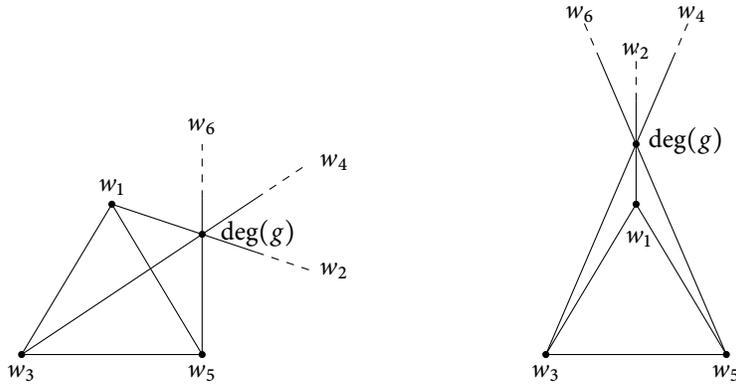


Figure 2

Lemma 5.8 *Let $X = X(q, t, m, u)$ be a standard intrinsic quadric of Picard number three. If there are pairwise different odd integers $1 \leq a, b, c \leq q - 1$ such that $\tau_0 := \gamma_{a,b,c}$ and $\tau_1 := \gamma_{a+1,b+1,c+1}$ are X -relevant, then X is not locally factorial.*

Proof Assume that X is locally factorial. Applying Proposition 2.8(iii) to τ_0 gives $\text{Cl}(X) \cong \mathbb{Z}^3$. Using K -homogeneity of g and suitable coordinates on $K = \text{Cl}(X)$, we achieve

$$[w_a, w_{a+1}, w_b, w_{b+1}, w_c, w_{c+1}] = \left[\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & d_2 & 1 & d_2 - 1 & 0 & d_2 \\ 0 & d_3 & 0 & d_3 & 1 & d_3 - 1 \end{array} \right],$$

where $(d_1, d_2, d_3) = \text{deg}(g)$. Consider $\tau_{i,j} := \text{cone}(e_i, e_{i+1}, e_j, e_{j+1})$, where $i, j \in \{a, b, c\}$ with $i \neq j$. For all three possibilities, we have

$$Q(\tau_0)^\circ \cap Q(\tau_1)^\circ \subseteq Q(\tau_{i,j})^\circ.$$

Thus, all the $\tau_{i,j}$ are X -relevant. Proposition 2.8(iii) says that $w_i, w_{i+1}, w_j, w_{j+1}$ generate K in all cases, which implies $d_1, d_2, d_3 \in \{-1, 1\}$. Consequently,

$$\det(w_{a+1}, w_{b+1}, w_{c+1}) = d_1 + d_2 + d_3 - 1 \in \{0, 2\}.$$

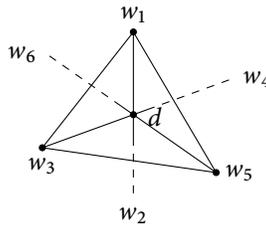
But Proposition 2.8(iii) applied to $\tau_1 = \gamma_{a+1,b+1,c+1}$ shows that this determinant should equal ± 1 , a contradiction. ■

Lemma 5.9 *Let $X = X(q, t, m, u)$ be a locally factorial full standard intrinsic quadric of Picard number three. Then $K = \text{Cl}(X) \cong \mathbb{Z}^3$ and $q \geq 6$ hold. Moreover, by a suitable reenumeration of variables, we achieve*

$$[w_1, \dots, w_6] = \left[\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array} \right],$$

where $d_1 \in \mathbb{Z}_{\geq 0}$, the faces $\gamma_{1,3,5}, \gamma_{1,4,6}, \gamma_{1,2,3,4}, \gamma_{1,2,5,6}$ are all X -relevant, and, moreover, $u \in \text{cone}(w_1, w_3, d) \cap Q(\gamma_{1,4,6})^\circ$ holds, where $d = (d_1, 1, 1) = \text{deg}(g)$. In particular, we have $t = 0$, and $n = q$ is even.

Proof Lemmas 5.5 and 5.7 show that $q \geq 6$ holds and that after suitably renumbering the variables, $\gamma_{1,3,5}$ is an X -relevant face with $d = \text{deg}(g) \in Q(\gamma_{1,3,5})$. By Proposition 2.8(iii), the cone $Q(\gamma_{1,3,5})$ is of dimension three and w_1, w_3, w_5 freely generate $K = \mathbb{Z}^3$.



Note that d might as well lie on the boundary of $Q(\gamma_{1,3,5})$. However, u lies in the relative interior of $Q(\gamma_{1,3,5})$, and, suitably renumbering the variables T_1, \dots, T_6 , we achieve that $\tau := \text{cone}(w_1, w_3, d)$ satisfies

$$\dim(\tau) = 3, \quad u \in \tau \setminus Q(\gamma_{1,3}).$$

Note that we have $w_2 \notin \text{cone}(w_1, d)$ and $w_4 \notin \text{cone}(w_3, d)$, because otherwise w_1 or w_3 would lie on $\text{cone}(d)$, contradicting $\dim(\tau) = 3$. We conclude that $u \in Q(\gamma_{1,2,3,4})^\circ$, and thus $\gamma_{1,2,3,4}$ is X -relevant. Observe that

$$\tau \subseteq Q(\gamma_{1,5,6}) \cup Q(\gamma_{3,5,6}) \cup Q(\gamma_{1,3,6}).$$

As X is locally factorial and $\gamma_{1,3,5}$ is X -relevant, Lemma 5.6 shows $u \notin Q(\gamma_{1,3,6})^\circ$. Thus, u lies in one of the other two r.h.s. cones in Figure 2. Suitably renumbering T_1, \dots, T_4 , we achieve $u \in Q(\gamma_{1,5,6})$. Then also $u \in Q(\gamma_{1,2,5,6})$ holds. Now,

$$\gamma_{1,2,5,6}, \gamma_{1,5}, \gamma_{1,6}, \gamma_{2,5}, \gamma_{2,6}, \gamma_1, \gamma_2, \gamma_5, \gamma_6$$

are \overline{X} -faces. Thus, Proposition 2.11 yields that u does not lie in any of the corresponding $Q(\gamma_i)$ and $Q(\gamma_{i,j})$. Consequently, $Q(\gamma_{1,2,5,6})$ is three-dimensional and contains u in its relative interior. That means that $\gamma_{1,2,5,6}$ is X -relevant. Observe that

$$u \in Q(\gamma_{1,5,6}) \cap \tau \subseteq Q(\gamma_{1,4,6}) \cup Q(\gamma_{2,4,6}).$$

Applying Lemma 5.8 to $\gamma_{1,3,5}$, we see that $\gamma_{2,4,6}$ is not X -relevant. Moreover, all faces of the two cones $\gamma_{1,4,6}$ and $\gamma_{2,4,6}$ are \overline{X} -relevant. We conclude $u \in Q(\gamma_{1,4,6})^\circ$ and thus $\gamma_{1,4,6}$ is X -relevant. A suitable choice of coordinates on $K = \mathbb{Z}^3$ yields

$$[w_1, \dots, w_6] = \left[\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & d_2 & 1 & d_2 - 1 & 0 & d_2 \\ 0 & d_3 & 0 & d_3 & 1 & d_3 - 1 \end{array} \right].$$

Because of $d \in Q(\gamma_{1,3,5}) = \mathbb{Q}_{\geq 0}^3$, we have $d_1, d_2, d_3 \geq 0$. Proposition 2.8(iii) together with the X -relevant faces $\gamma_{1,2,3,4}$ and $\gamma_{1,2,5,6}$ show that $d_2 = d_3 = 1$ holds. ■

Proof of Theorem 1.3 Proposition 3.1 and Theorem 1.1 settle the case of Picard number at most two. Proposition 5.1 settles the case of Picard number at least four. The remaining task is to consider smooth full intrinsic quadrics X of Picard number three. By Proposition 2.5, we can assume that $X = X(q, t, m, u)$ is a standard intrinsic quadric. Moreover, Lemma 5.9 says $\text{Cl}(X) = \mathbb{Z}^3$ and that by a suitable choice of coordinates, we have

$$[w_1, \dots, w_6] = \left[\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array} \right],$$

where $d_1 \in \mathbb{Z}_{\geq 0}$ and the ample cone of X is contained in $Q(\gamma_{146})^\circ$. Proposition 4.2 tells us that $-\mathcal{K}_X$ is a multiple of $\text{deg}(g) = (d_1, 1, 1)$. But $(d_1, 1, 1)$ cannot be represented as a strict positive combination over w_1, w_4 and w_6 . Thus, $\text{deg}(g)$ is not contained in $Q(\gamma_{146})^\circ$. Consequently, $-\mathcal{K}_X$ is not ample and hence X is not Fano. ■

6 Proof of Theorem 1.4

A detailed analysis of the combinatorics of the X -relevant faces together with the resulting conditions on determinants provided by Proposition 2.8 lead to the normal form asserted in the theorem. This is the first part of the proof. The second one establishes the geometric supplements. At the end of the section, we prove Corollary 6.1.

Proof of Theorem 1.4, Part I According to Proposition 2.5, we can assume that $X = X(q, t, m, u)$ is a standard intrinsic quadric. Lemma 5.9 tells us $n = q \geq 6$ and $K = \mathbb{Z}^3$. Moreover, choosing suitable coordinates on K , we achieve that

$$[w_1, \dots, w_6] = \left[\begin{array}{cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{array} \right]$$

holds with $d_1 \in \mathbb{Z}_{\geq 0}$, the faces $\gamma_{1,3,5}, \gamma_{1,4,6}, \gamma_{1,2,3,4}, \gamma_{1,2,5,6}$ of γ are all X -relevant, and the ample class u of X satisfies

$$u \in \text{cone}(w_1, w_3, d) \cap Q(\gamma_{1,4,6})^\circ,$$

where $d = (d_1, 1, 1)$ denotes the degree of the relation g . Depending on d_1 , the situation looks as in Figure 3.

We claim that $n \geq 8$ holds. Otherwise, $n = 6$, and according to Proposition 2.10, we have

$$\text{Mov}(X) \subseteq \text{cone}(w_1, w_3, w_5) \cap \text{cone}(w_2, w_4, w_6).$$

By Lemma 5.8, this contradicts smoothness of X . Thus, we obtain $n \geq 8$, which implies in particular that $\dim(X) \geq 4$.

We specify the possible positions of the weights w_ℓ , where $\ell = 7, \dots, n$. For $i = 1, \dots, 6$ choose linear forms l_i on $K_{\mathbb{Q}}$ such that

$$l_i(w_i) = l_i(u) = 0, \quad i = 1, \dots, 6, \\ l_i(w_1) > 0, \quad i = 3, \dots, 6, \quad l_2(w_4) > 0, \quad l_1(w_3) > 0.$$

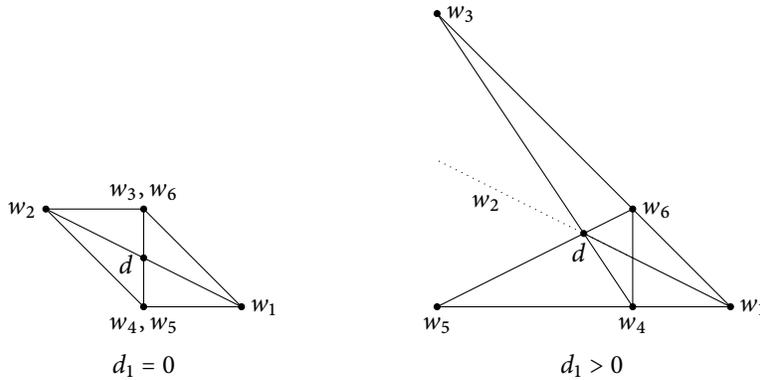


Figure 3

Each of the linear forms l_1, \dots, l_6 defines a negative half space and a positive half space:

$$H_i^- := \{x \in K_{\mathbb{Q}}; l_i(x) \leq 0\}, \quad H_i^+ := \{x \in K_{\mathbb{Q}}; l_i(x) \geq 0\}.$$

Note that $\gamma_{i,\ell}$ is an \bar{X} -face for all $i = 1, \dots, 6$ and $\ell = 7, \dots, n$. Thus, Proposition 2.11 yields that u cannot lie in $Q(\gamma_{i,\ell})$. In other words, for all $i = 1, \dots, 6$ and $\ell = 7, \dots, n$, we have

$$w_\ell \notin H_i := \mathbb{Q}u - \mathbb{Q}_{\geq 0}w_i \subseteq H_i^- \cap H_i^+.$$

The half planes H_1, \dots, H_6 define a subdivision of $K_{\mathbb{Q}} = \mathbb{Q}^3$ into the following three-dimensional cones, all having $\mathbb{Q}u$ as a common line:

$$\begin{aligned} M_a &:= H_1 + H_4, & M_b &:= H_4 + H_5, & M_c &:= H_5 + H_2, \\ M_d &:= H_2 + H_3, & M_e &:= H_3 + H_6, & M_f &:= H_6 + H_1. \end{aligned}$$

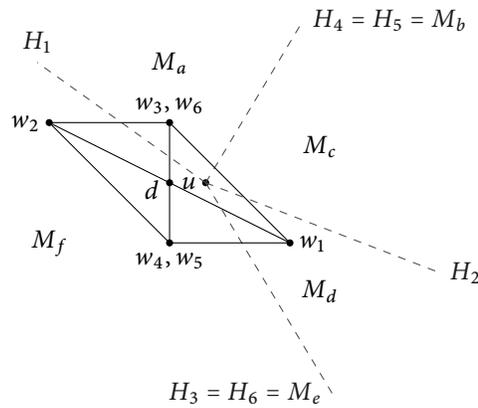
As observed before, the degrees w_ℓ , where $\ell = 7, \dots, n$, are distributed over the relative interiors $M_a^\circ, \dots, M_f^\circ$. According to the cases $d_1 = 0$ and $d_1 > 0$, the situation looks as in Figure 4.

We show $w_\ell \notin M_b$ for $\ell = 7, \dots, n$. Otherwise, $w_\ell = (x, y, z) \in M_b^\circ$ holds. Then $\gamma_{1,\ell,5}, \gamma_{4,\ell,5}, \gamma_{2,4,\ell}$ and $\gamma_{6,4,\ell}$ are X -relevant. The way we list the indices i, j, k for the $\gamma_{i,j,k}$ ensures that $\det(w_i, w_j, w_k)$ is positive and thus, by Proposition 2.8(iii), equals one. For $1, \ell, 5$, this implies that $y = 1$. Looking at $4, \ell, 5$ yields $d_1 = 1$. Taking $2, 4, \ell$ gives $z = x$. But this leads to $\det(w_6, w_4, w_\ell) = -1$, a contradiction.

We show that $w_\ell \notin M_e$ for $\ell = 7, \dots, n$. Otherwise, $w_\ell = (x, y, z) \in M_e^\circ$ holds and thus $\gamma_{1,3,\ell}, \gamma_{6,3,\ell}, \gamma_{6,2,\ell}, \gamma_{6,4,\ell}$ are X -relevant. Again the indices are listed in a way that $\det(w_i, w_j, w_k) = 1$ holds. For $1, 3, \ell$, this means that $z = 1$. Then $6, 3, \ell$ brings us to $d_1 = 1$. Taking $6, 2, \ell$ yields $y = x$. But then $\det(w_6, w_4, w_\ell) = -1$ holds, a contradiction.

Next observe that $\{w_\ell, w_{\ell+1}\} \notin M_f$ holds for all odd $\ell \geq 7$, because otherwise we have the X -relevant faces $\gamma_{1,3,\ell}$ and $\gamma_{1,3,\ell+1}$, contradicting Lemma 5.6. Hence, suitably

$(d_1 = 0)$



$(d_1 > 0)$

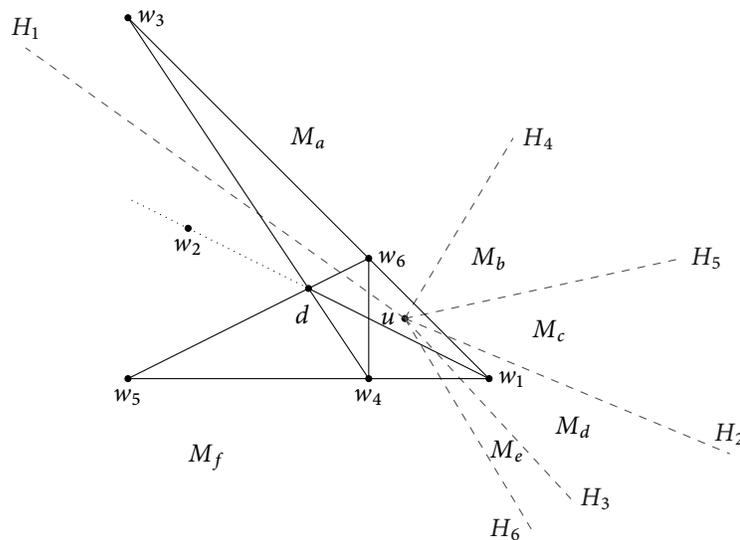


Figure 4

renumbering the variables T_7, \dots, T_n , we achieve $w_\ell \in M_a^\circ \cup M_c^\circ \cup M_d^\circ$ for all odd $7 \leq \ell < n$. Thus, for a given odd $\ell \geq 7$, Table 1 shows the possibilities for the positions of the pair $w_\ell, w_{\ell+1}$.

Here, the position of $w_{\ell+1}$ is determined by $w_\ell + w_{\ell+1} = \deg(g)$. Moreover, the $\gamma_{i,j,k}$ occurring in the table are some but not necessarily all X -relevant faces containing e_ℓ or $e_{\ell+1}$ and the indices i, j, k are listed in such a manner that $\det(w_i, w_j, w_k) = 1$ holds. We now discuss each of these cases.

Case	Pos. of w_ℓ	Pos. of $w_{\ell+1}$	Resulting X -relevant faces
(1)	M_a°	M_d°	$\gamma_{1,\ell,5}, \gamma_{3,5,\ell+1}, \gamma_{6,4,\ell+1}$
(2)	M_a°	M_f°	$\gamma_{1,\ell,4}, \gamma_{1,6,\ell+1}$
(3)	M_c°	M_f°	$\gamma_{\ell,3,5}, \gamma_{1,3,\ell+1}, \gamma_{\ell,2,4}$
(4)	M_d°	M_f°	$\gamma_{\ell,3,5}, \gamma_{1,6,\ell+1}, \gamma_{\ell,6,4}, \gamma_{\ell,3,2}, \gamma_{\ell+1,3,1}$

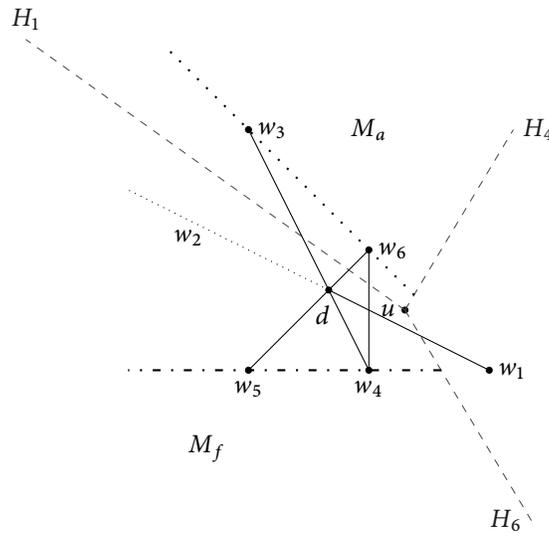
Table 1

Case (1). Write $w_\ell = (x, y, z)$. Then $\det(w_1, w_\ell, w_5) = 1$ implies that $y = 1$ and $\det(w_3, w_5, w_{\ell+1}) = 1$ gives $x = d_1 - 1$. Moreover, $\det(w_6, w_4, w_{\ell+1}) = 1$ leads to $z = 1$. Thus, we arrive at $w_1 = w_{\ell+1}, w_2 = w_\ell$, which contradicts $w_\ell \in M_a^\circ$.

Case (2). Write $w_\ell = (x, y, z)$. Then $\det(w_1, w_\ell, w_4) = 1$ implies $y = 1$. From $\det(w_1, w_6, w_{\ell+1})$, we derive $z = 0$. Thus, we obtain

$$[w_1, w_2, w_3, w_4, w_5, w_6, w_\ell, w_{\ell+1}] = \left[\begin{array}{cc|cc|cc|cc} 1 & d_1 - 1 & 0 & d_1 & 0 & d_1 & x & d_1 - x \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right].$$

The weights are arranged as follows, where w_2 lies on the dotted line, w_ℓ on the thickly dotted line, and $w_{\ell+1}$ on the dash-dotted line.



We remark that $n = 8$ with w_7, w_8 as in Case (2) is not possible. Indeed, otherwise, we have $u \notin \text{cone}(w_2, \dots, w_6, w_7, w_8)$, where by Proposition 2.10, the latter cone contains the moving cone X and thus u ; a contradiction. Moreover, we note that for any $w_\ell, w_{\ell+1}$ of Case (2), we have

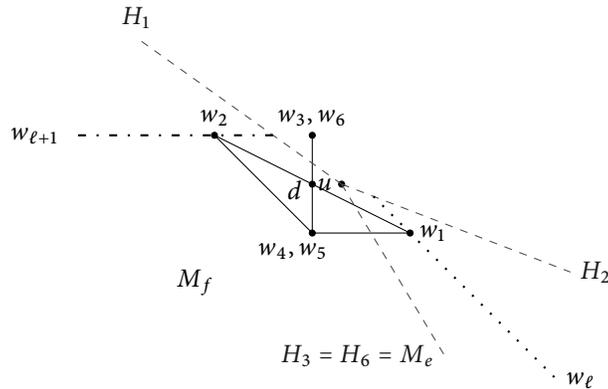
$$\text{Sample}(X) \subseteq Q(\gamma_{1,6,4}) \cap Q(\gamma_{1,\ell,4}) \cap Q(\gamma_{1,6,\ell+1}).$$

Case (3). Write $w_\ell = (x, y, z)$. Then $\det(w_\ell, w_3, w_5) = 1$ shows $x = 1$. Moreover, $\det(w_\ell, w_2, w_4) = 1$ implies $y = d_1 z$. Finally, $\det(w_1, w_3, w_{\ell+1}) = 1$ leads to $z = 0$. We arrive at $w_\ell = w_1$ and thus $w_\ell \notin M_c^\circ$, a contradiction.

Case (4). Write $w_\ell = (x, y, z)$. Then $\det(w_\ell, w_3, w_5) = 1$ and $\det(w_1, w_6, w_{\ell+1}) = 1$ show $x = 1$ and $z = 0$. Now $\det(w_\ell, w_6, w_4) = 1$ yields $d_1 y = 0$. We distinguish the cases $d_1 = 0$ and $d_1 > 0$.

Case (4.1): We have $d_1 = 0$. Here, we have the following situation, where, in the figure, $w_\ell = (1, y, 0)$ lies on the dotted line and $w_{\ell+1}$ on the dash-dotted line.

$$[w_1, w_2, w_3, w_4, w_5, w_6, w_\ell, w_{\ell+1}] = \left[\begin{array}{cc|cc|cc|cc} 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 & y & 1-y \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right],$$



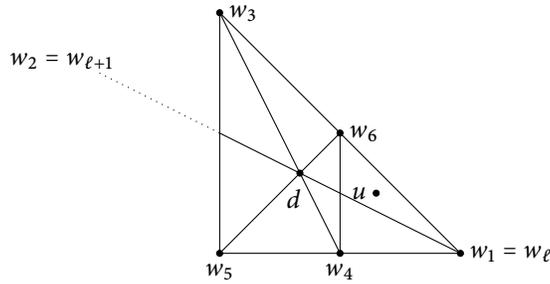
Applying Proposition 2.10 to the resulting X -relevant faces of the present case, we arrive at

$$\text{Sample}(X) \subseteq Q(\gamma_{1,3,5}) \cap Q(\gamma_{\ell,3,5}) \cap Q(\gamma_{\ell,3,2}) \cap Q(\gamma_{\ell+1,3,1}).$$

Case (4.2): We have $d_1 > 0$. Then $y = 0$ must hold. This implies $w_\ell = w_1$ and $w_{\ell+1} = w_2$. For the semiample cone, we have

$$\text{Sample}(X) \subseteq Q(\gamma_{1,4,6}) \cap Q(\gamma_{1,2,6}).$$

Moreover, the weights are arranged as in the figure below, where $w_2 = w_{\ell+1}$ lies on the dotted line.



Subsuming the discussion so far, we see that only the Cases (2) and (4) allow weights w_i , where $i \geq 7$. The remaining task is to check in which ways these cases can be combined. So, let us go through the possible constellations of the pairs w_i, w_{i+1} for $i = 7, 9, \dots, n - 1$.

(a) All pairs w_i, w_{i+1} , where $i = 7, 9, \dots, n - 1$, are from Case (2). In the discussion of Case (2), we have seen that $n \geq 10$ must hold. For odd $i \geq 7$, we have $w_i = (x_i, 1, 0)$ and $w_{i+1} = (d_1 - x_i, 0, 1)$, where we can assume $x_7 \geq x_9 \geq \dots \geq x_{n-1}$. Now, the ample class u lies in the moving cone. Proposition 2.10 yields

$$u \in \text{cone}(w_2, w_3, \dots, w_{n-1}, w_n)^\circ.$$

We conclude $w_7 \in Q(\gamma_{1,6})^\circ$ and $w_n \in Q(\gamma_{1,4})^\circ$. This in turn implies that $x_7 > d_1$ and $x_{n-1} < 0$. Moreover, $Q(\gamma_{7,n})$ is a bounding face of the moving cone. Thus, we obtain

$$u \in Q(\gamma_{1,7,4})^\circ \cap Q(\gamma_{1,6,n})^\circ \setminus Q(\gamma_{1,7,n}) \subseteq Q(\gamma_{7,2,n})^\circ.$$

We conclude that $\gamma_{7,2,n}$ is X -relevant. Applying Proposition 2.8(iii) and the estimates for x_7 and x_n just obtained, we arrive at a contradiction, showing that the present setting cannot occur:

$$1 = \det(w_7, w_2, w_n) = d_1 - x_{n-1} + x_7 - d_1 + 1 \geq 2.$$

(b) There is an even $7 < k \leq n$ such that for $i = 7, \dots, k - 1$, the pairs w_i, w_{i+1} are from Case (4.1) and for all odd $j = k + 1, \dots, n - 1$, we have $w_j = w_3$ and $w_{j+1} = w_4$. Then, for the odd $i = 7, \dots, k - 1$, we have $w_i = (1, y_i, 0)$ and $w_{i+1} = (-1, 1 - y_i, 1)$, where we can assume $y_7 \geq \dots \geq y_{k-1}$. Set

$$\alpha := \max(0, y_7), \quad \tilde{w}_1 := (1, \alpha, 0), \quad \beta := \min(0, y_{k-1}), \quad \tilde{w}_k := (-1, 1 - \beta, 1).$$

Then \tilde{w}_1 and \tilde{w}_k are degrees of variables, and they are closest to w_3 in the sense that $\tilde{w}_1 \in \text{cone}(w_3, w_i)$ and $\tilde{w}_k \in \text{cone}(w_3, w_{i+1})$ holds for $i = 1$ and $i = 7, \dots, k - 1$; see the figure in Case (4.1). Recall that the semiample cone is contained in the intersection of $Q(\gamma_{1,3,5})$ and $\text{cone}(d, w_1, w_3)$. We even claim that

$$\text{SAmple}(X) = \text{cone}(\tilde{w}_1, w_3, w_5) \cap \text{cone}(\tilde{w}_1, w_3, \tilde{w}_k).$$

By the definition of \tilde{w}_1 and \tilde{w}_k , we only have to show that both cones are images of X -relevant faces. For the first one this is clear. We discuss the second one. Observe that we have

$$u \in \text{cone}(\tilde{w}_1, w_3, \tilde{w}_k) \cup \text{cone}(\tilde{w}_1, w_4, \tilde{w}_k)$$

Thus, according to Proposition 2.11, the task is to show that $\text{cone}(\tilde{w}_1, w_4, \tilde{w}_k)$ is not the image of an X -relevant face. Indeed, this would contradict Lemma 5.8 applied to an X -relevant face projecting onto

$$\text{cone}(\tilde{w}_2, w_3, \tilde{w}_{k-1}), \quad \tilde{w}_2 := (-1, 1 - \alpha, 1), \quad \tilde{w}_{k-1} := (1, \beta, 0).$$

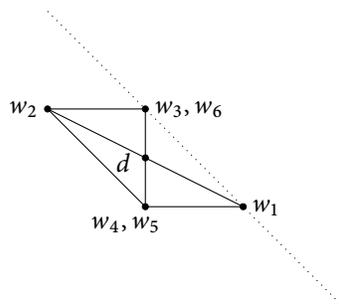
Now, the coordinate change on $K = \mathbb{Z}^3$ given by the following unimodular matrix and suitably renumbering of variables leads to the setting of Theorem 1.4:

$$\begin{bmatrix} 1 & 0 & 1 \\ -\beta & 1 & \alpha - \beta - 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) *There is an even $7 < k < n$ such that for $i = 7, \dots, k - 1$, the pairs w_i, w_{i+1} are from Case (4.1) and for all odd $j = k + 1, \dots, n - 1$ the pair w_j, w_{j+1} is from Case (2). Note that we have $d_1 = 0$ and the weights are of the form*

$$w_i = (1, y_i, 0), \quad w_{i+1} = (-1, 1 - y_i, 1), \quad w_j = (x_j, 1, 0), \quad w_{j+1} = (-x_j, 0, 1),$$

where we can assume that $y_7 \geq \dots \geq y_{k-1}$ and $x_{k+1} \geq \dots \geq x_{n-1}$. The weights are arranged as follows, where the w_i for $i = 7, \dots, k - 1$ and the w_j for $j = k + 1, \dots, n - 1$ lie on the dotted line:



The discussion on the Cases (4.1) and (2) performed so far shows that for all odd $i = 7, \dots, k - 1$ and $j = k + 1, \dots, n - 1$, the semisample cone of X satisfies

$$\text{Sample}(X) \subseteq Q(\gamma_{3,5,i}) \cap Q(\gamma_{1,4,j}) \subseteq Q(\gamma_{5,i,j}).$$

Since the semisample cone is full-dimensional, we see that $\gamma_{5,i,j}$ is X -relevant. Thus, we can apply Proposition 2.8 and obtain

$$1 = \det(w_5, w_i, w_j) = 1 - y_i x_j.$$

This leaves us with $y_i = 0$ for $i = 7, \dots, k - 1$ or $x_j = 0$ for $j = k + 1, \dots, n - 1$. If all the x_j vanish, then we are in the case just treated. So, assume that all the y_i vanish. Then we have $w_i = w_1$ and $w_{i+1} = w_2$ for $i = 7, \dots, k - 1$. Set

$$\alpha := \max(0, x_{k+1}), \quad \tilde{w}_3 := (\alpha, 1, 0), \quad \beta := \min(0, x_{n-1}), \quad \tilde{w}_n := (-\beta, 0, 1).$$

Then \tilde{w}_3 and \tilde{w}_n are the degrees of the variables sitting closest to w_1 and among the w_j and w_{j+1} with $j = 3$ or $j = k + 1, \dots, n - 1$. We claim that the semisample cone is given by

$$\text{Sample}(X) = \text{cone}(w_1, \tilde{w}_3, w_2) \cap \text{cone}(w_1, \tilde{w}_3, \tilde{w}_n).$$

As in the preceding case, we only have to show that both cones are projected X -relevant faces. For the first one this is clear. We turn to the second one. For sure we have

$$u \in \text{cone}(w_1, \tilde{w}_3, \tilde{w}_n) \cup \text{cone}(\tilde{w}_3, w_2, \tilde{w}_n).$$

We verify $u \notin \text{cone}(\tilde{w}_3, w_2, \tilde{w}_n)$. Otherwise, because of $\det(\tilde{w}_3, w_2, \tilde{w}_n) = 1 + \alpha - \beta$, Proposition 2.8(iii) yields $\alpha = \beta = 0$, a contradiction. Thus, $\text{cone}(w_1, \tilde{w}_3, \tilde{w}_n)$ is the image of an X -relevant face.

Now, the coordinate change on $K = \mathbb{Z}^3$ given by the following unimodular matrix and suitably renumbering of variables leads to the setting of Theorem 1.4:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -\beta & \alpha \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) *There is an even $7 < k < n$ such that for $i = 7, \dots, k - 1$, the pairs w_i, w_{i+1} are from Case (4.2) and for all odd $j = k + 1, \dots, n - 1$ the pair w_j, w_{j+1} is from Case (2).* This means

$$w_i = (1, 0, 0), \quad w_{i+1} = (d_1 - 1, 1, 1), \quad w_j = (x_j, 1, 0), \quad w_{j+1} = (d_1 - x_j, 0, 1),$$

A coordinate change on $K = \mathbb{Z}^3$ given by the following unimodular matrix and suitably renumbering of variables leads to the preceding case:

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 - d_1 \\ 0 & 0 & 1 \end{bmatrix}. \quad \blacksquare$$

Proof of Theorem 1.4, Part II Let X arise from Construction 2.4 with the input data specified in Theorem 1.4. Consider the toric embedding $X \subseteq Z$ provided by Construction 2.4. From Remark 2.7, we infer $\tau_X = \tau_Z$ for the semiample cones. Thus, for the divisor class $w = (1, a + 1, 1)$, a representing toric divisor E on Z and its restriction D on X , we obtain a commutative diagram

$$\begin{array}{ccc} X & \subseteq & Z \\ \varphi_D \downarrow & & \downarrow \varphi_E \\ X(D) & \subseteq & Z(E) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{l-1}}(b_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^{l-1}}(b_k)), \end{array}$$

where the inclusions are closed embeddings, l is the number of coordinates of $\bar{Z} = \mathbb{K}^n$ of degree $w_1 = (0, 1, 0)$, call them f_1, \dots, f_l , and k is the number of coordinates whose degree is located on the line segment $\text{cone}(w_5, w_8)$, call them h_1, \dots, h_k . Then we have $n = 2l + 2k$ and $\text{deg}(h_i) = (1, b_i, 0)$ with $0 = b_1 \leq b_2 \leq \dots \leq b_k = a$. Using local trivializations, we see that X projects onto the base $Z(E)$, which means $X(D) = Z(E)$. Moreover, on each fiber $\varphi_E^{-1}([z])$, the relation g becomes a linear form in the coordinates T_i different from f_i and h_j and thus cuts out a hyperplane of $\varphi_E^{-1}([z]) \cong \mathbb{P}_{l+k-1}$. Consequently, $\varphi_D: X \rightarrow X(D)$ is as claimed. \blacksquare

Corollary 6.1 *Let X be a Fano smooth full intrinsic quadric of Picard number three. Then every numerically effective divisor on X is base point free. In particular, X fulfills Fujita’s freeness conjecture.*

Proof We can assume that X arises from Construction 2.4 with the input data specified in Theorem 1.4. As in the proof of Corollaries 4.3 and 4.4, we consider the monoid of base point free divisor classes and its combinatorial description:

$$\text{BPF}(X) = \bigcap_{\gamma_0 \in \text{cov}(X)} Q(\gamma_0 \cap \mathbb{Z}^n).$$

Again the task is to show that for all $\gamma_0 \in \text{cov}(X)$, the monoid

$$Q(\gamma_0 \cap E) \subseteq Q(\text{lin}(\gamma_0) \cap E)$$

is saturated. For three-dimensional γ_0 , this is due to Proposition 2.8(iii). If $\gamma_0 \in \text{cov}(X)$ is not three-dimensional, then we have $\gamma_0 = \gamma_{i,i+1,j,j+1}$ with $w_i = (0, 1, 0)$, $w_{i+1} = (1, a-1, 1)$, and $w_j = (0, b, 1)$, $w_{j+1} = (1, a-b, 0)$ for some $0 \leq b \leq a$. One directly checks that the corresponding monoid $Q(\gamma_0 \cap E) \subseteq Q(\text{lin}(\gamma_0) \cap E)$ is saturated. ■

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