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A family of groups with a countable infinity of full orders

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We construct a family of groups with precisely $\aleph_0$ full orders.

1. Introduction

In Fuchs [1], B.H. Neumann is reported as asking if, when a orderable group has infinitely many full orders, the total number of these is a power of 2. We show that there are groups with precisely a countable infinity of orders.

2. Notation and preliminary results

Group operations are written multiplicatively. Elementary results about ordered groups are assumed and both these and relevant notation is found in Fuchs [1]. The following easy results are also assumed.

Let G be a fully ordered group with order denoted by $\succ$ and

$$x, y_i \in G , \quad i \in Z .$$

2.1. If

$$x^{-1}y_i^{k_i}x = y_i^{l_i} , \quad k_i, l_i \in Z , \quad k_i \neq l_i ,$$

then

$$|x| \succ\succ |y_i| ,$$

and

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$$|y_i| \sim |y_j| \text{ if } k_i l_j = k_j l_i .$$

2.2. If $x^{-1}y_i x = y_{i+1}$ and if $|y_i| \sim |y_{i+1}|$ does not hold then either

$$|x| \gg \ldots \gg |y_{i+1}| \gg |y_i| \gg |y_{i-1}| \gg \ldots$$

or

$$|x| \gg \ldots \gg |y_{i-1}| \gg |y_i| \gg |y_{i+1}| \gg \ldots .$$

3. The family of groups

Our main result is:

THEOREM 3.1. *There are* O*-groups with precisely a countable infinity of distinct full orders.*

We prove this by producing a family of such groups. First we need a few definitions.

DEFINITION 3.2. X is that subgroup of the rational numbers under addition whose elements are just those with denominators a power of 2 .

DEFINITION 3.3. X_1 is a subset of X given by

$$X_1 = \{x \mid x \in X , \ 0 \le x < 1\}$$

so that

$$X_1 = \left\{\frac{m}{2^n} \mid m < 2^n , \ m, n \in N\right\} \cup \{0\} ,$$

where N denotes the positive integers.

Z is the integers under addition.

DEFINITION 3.4. The groups $H_{z,x}$ and $K_{z,x}$: $z \in Z$, $x \in X_1$ are all copies of Q , the rational numbers under addition.

DEFINITION 3.5.

$$H_z = \prod_{x \in X_1} H_{z,x} ,$$

and

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$$K_z = \prod_{x \in X_1} K_{z,x} ,$$

are (restricted) direct products of copies of Q .

$$H = \prod_{z \in Z} H_z = \prod_{z \in Z} \prod_{x \in X_1} H_{z,x} ,$$

$$K = \prod_{z \in Z} K_z = \prod_{z \in Z} \prod_{x \in X_1} K_{z,x} ,$$

and

$$L = H \times K .$$

Next we define a semidirect product of X and L . To accomplish this we specify the transformations of the basic components, $H_{z,x}$ and $K_{z,x}$ by each element of X .

DEFINITION 3.6. Let $h_{z,x} \in H_{z,x}$ be a distinguished element for each $H_{z,x}$ and similarly

$$k_{z,x} \in K_{z,x} .$$

Likewise $\xi \in X$ is distinguished and so is $\zeta \in Z$.

Thus arbitrary members of $H_{z,x}$ and of X can be expressed as $h^r_{z,x}$ and ξ^α respectively, where $r \in Q$ and $\alpha \in X$.

Our transformations are given by

$$(1) \qquad \xi^{-\alpha} h^r_{z,x} \xi^\alpha = h^{rp^n}_{z,x+\alpha 2^z - n}$$

where $n \le x + \alpha 2^z < n + 1$, $n \in N$, and

$$(2) \qquad \xi^{-\alpha} k^r_{z,x} \xi^\alpha = k^{rq^n}_{z,x+\alpha 2^z - n}$$

where

$$n \le x + \alpha 2^z < n + 1$$

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and where $p, q \in N$ are square-free with

$$p \neq q .$$

LEMMA 3.7. *These transformations form a subgroup of the automorphism group of* L *isomorphic to* X *, so we have an associated semidirect product,* M *, of* L *by* X .

Finally we define transformations of M by Z .

DEFINITION 3.8.

$$(3) \qquad \zeta^{-\beta} h^{r}_{z,x} \zeta^{\beta} = h^{r}_{z+\beta,x} ,$$

$$(4) \qquad \zeta^{-\beta} k^{r}_{z,x} \zeta^{\beta} = k^{r}_{z+\beta,x} ,$$

and

$$(5) \qquad \zeta^{-\beta} \xi^{\alpha} \zeta^{\beta} = \xi^{\frac{\alpha}{2^{\beta}}} .$$

LEMMA 3.9. *These transformations form a subgroup of the automorphism group of* M *isomorphic to* Z *, so we have an associated semidirect product,* $G(p, q)$ *of* M *by* Z .

LEMMA 3.10. *In any full order of* M *(and hence of* $G(p, q)$*) the order of each group* H_z *, and* K_z *is archimedean, and in fact unique up to duals.*

Proof. If $x_1, x_2 \in X_1$ with (say)

$$x_2 > x_1 ,$$

we may put

$$x_2 - x_1 = \frac{m}{2^n} , \quad m, n \in N , \quad m < 2^n .$$

Thus we define

$$\alpha = \frac{m}{2^{n+z}} ,$$

so that

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(6) $$\xi^{-\alpha 2^n} h_{z,x_1} \xi^{\alpha 2^n} = h^{p^m}_{z,x_1}$$

according to (1).

But

(7) $$\xi^{-\alpha} h_{z,x_1} \xi^{\alpha} = h_{z,x_2}$$

again from (1).

(6) and (7) together show that all elements $h_{z,x}$ for fixed z belong to the same archimedean class, showing H_z to be archimedean.

Clearly analogous results hold for K_z. Further we deduce from (6) and (7) that

$$h^{r_1}_{z,x_1} h^{r_2}_{z,x_2} \in P \iff r_1 p^{x_1} + r_2 p^{x_2} \geq 0$$

under the condition $h_{z,x_1} \in P$, where p^{x_1} is a real number taken positive whenever ambiguity might arise.

This follows since H_z is archimedean so that it is isomorphic to a subgroup of the real numbers (Fuchs [1], p. 45) whose only automorphisms are given by multiplication by real numbers (Fuchs [1], p. 46); from (6), the number in question for transformation by ξ^{α} is seen to satisfy

$$(x)^{2^n} = p^m$$

so

$$x = p^{\frac{m}{2^n}}$$
$$= p^{x_2 - x_1} .$$

Our result follows by including in the automorphisms the raising to rational powers r_1 and r_2. This determines the order of H_z, while

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its dual occurs if we impose

$$h_{z,x_1} \in -P \ .$$

With the similar results for K_z, the lemma is proved.

LEMMA 3.11. *In any full order of* $G(p, q)$ *, either*

$$|\zeta| \gg |\xi| \gg \ldots \gg |h_{z+n,0}| \gg |h_{z+n+1,0}| \gg |h_{z+n+2,0}| \gg \ldots$$

or

$$|\zeta| \gg |\xi| \gg \ldots \gg |h_{z+n,0}| \gg |h_{z+n-1\ 0}| \gg |h_{z+n-2,0}| \gg \ldots$$

holds, and either

$$|\zeta| \gg |\xi| \gg \ldots \gg |k_{z+n,0}| \gg |k_{z+n+1,0}| \gg |k_{z+n+2,0}|$$

or

$$|\zeta| \gg |\xi| \gg \ldots \gg |k_{z+n,0}| \gg |k_{z+n-1,0}| \gg |k_{2+n-2,0}|$$

is true.

Proof. The results follow from the relations

$$\xi^{-1}h_{z,0}\xi = h_{z,0}^{p^{2^z}} ,$$

$$\xi^{-1}k_{z,0}\xi = k_{z,0}^{q^{2^z}} ,$$

$$\zeta^{-1}h_{z,0}\zeta = h_{z+1,0} ,$$

and

$$\zeta^{-1}k_{z,0}\zeta = k_{z+1,0} ,$$

by applying the preliminary results 2.1 and 2.2.

We are now able to prove the theorem.

Proof of Theorem 3.1. Since all elements of $G(p, q)$ are uniquely expressible in the form

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$$\zeta^a\xi^b\eta\theta$$

where ζ and ξ are as before, and $\eta \in H$, $\theta \in K$, Lemma 3.11 ensures that the order of $G(p, q)$ is completely determined by the orders of Z , X , H and K .

From Lemma 3.10, the "signs" of $h_{z,0}$ and $k_{z,0}$ completely determine the order on the groups H_z and K_z respectively.

From this, the relations

$$\zeta^{-z}h_{0,0}\zeta^z = h_{z,0} , \quad z \in Z$$

and

$$\zeta^{-z}k_{0,0}\zeta^z = k_{z,0} , \quad z \in Z$$

show that the "sign" of $h_{0,0}$ and $k_{0,0}$ determine the orders of each H_z and K_z . Three cases arise:

(1) $|x| \succ\succ |y|$, $x \in H$, $y \in K$;

(2) $|y| \succ\succ |x|$, $x \in H$, $y \in K$;

(3) neither of these hold.

In each of (1) and (2) there are only a finite number of orders possible, 2^4 in all, determined by the choice of "sign" for the elements ζ , ξ , h and k_{00} .

In case (3) there is an integer m such that either

$$\cdots \succ\succ |h_{0,0}| \succ\succ |k_{m,0}| \succ\succ |h_{1,0}| \succ\succ |k_{m+1,0}| \succ\succ \cdots$$

or

$$\cdots \succ\succ |h_{00}| \succ\succ |k_{m,0}| \succ\succ |h_{-1,0}| \succ\succ |k_{m-1,0}| \gg \cdots$$

depending on the possibilities of Lemma 2.

There are countably many such choices of ordering and since the first two cases give only 32 orders, the theorem is proved.

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Reference

[1] L. Fuchs, *Partially ordered algebraic systems* (Pergamon Press, Oxford, London, New York, Paris, 1963).

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