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ON THE ABSOLUTE NÖRLUND SUMMABILITY OF A FOURIER SERIES

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1. Let $\sum A_n$ be a given infinite series and $\{s_n\}$ the sequence of its partial sums. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0+p_1+\cdots+p_n. \tag{1.1}$$

If

$$\sigma_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k \to \sigma \tag{1.2}$$

as $n\to\infty$, we say that the series $\sum A_n$ is summable by the Nörlund method (N, p_n) to σ. The series $\sum A_n$ is said to be absolutely summable (N, p_n) or summable $|N, p_n|$ if σ_n is of bounded variation, i.e.,

$$\sum_{n=1}^{\infty}|\Delta\sigma_n| = \sum_{n=1}^{\infty}|\sigma_n-\sigma_{n-1}| < \infty. \tag{1.3}$$

2. Let f be a periodic function with period 2π, and integrable in the sense of Lebesgue. The Fourier series associated with f, at the point x, is

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \equiv \sum_{n=0}^{\infty}A_n(x), \text{ say.} \tag{2.1}$$

We write

$$\phi(t) = \phi_x(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}.$$

The following theorem has been proved by Hsiang [3].

THEOREM A. *Let $\{p_n\}$ be a sequence of positive constants. If (p_n-p_{n-1}) is monotonic and bounded,*

$$\sum_{n=2}^{\infty}\frac{n}{P_n(\log n)^a} < \infty \tag{2.2}$$

for some $a>0$, and

$$\left(\log\frac{1}{t}\right)^a|\phi_x(t)| = O(1), \qquad \text{as } t\to 0+ \tag{2.3}$$

then the Fourier series of f is summable $|N, p_n|$ at x.

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The object of the present paper is to generalize the above theorem of Hsiang and to give an alternate simple proof.

We shall prove the following theorem.

THEOREM. *Let p_n be a sequence of positive constants. If (p_n-p_{n-1}) is monotonic and bounded, and if*

$$\sum_{n=1}^{\infty} \frac{n}{P_n H(n)} < \infty \tag{2.4}$$

where $H(u)$ is a positive increasing function such that

$$\int_1^n \frac{1}{H(u)}\,du = 0\left(\frac{n}{H(n)}\right) \tag{2.5}$$

and

$$H\left(\frac{1}{t}\right)\,|\phi_x(t)| = 0(1) \quad \text{as} \quad t \to 0 \tag{2.6}$$

then the Fourier series of f is summable $|N, p_n|$ at x.

3. We shall require the following lemmas for the proof of the theorem.

LEMMA 1. [2] *If $p_n>0$, $\{p_n-p_{n-1}\}$ is monotonic and bounded, and if the series*

$$\sum \frac{|t'_n|}{P_n} < \infty$$

where $t'_n=\sum_{k=0}^{n}(n-k+1)A_k$, then $\sum A_n$ is summable $|N, p_n|$.

LEMMA 2. *If* (2.5) *and* (2.6) *are satisfied, then*

$$t_n \equiv t_n(x) = \sum_{k=0}^{n}(n-k+1)A_k(x) = 0\left(\frac{n}{H(n)}\right) \quad \text{as} \quad n \to \infty.$$

Proof. We have [1, p. 19]

$$\pi t_n = \int_0^{\pi} \phi(t) \frac{\sin^2 (n+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}\,dt$$

$$= \left\{\int_0^{1/n} + \int_{1/n}^{\pi}\right\} \phi(t) \frac{\sin^2 (n+1)\frac{t}{2}}{\sin^2 \frac{t}{2}}\,dt$$

$$= I_1 + I_2.$$

Now,

$$|I_1| \le \int_0^{1/n} |\phi(t)| \frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

$$\le \sup_{0<t<(1/n)} |\phi(t)| \int_0^{\pi} \frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt$$

$$= 0(n+1)0\left(\frac{1}{H(n)}\right) = 0\left(\frac{n}{H(n)}\right) \text{ as } n \to \infty \text{ by (2.6)}$$

because

$$\frac{1}{\pi(n+1)} \int_0^{\pi} \frac{\sin^2(n+1)\frac{t}{2}}{\sin^2\frac{t}{2}}\,dt = 1,$$

and

$$\tfrac{1}{4}I_2 = \int_{1/n}^{\pi} \phi(t) \frac{\sin^2(n+1)\frac{t}{2}}{t^2}\,dt + 0(1),$$

$$\left| \int_{1/n}^{\pi} \phi(t) \frac{\sin^2(n+1)\frac{t}{2}}{t^2}\,dt \right| \le A \int_{1/n}^{\pi} |\phi(t)| \frac{1}{t^2}\,dt, \qquad (A \text{ is a constant})$$

$$\le A \int_{1/n}^{\pi} \frac{1}{H\left(\frac{1}{t}\right)t^2}\,dt$$

$$= 0\left(\frac{n}{H(n)}\right) \text{ by condition (2.5)}$$

therefore

$$I_2 = 0\left(\frac{n}{H(n)}\right).$$

Hence

$$t_n = 0\left(\frac{n}{H(n)}\right).$$

4. **Proof of the theorem.** By Lemma 2 and (2.4) we have

$$\frac{|t_n|}{P_n} = 0\left(\frac{n}{P_n H(n)}\right),$$

where $\sum n/(p_n H(n) < \infty$.

The theorem now follows by Lemma 1.

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