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AN OSCILLATION THEOREM FOR SUBLINEAR ELLIPTIC DIFFERENTIAL INEQUALITIES

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Sublinear elliptic differential inequalities with variable coefficients are studied. Sufficient conditions are given that all solutions are oscillatory in exterior domains. Riccati inequalities are used to establish sublinear oscillation criteria.

Oscillation of solutions of sublinear elliptic differential equations has been discussed by several authors, see e.g. [2,3,5,6,8,9] and the references contained therein. In particular, sublinear elliptic differential operators with variable coefficients have been studied by the author [9]. In [5] Noussair and Swanson established an oscillation criterion by using Riccati transformations. The purpose of this paper is to solve an open question stated in [5, p.922, (3)] by employing a Riccati inequality.

We are concerned with the oscillatory behaviour of solutions of the sublinear elliptic differential operator L defined by

$$L[u] \equiv \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_ju) + c(x,u), \quad x \in \Omega ,$$

where Ω is an exterior domain in R^n, i.e. Ω contains the complement of some n-ball in R^n. Points in R^n will be denoted by $x = (x_1,\ldots,x_n)$, and differentiation with respect to x_i by D_i,

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$i = 1,\dots,n$. The notation $|x|$ will be used for the Euclidean length of $x \in R^n$. The domain $D_L(\Omega)$ of L is defined to be the set of all real-valued functions of class $C^2(\Omega)$. We assume that the following conditions hold throughout this paper:

(A-I) $c(x,\xi)$ is a real-valued continuous function in $\Omega \times R^1$;

(A-II) $D_k D_\ell a_{ij}(x)$ are locally uniformly Hölder-continuous in Ω $(i,j,k,\ell = 1,\dots,n)$;

(A-III) the matrix $A(x) \equiv (a_{ij}(x))_{i,j=1}^n$ is real symmetric and strictly positive definite in Ω (ellipticity condition);

(A-IV) $c(x,\xi) \geqq p_1(x)\phi_1(\xi)$ for all $(x,\xi) \in \Omega \times [0,\infty)$, where p_1 is continuous in Ω , $\phi_1 \in C^1(0,\infty)$ and $\phi_1(\xi) > 0$ in $(0,\infty)$;

(A-V) $c(x,\xi) \leqq -p_2(x)\phi_2(-\xi)$ for all $(x,\xi) \in \Omega \times (-\infty,0)$, where p_2 is continuous in Ω , $\phi_2 \in C^1(0,\infty)$ and $\phi_2(\xi) > 0$ in $(0,\infty)$;

(A-VI) $\phi_i'(\xi) \geqq 0$ for all $\xi > 0$ and $\int_0^t \frac{d\xi}{\phi_i(\xi)} < \infty$ for all $t > 0$, $i = 1,2$.

The n-dimensional sublinear Emden-Fowler equation

$$\sum_{i,j=1}^n D_i(a_{ij}(x)D_j u) + p(x)|u|^\gamma \operatorname{sgn} u = 0 \ , \ 0 < \gamma < 1 \ ,$$

is an important special case of $L[u] = 0$ which satisfies (A-IV)-(A-VI).

DEFINITION. A function $u \in D_L(\Omega)$ is said to be *oscillatory* in Ω if it has arbitrarily large zeroes. The inequality $uL[u] \leqq 0$ is said to be *oscillatory* in Ω whenever every solution u of the inequality is oscillatory in Ω .

Since Ω is an exterior domain in R^n , there exists a number $\varepsilon > 0$ such that $R^n(\varepsilon) \subset \Omega$, where $R^n(\varepsilon) \equiv \{x \in R^n : |x| > \varepsilon\}$. Let x_0 be a fixed point in $R^n(\varepsilon)$. Then, there exists a fundamental solution

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$E(x) \in C^2(R^n(\varepsilon)\backslash\{x_0\})$ of the operator $P \equiv \sum_{i,j=1}^{n} D_i a_{ij}(x) D_j$ with a singularity at the point x_0 (see Itô [1, p.84]). We assume that $\lim_{|x|\to\infty} E(x) = -\infty$ $(n = 2)$ and $\lim_{|x|\to\infty} E(x) = 0$ $(n \geqq 3)$. In the case that $P = \Delta$ (Laplacian), one takes $E(x) = (2\pi)^{-1}\log(|x|^{-1})$ $(n = 2)$ and $E(x) = (\sigma_n(n-2))^{-1}|x|^{2-n}$ $(n \geqq 3)$, where σ_n denotes the surface area of the unit sphere in R^n. We define

$$\rho(x) = \begin{cases} \exp(-2\pi E(x)), & n = 2, \\ (\sigma_n(n-2)E(x))^{1/(2-n)}, & n \geqq 3. \end{cases}$$

We note that $\rho(x) = |x|$ if $P = \Delta$. The distance of each point on $S_r \equiv \{x \in \Omega : \rho(x) = r\}$ from the origin tends to infinity as $r \to \infty$, and therefore there exists a number $r_0 > 0$ such that $\{x \in R^n : \rho(x) > r_0\} \subset R^n(\varepsilon) \subset \Omega$. We find that each level surface S_r represents a smooth closed $n-1$ dimensional surface in Ω and $\Omega(r_1, r_2) \equiv \{x \in \Omega : r_1 < \rho(x) < r_2\}$ $(r_0 < r_1 < r_2)$ is a bounded domain with boundary $S_{r_1} \cup S_{r_2}$ (cf. Levine and Payne [4]).

Associated with every positive function $u \in D_L(\Omega)$, we define the n-vector function $w(x)$ by

$$w(x) = -\frac{1}{\phi_1(u)} A(x)\nabla u, \tag{1}$$

where ∇u denotes the gradient of u. The following notation will be used:

$$p(x) = \min\{p_1(x), p_2(x)\}, \quad x \in \Omega,$$

$$\tilde{p}(r) = (\sigma_n r^{n-1})^{-1} \int_{S_r} p(x)|\nabla\sigma|^{-1} d\sigma .$$

By the same arguments as were used by Noussair and Swanson [5], we obtain the following result.

THEOREM. *Under assumptions (A-I)-(A-VI), the sublinear elliptic differential inequality* $uL[u] \leqq 0$ *is oscillatory in an exterior domain* Ω *of* R^n *if*

$$\limsup_{r\to\infty} \int_{\tilde{r}}^{r} \left(1 - \frac{\theta_n(t)}{\theta_n(r)}\right) t\tilde{p}(t)dt = \infty \tag{2}$$

for some $\tilde{r} > r_0$, *where* $\theta_2(r) = \log r$ *and* $\theta_n(r) = r^{n-2}$ $(n \geqq 3)$.

Proof. Suppose to the contrary that there exists a solution u of $uL[u] \leqq 0$ which has no zero in $\Omega(r_1,\infty)$ for some $r_1 > r_0$. First we assume that $u > 0$ in $\Omega(r_1,\infty)$. It follows from a result of Noussair and Swanson [5, Lemma 1] that

$$\operatorname{div} w(x) \geqq p_1(x) . \tag{3}$$

Integrating (3) over $\Omega(r_1,r)$, we obtain

$$\int_{\Omega(r_1,r)} \operatorname{div} w(x)\, dx \geqq \int_{\Omega(r_1,r)} p_1(x)dx \geqq \int_{\Omega(r_1,r)} p(x)dx .$$

An application of the Divergence Theorem shows that

$$\int_{S_r} w^*(\nabla\rho)|\nabla\rho|^{-1}d\sigma - \int_{S_{r_1}} w^*(\nabla\rho)|\nabla\rho|^{-1}d\sigma \geqq \int_{\Omega(r_1,r)} p(x)dx , \tag{4}$$

where * denotes the transpose. We define

$$Q[u](r) \equiv \int_{S_r} (\nabla G(u))^* A(x)(\nabla\rho)|\nabla\rho|^{-1}d\sigma ,$$

where $G(u) = \int_0^u \phi_1(\xi)^{-1}d\xi$. Since

$$w^*(\nabla\rho)|\nabla\rho|^{-1} = -\phi_1(u)^{-1}(\nabla u)^* A(x)(\nabla\rho)|\nabla\rho|^{-1} = -(\nabla G(u))^* A(x)(\nabla\rho)|\nabla\rho|^{-1} ,$$

we have

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$$Q[u](r) = -\int_{S_r} w^*(\nabla\rho)|\nabla\rho|^{-1}d\sigma .$$

Hence, (4) is equivalent to

(5) $$Q[u](r) - Q[u](r_1) \leqq -\int_{\Omega(r_1,r)} p(x)dx .$$

We easily obtain

$$\int_{\Omega(r_1,r)} p(x)dx = \sigma_n\int_{r_1}^{r} t^{n-1}dt\left\{(\sigma_n t^{n-1})^{-1}\int_{S_t} p(x)|\nabla\rho|^{-1}d\sigma\right\}$$

(6) $$= \sigma_n\int_{r_1}^{r} t^{n-1}\tilde{p}(t)dt .$$

Hence, it follows from (5) and (6) that

$$Q[u](r) - Q[u](r_1) \leqq -\sigma_n\int_{r_1}^{r} t^{n-1}\tilde{p}(t)dt ,$$

and therefore

(7) $$\int_{r_1}^{r} \frac{Q[u](\eta)}{\sigma_n\eta^{n-1}}d\eta - \int_{r_1}^{r} \frac{Q[u](r_1)}{\sigma_n\eta^{n-1}}d\eta \leqq -\int_{r_1}^{r}\frac{d\eta}{\eta^{n-1}}\int_{r_1}^{\eta} t^{n-1}\tilde{p}(t)dt .$$

Using a result of Suleimanov [7, Lemma 2], we obtain

(8) $$\frac{Q[u](r)}{\sigma_n\eta^{n-1}} = \frac{d}{d\eta}M[u](\eta) ,$$

where

$$M[u](\eta) = (\sigma_n\eta^{n-1})^{-1}\int_{S_\eta} G(u)(\nabla\rho)^*A(x)(\nabla\rho)|\nabla\rho|^{-1}d\sigma ,$$

(cf. the author [9, the proof of Lemma 2.1]). It is easy to see that

$$\int_{r_1}^{r} \frac{d\eta}{\eta^{n-1}} \int_{r_1}^{\eta} t^{n-1}\tilde{p}(t)dt = \int_{r_1}^{r} t^{n-1}\tilde{p}(t)dt \int_{t}^{r} \frac{d\eta}{\eta^{n-1}}$$

$$(9) \qquad = k_n(r)\int_{r_1}^{r} t\tilde{p}(t)\left[1 - \frac{\theta_n(t)}{\theta_n(r)}\right]dt \ ,$$

where $k_2(r) = \log r$ and $k_n(r) = (n - 2)^{-1}$ $(n \geqq 3)$. Combining (7)-(9) yields

$$(10) \qquad M[u](r) - M[u](r_1) + Q[u](r_1)\sigma_2^{-1} \log r_1$$

$$\leqq - (\log r)\left(\int_{r_1}^{r} \left[1 - \frac{\theta_2(t)}{\theta_2(r)}\right]t\tilde{p}(t)dt - \sigma_2^{-1}Q[u](r_1)\right) \ (n = 2) \ ,$$

and

$$(11) \qquad M[u](r) - M[u](r_1) - Q[u](r_1)\sigma_n^{-1}(n-2)^{-1}r_1^{-n+2}$$

$$\leqq - (n-2)^{-1}\left(\int_{r_1}^{r} \left[1 - \frac{\theta_n(t)}{\theta_n(r)}\right]t\tilde{p}(t)dt + \sigma_n^{-1}Q[u](r_1)r^{-n+2}\right) \ (n \geqq 3) \ .$$

Hence, condition (2) implies that the right hand side of (10) and (11) are not bounded from below, and consequently $M[u](r)$ would take negative values for sufficiently large r . This contradicts the positivity of $M[u](r)$.

If $u < 0$ in $\Omega(r_1,\infty)$, $v \equiv - u$ is a positive solution of $v(-L[-v]) \leqq 0$. In view of the assumption (A-V) we conclude that

$$\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_jv) + p_2(x)\phi_2(v) \leqq \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_jv) - c(x,-v)$$

$$(12) \qquad = - L[-v] \leqq 0 \ .$$

Hence, v is a positive solution of (12). Repeating the same arguments as in the case where $u > 0$, we are led to a contradiction. This completes the proof.

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The proof of the following result is quite similar to that of Corollary of the author [9, p.716], and hence will be omitted.

COROLLARY. *Under assumptions (A-I)-(A-VI), the sublinear elliptic differential inequality* $uL[u] \leqq 0$ *is oscillatory in an exterior domain* Ω *of* R^n *if*

$$\int_{\tilde{r}}^{\infty} t\tilde{p}(t)dt = \infty \quad \text{for some } \tilde{r} > r_0 . \tag{13}$$

REMARK. In the case where $n = 2$, $\phi_1 \equiv \phi_2$ and $a_{ij}(x) = \delta_{ij}$ (the Kronecker delta), Corollary reduces to a result of Noussair and Swanson [5, Theorem 15]. Let $\Phi(\xi)$ be defined by

$$\Phi(\xi) = \begin{cases} \phi_1(\xi) & \text{for } \xi \geqq 0 , \\ -\phi_2(-\xi) & \text{for } \xi < 0 , \end{cases}$$

where $\phi_1(0) = \lim_{\xi \to +0} \phi_2(\xi) = 0$. If $p_1(x) = p_2(x) = p(x)$ in Ω and $c(x,\xi) = p(x)\Phi(\xi)$, then Theorem reduces to a result of the author [9, Theorem 4.1]. For the sublinear equation $\Delta u + p(x)\Phi(u) = 0$ oscillation criteria (2) and (13) were established by Kura [3] and Kitamura and Kusano [2], respectively.

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