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ON KLEINIAN GROUPS WITH THE SAME SET OF AXES

BAOHUA XIE and YUEPING JIANG

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Abstract

J. W. Anderson (1996) asked whether two finitely generated Kleinian groups $G_1, G_2 \subset \mathrm{Isom}(\mathbb{H}^n)$ with the same set of axes are commensurable. We give some partial solutions.

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1. Introduction

In 1990, Mess [7] showed that if G_1, G_2 are finitely generated nonelementary Fuchsian groups having the same nonempty set of simple axes, then G_1, G_2 are commensurable. By using some technical results of arithmetic Kleinian groups, Long and Reid [6] gave an affirmative answer to this question in the case G_1, G_2 are arithmetic Fuchsian groups. The general case of this question is still open so far. We consider the following analogue of the question proposed by Anderson [4] for higher-dimensional Kleinian groups.

QUESTION. If G is a group of isometries of $\mathbb{H}^n$, denote by $\mathrm{Ax}(G)$ the set of axes of elements of G. If G_1 and G_2 are finitely generated and discrete, does $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2)$ imply that G_1 and G_2 are commensurable?

A very simple example shows that the answer to this question is negative in general. That is, G_1 and G_2 having the same set of axes cannot guarantee they are commensurable.

EXAMPLE. Let $A, B \in \mathrm{PSL}(2, \mathbb{C})$ be given by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}.$$

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It is clear A and B have the same axis. So $\mathrm{Ax}(\langle A\rangle) = \mathrm{Ax}(\langle B\rangle)$. Obviously, one has $\langle A\rangle \cap \langle B\rangle = I$, $[\langle A\rangle : I] = \infty$ and $[\langle B\rangle : I] = \infty$.

In the above example, $\langle A\rangle$ and $\langle B\rangle$ are finitely generated (and even geometrically finite) Kleinian groups. The groups $\langle A\rangle$ and $\langle B\rangle$ have the same set of axes, but they are not commensurable. Following on from the above example, it is interesting to explore the conditions which imply two Kleinian groups are commensurable. The main purpose of this note is to prove that G_1 and G_2 are commensurable under some stronger conditions.

The main results of this note are the following two theorems.

THEOREM 1.1. *Let $G_1, G_2 \subset \mathrm{Isom}(\mathbb{H}^n)$ be finitely generated, torsion-free Kleinian groups. Then G_1 and G_2 are commensurable if and only if* $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$.

THEOREM 1.2. *Let $G_1, G_2 \subset \mathrm{Isom}(\mathbb{H}^n)$ be geometrically finite, purely hyperbolic Kleinian groups. Suppose $\langle G_1, G_2\rangle$ is also Kleinian. Then G_1 and G_2 are commensurable if and only if $\Lambda(G_1) = \Lambda(G_2)$.*

The note is organized as follows. In Section 2, we gather some well-known facts about Kleinian groups. In Section 3, we prove Theorems 1.1 and 1.2.

2. Preliminaries

First, we recall some terminologies. For more details see [3, 8].

Let $\mathbb{B}^n$ denote the closed ball $\mathbb{H}^n \cup \mathbb{S}^{n-1}$, whose boundary $\mathbb{S}^{n-1}$ is identified via the stereographic projection with $\overline{\mathbb{R}^{n-1}} = \mathbb{R}^{n-1} \cup \infty$. Let $\mathrm{Mob}(\mathbb{S}^{n-1})$ denote the group of all Möbius transformations of the $n-1$-sphere $\mathbb{S}^{n-1}$, that is, compositions of inversions in $\mathbb{S}^{n-1}$. The group $\mathrm{Mob}(\mathbb{S}^{n-1})$ admits an extension to the hyperbolic n-space $\mathbb{H}^n$, so that $\mathrm{Mob}(\mathbb{S}^{n-1}) = \mathrm{Isom}(\mathbb{H}^n)$, the isometry group of $\mathbb{H}^n$. A discrete subgroup $G \subset \mathbb{H}^n$ is called a *Kleinian group*. The *discontinuity set* $\Omega(G)$ of a group $G \subset \mathbb{H}^n$ is the largest open subset in $\partial\mathbb{H}^n$ on which G acts properly discontinuously. Its complement $\partial\mathbb{H}^n \setminus \Omega(G)$ is the *limit set* $\Lambda(G)$ of the group G. A discrete group whose limit set contains fewer than three points is called *elementary* and *nonelementary* otherwise. The elements of $\mathrm{Isom}(\mathbb{H}^n)$ are classified in terms of their fixed-point sets. An element $g \neq \mathrm{id}$ in $\mathrm{Isom}(\mathbb{H}^n)$ is *elliptic* if it has a fixed point in $\mathbb{H}^n$, *parabolic* if it has exactly one fixed point which lies in $\partial\mathbb{H}^n$, *hyperbolic* if it has exactly two fixed points which lie in G. G is *purely hyperbolic* if G contains neither parabolic nor elliptic elements. The unique geodesic joint of two fixed points of the hyperbolic element g, which is invariant under g, is called the axis of the hyperbolic element and is denoted by $\mathrm{Ax}(g)$. For a discrete group G, let $\mathrm{Ax}(G)$ denote the set of all the axes of hyperbolic elements of G. Two Kleinian groups G_1 and G_2 are said to be *commensurable*, if both the indices

$$[G_1 : G_1 \cap G_2], \quad [G_1 : G_1 \cap G_2]$$

are finite.

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For a nonelementary Kleinian group G, $\mathrm{Hull}(\Lambda(G)) \subset \mathbb{H}^n$ is defined as the convex hull of the limit set $\Lambda(G)$, that is $\mathrm{Hull}(\Lambda(G))$ is the smallest convex set which is invariant under G. Let $M^n = \mathbb{H}^n/G$ be the quotient orbifold, and $\overline{M^n} = (\mathbb{H}^n \cup \Omega(G))/G$ its closure. The quotient $\mathrm{Hull}(\Lambda(G))/G$ is called the convex core of M^n. A Kleinian group is called *geometrically finite* if it has a finite-sided fundamental polyhedron $P \subset \mathbb{H}^n$, *algebraically finite* if it is finitely generated. It is clear that geometrical finiteness implies algebraical finiteness and geometrical finiteness is preserved for a finite-index subgroup. For a nonelementary Kleinian group G, geometrical finiteness is equivalent to the fact that, for some $\varepsilon > 0$, the ε-neighborhood of convex core $\mathrm{Hull}(\Lambda(G))/G$ has finite volume. Here we call the subset

$$\mathrm{Hull}_\varepsilon(\Lambda(G)/G) = \{p \in M^n \mid d(p, \mathrm{Hull}(\Lambda(G))/G) \le \varepsilon\}$$

the ε-neighborhood of convex core.

3. Proof of the theorems

PROOF OF THEOREM 1.1. We only need to show that $[G_1 : G_1 \cap G_2] < \infty$, and the proof of $[G_2 : G_1 \cap G_2] < \infty$ is similar. We claim that for every element $g \in G_1$, there exists a positive integer N such that $g^N \in G_1 \cap G_2$. Indeed, let $l \in \mathrm{Ax}(G_1)$ be an axis of some element $g \in G_1$. Since $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_1 \cap G_2)$, l is also the axis of some element $h \in G_1 \cap G_2$. Therefore h and g have the same axis l. By the discreteness of elementary groups, $\langle h, g\rangle$ is a cyclic group. Then there exists a primitive element f of G_1 such that $h = f^{N_1}$ and $g = f^{N_2}$. It follows that $g^{N_1} = f^{N_1 N_2} \in G_1 \cap G_2$. If every element of G is of finite order, then G is finite. By the above arguments, it follows that $G_1 \cap G_2$ is a normal subgroup of G_1. Every element $g(G_1 \cap G_2)$ is of finite order in $G_1/G_1 \cap G_2$, then $G_1/G_1 \cap G_2$ is a finite group. That is, $[G_1 : G_1 \cap G_2] < \infty$.

Conversely, if G_1 and G_2 are commensurable, then it is easy to see that $\mathrm{Ax}(G_1) = \mathrm{Ax}(G_2) = \mathrm{Ax}(G_1 \cap G_2)$. □

REMARK 3.1. Susskind and Swarup [10] proved that if G_1 is a geometrically finite subgroup of a Kleinian group G, $\Lambda(G_1) = \Lambda(G)$, then $[G : G_1] < \infty$. In the proof of Theorem 1 we actually prove that if G_1 is a subgroup of finitely generated Kleinian group G and $\mathrm{Ax}(G_1) = \mathrm{Ax}(G)$, then G_1 is a normal subgroup of G and $[G : G_1] < \infty$.

In the proof Theorem 1.2, we consider geometrically finite Kleinian groups. Although this finiteness is equivalent to algebraical finiteness in the two-dimensional case, the concepts do not coincide in the three-dimensional case.

PROOF OF THEOREM 1.2. In order to to prove this theorem, it suffices to make use of a main result of Susskind and Swarup [10] (see also the discussion in Anderson [1]) which says if G_1 and G_2 are geometrically finite, purely hyperbolic subgroups of a Kleinian group, then $\Lambda(G_1) \cap \Lambda(G_2) = \Lambda(G_1 \cap G_2)$. We have $\Lambda(G_1) = \Lambda(G_2) = \Lambda(G_1 \cap G_2)$. Since we have $\Lambda(G_1) = \Lambda(G_2)$ in this theorem, then

$$\mathrm{Hull}_\varepsilon(\Lambda(G_1))/G_1 = \mathrm{Hull}_\varepsilon(\Lambda(G_1 \cap G_2))/G_1.$$

Observe that

$$[G_1 : G_1 \cap G_2]\, \mathrm{Vol}(\mathrm{Hull}_\varepsilon(\Lambda(G_1))/G_1) = \mathrm{Vol}(\mathrm{Hull}_\varepsilon(\Lambda(G_1 \cap G_2))/G_1 \cap G_2).$$

The group $\langle G_1, G_2 \rangle$ is Kleinian by the assumption, G_1 and G_2 are two geometrically finite discrete subgroups of $\langle G_1, G_2 \rangle$, and so $G_1 \cap G_2$ is finitely generated. This follows from the main result of Hempel [5]. The group $G_1 \cap G_2$ is a finitely generated subgroup of a geometrically finite group G_1 or G_2. It follows that $G_1 \cap G_2$ is geometrically finite by a result of Thurston which says that every finitely generated subgroup of a geometrically finite Kleinian group of the second kind is geometrically finite [2]. Therefore, we can conclude that

$$\mathrm{Vol}(\mathrm{Hull}_\varepsilon(\Lambda(G_1))/G_1) < \infty$$

and

$$\mathrm{Vol}(\mathrm{Hull}_\varepsilon(\Lambda(G_1 \cap G_2))/G_1 \cap G_2) < \infty.$$

Hence

$$[G_1 : G_1 \cap G_2] < \infty.$$

For the same reasons,

$$[G_2 : G_1 \cap G_2] < \infty.$$

This completes the proof of Theorem 1.2. □

REMARK 3.2. Susskind [9] provided an example of an infinitely generated intersection of geometrically finite hyperbolic groups. That is to say, there exists two geometrical finite Kleinian groups J and H such that $J \cap H$ is infinitely generated; moreover, $\langle J, H \rangle$ is not discrete. In his example, J and H are 'elementary' hyperbolic groups in the sense that they both leave invariant the same two-dimensional hyperplane. Then he asked that whether there exist examples of co-compact, finite volume or nonelementary groups which produce a similar result concerning the need for $\langle J, H \rangle$ to be discrete. If G_1 and G_2 are not 'elementary' hyperbolic groups, it is possible that there is no need for the discreteness of $\langle G_1, G_2 \rangle$ in Theorem 1.2.

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BAOHUA XIE, College of Mathematics and Econometrics, Hunan University, Changsha, 410082, People's Republic of China
e-mail: xiexbh@gmail.com

YUEPING JIANG, College of Mathematics and Econometrics, Hunan University, Changsha, 410082, People's Republic of China
e-mail: ypjiang731@163.com

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