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COCOMPACT LATTICES ON $\tilde{A}_n$ BUILDINGS

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Abstract. We construct cocompact lattices $\Gamma_0' < \Gamma_0$ in the group $G = \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$ which are type-preserving and act transitively on the set of vertices of each type in the building Δ associated to G. These lattices are commensurable with the lattices of Cartwright–Steger *Isr. J. Math.* **103** (1998), 125–140. The stabiliser of each vertex in Γ_0' is a Singer cycle and the stabiliser of each vertex in Γ_0 is isomorphic to the normaliser of a Singer cycle in $\mathrm{PGL}_d(q)$. We show that the intersections of Γ_0' and Γ_0 with $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ are lattices in $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$, and identify the pairs (d, q) such that the entire lattice Γ_0' or Γ_0 is contained in $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. Finally we discuss minimality of covolumes of cocompact lattices in $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$. Our proofs combine the construction of Cartwright–Steger *Isr. J. Math.* **103** (1998), 125–140 with results about Singer cycles and their normalisers, and geometric arguments.

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1. Introduction. Let $\mathbb{F}_q$ be the finite field of order q where q is a power of a prime p, and let K be the field $\mathbb{F}_q(\!(t)\!)$ of formal Laurent series over $\mathbb{F}_q$, with discrete valuation $\nu : K^\times \to \mathbb{Z}$. Let Δ be the building $\tilde{A}_n(K, \nu)$, as constructed in, for example [**25**, Chapter 9] (see also Section 2.2 below). Then Δ is an affine building of type $\tilde{A}_n$, meaning that the apartments of Δ are isometric images of the Coxeter complex of type $\tilde{A}_n$. The link of each vertex of Δ may be identified with the n–dimensional projective space $\mathrm{PG}(n, q)$ over $\mathbb{F}_q$.

Let $d = n + 1$ and let G be the group $G = \mathcal{G}(K)$, where $\mathcal{G}$ is in the set $\{\mathrm{GL}_d, \mathrm{PGL}_d, \mathrm{SL}_d, \mathrm{PSL}_d\}$. Then G is a totally disconnected, locally compact group which acts on Δ with kernel $Z(G)$. It follows from a theorem of Tits [**28**] that $G/Z(G)$

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is cocompact in the full automorphism group of Δ. If $\mathcal{G}$ is GL_d or PGL_d, then the G–action is type-rotating and transitive on the vertex set of Δ, while if $\mathcal{G}$ is SL_d or PSL_d, then the G–action is type-preserving and transitive on each type of vertex. See Section 2 below for definitions of these terms.

By definition, a subgroup $\Gamma \leq G$ is a *lattice* if it is a discrete subgroup such that $\Gamma\backslash G$ admits a finite G–invariant measure, and a lattice Γ is *cocompact* if $\Gamma\backslash G$ is compact. In the cases $\mathcal{G} = \mathrm{PGL}_d$, SL_d and PSL_d, the centre of $G = \mathcal{G}(K)$ is compact, hence G acts on Δ with compact vertex stabilisers. A subgroup $\Gamma \leq G$ is then discrete if and only if Γ acts on Δ with finite vertex stabilisers, and if $\Gamma \leq G$ is discrete then Γ is a cocompact lattice if and only if, in addition, Γ acts cocompactly on Δ. Given any lattice Γ and a set A of vertices of Δ which represent the orbits of Γ, the Haar measure μ on G may be normalised so that $\mu(\Gamma\backslash G)$, the covolume of Γ in G, is given by the series $\sum_{a\in A} |\operatorname{Stab}_\Gamma(a)|^{-1}$ (see [**2**]). This is a finite sum if and only if Γ is cocompact.

The existence of an arithmetic cocompact lattice in $G = \mathcal{G}(K)$ is due to Borel–Harder [**3**]. By Margulis' Arithmeticity Theorem [**21**], if $d \geq 3$ then every lattice in such G is arithmetic. In the rank 1 case, that is, for $d = 2$, the building Δ is a tree of valence $q + 1$, and there are several additional known constructions of cocompact lattices in G. For example, Figá-Talamanca and Nebbia [**12**] constructed lattices in $G = \mathrm{PGL}_2(\mathbb{F}_q((t)))$ which act simply transitively on the set of vertices of the tree Δ. Such lattices are necessarily free products of s copies of the cyclic group of order 2, and t copies of the infinite cyclic group, where $s + t = q + 1$. The cocompact lattices of minimal covolume in $G = \mathrm{SL}_2(\mathbb{F}_q((t)))$ were constructed in [**18**, **20**]. These lattices are fundamental groups of finite graphs of finite groups which, using Bass' covering theory for graphs of groups [**1**], are embedded in G. Lubotzky [**19**] also constructed a moduli space of cocompact lattices in $\mathrm{SL}_2(\mathbb{F}_q((t)))$ which are finitely generated free groups, using a Schottky-type construction.

If $d = 3$, then additional constructions of lattices in G may be complicated by the fact that there exist uncountably many "exotic" $\tilde{A}_2$–buildings, that is, buildings of type $\tilde{A}_2$ which are not of the form $\tilde{A}_2(K, \nu)$ for any field K, not necessarily commutative, with discrete valuation ν (Tits [**29**]). On the other hand for $d \geq 4$, that is, for $n \geq 3$, there are no exotic buildings of type $\tilde{A}_n$ (Tits [**30**]).

For $d \geq 3$, there exists a chamber-transitive lattice in $\mathrm{PSL}_d(\mathbb{F}_q((t)))$ if and only if $d = 3$ and $q = 2$ or $q = 8$ (see [**16**] and its references). Lattices in the group $G = \mathrm{PGL}_d(\mathbb{F}_q((t)))$ which act simply transitively on the vertex set of the associated building Δ were constructed for the case $d = 3$ in [**6**], and for $d > 3$ in [**7**]. We will describe the work of [**6**] and [**7**] further below. In addition, in the case $d = 3$, Ronan [**24**] constructed lattices acting simply transitively on the set of vertices of the same type in some, possibly exotic, $\tilde{A}_2$–building, and Essert [**11**] constructed lattices acting simply transitively on the set of panels of the same type in some, again possibly exotic, $\tilde{A}_2$–building. Essert's construction used complexes of groups (see [**5**]), and had vertex stabilisers cyclic groups acting simply transitively on the set of points and lines of $\mathrm{PG}(2, q)$, the projective plane over $\mathbb{F}_q$. Our work resolves some open questions of [**11**], as we explain below.

Our main results are Theorems 1 and 2 below. See Section 2.1 below for the definition of a Singer cycle in $\mathrm{PGL}_d(q)$; such a group acts simply transitively on the set of points of $\mathrm{PG}(d-1, q)$. We first construct lattices in $\mathrm{PGL}_d(\mathbb{F}_q((t)))$.

THEOREM 1. *Let $G = \mathrm{PGL}_d(\mathbb{F}_q((t)))$ and let Δ be the building associated to G. Then G admits cocompact lattices $\Gamma_0' \leq \Gamma_0$ such that:*

- *the action of Γ_0' and of Γ_0 on Δ is type-preserving and transitive on each type of vertex;*
- *the stabiliser of each vertex in Γ_0' is isomorphic to a Singer cycle in $\mathrm{PGL}_d(q)$; and*

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- *the stabiliser of each vertex in* Γ_0 *is isomorphic to the normaliser of a Singer cycle in* $\mathrm{PGL}_d(q)$.

Moreover Γ_0' *and* Γ_0 *are generated by their d subgroups which are the stabilisers of the vertices of the standard chamber in* Δ.

In fact, the stabiliser of each vertex in Γ_0' is always contained in a finite subgroup of G isomorphic to $\mathrm{PGL}_d(q)$. However for the vertex stabilisers of Γ_0 the situation is trickier. If $(p, d) = 1$, then the stabiliser of each vertex in Γ_0 is indeed contained in a finite subgroup of G isomorphic to $\mathrm{PGL}_d(q)$. On the other hand, as we discuss in Section 3.2, if p divides d, then the stabiliser of each vertex in Γ_0 intersects a finite subgroup of G isomorphic to $\mathrm{PGL}_d(q)$ in a subgroup of index p^a, where $d = p^a b$ and $(p, b) = 1$.

We then construct lattices in $\mathrm{PSL}_d(\mathbb{F}_q((t)))$, where we identify the group $\mathrm{PSL}_d(\mathbb{F}_q((t)))$ with a subgroup of $\mathrm{PGL}_d(\mathbb{F}_q((t)))$. Our notation continues from Theorem 1.

THEOREM 2. *The groups*

$$\Lambda_0' := \Gamma_0' \cap \mathrm{PSL}_d(\mathbb{F}_q((t))) \quad and \quad \Lambda_0 := \Gamma_0 \cap \mathrm{PSL}_d(\mathbb{F}_q((t)))$$

are cocompact lattices in $\mathrm{PSL}_d(\mathbb{F}_q((t)))$*, necessarily type-preserving. Moreover:*

(1) *Suppose that* $(d, q-1) = 1$.
  (a) *If p does not divide d, then* $\Lambda_0' = \Gamma_0'$ *and* $\Lambda_0 = \Gamma_0$.
  (b) *If p divides d, then* $\Lambda_0' = \Gamma_0'$ *and* Λ_0 *is a proper subgroup of* Γ_0.

(2) *If* $(d, q-1) \neq 1$*, then* Λ_0' *is a proper subgroup of* Γ_0' *and* Λ_0 *is a proper subgroup of* Γ_0.

In all cases where $\Lambda_0' = \Gamma_0'$ *(respectively,* $\Lambda_0 = \Gamma_0$*), it follows that* Γ_0' *(respectively,* Γ_0*) is a cocompact lattice in* $\mathrm{PSL}_d(\mathbb{F}_q((t)))$ *with properties as described in Theorem 1.*

In particular, in Section 4 we give the precise structure of the vertex stabilisers in Λ_0 and Λ_0', and we describe the cases in which these lattices can be generated by their vertex stabilisers.

Since the centre of $\mathrm{SL}_d(\mathbb{F}_q((t)))$ is finite and fixes Δ pointwise, if Γ is any lattice in $\mathrm{PSL}_d(\mathbb{F}_q((t)))$ then the full pre-image of Γ under the canonical epimorphism is a cocompact lattice in $\mathrm{SL}_d(\mathbb{F}_q((t)))$. We thus obtain lattices in $\mathrm{SL}_d(\mathbb{F}_q((t)))$ as well. Of course if $(d, q-1) = 1$, then the centre of $\mathrm{SL}_d(\mathbb{F}_q((t)))$ is trivial, and so, for example, $\Gamma_0' = \Lambda_0'$ itself is a lattice in $\mathrm{SL}_d(\mathbb{F}_q((t)))$.

The question of minimality of covolumes for lattices in $\mathrm{SL}_n(\mathbb{F}_q((t)))$ was pioneered for $n = 2$ by Lubotzky [**18**]. For $n \geq 3$ it has been studied by Golsefidy [**27**]. They have shown that in $\mathrm{SL}_n(\mathbb{F}_q((t)))$ the minimal covolume is attained on non-cocompact lattices.

Lubotzky also posed the question of determining the cocompact lattices of minimal covolume in $\mathrm{SL}_2(\mathbb{F}_q((t)))$ (that is, the cocompact lattices whose covolume is the smallest among all cocompact lattices). In general this question is a bit delicate, as there is very little known about cocompact lattices in $\mathrm{SL}_n(\mathbb{F}_q((t)))$. Our original motivation was to find cocompact lattices of minimal covolume in $\mathrm{SL}_3(\mathbb{F}_q((t)))$. For this, it is natural to consider vertex stabilisers which are Singer cycles or normalisers of Singer cycles, since these are the vertex stabilisers of the cocompact lattices of minimal covolume in $\mathrm{SL}_2(\mathbb{F}_q((t)))$ (cf. [**18**, **20**]) and more generally in topological rank 2 Kac–Moody groups G over $\mathbb{F}_q$ [**9**], where the minimality result holds under the conjecture that cocompact lattices in such G do not contain p–elements. In Section 5.1 we show that a lattice $\Gamma < \mathrm{SL}_d(\mathbb{F}_q((t)))$ is cocompact if and only if it does not contain any p–elements. This is an analogue of Godement's Compactness Criterion (or the

Kazhdan–Margulis Theorem in the case of real Lie groups [**17**]). We were not able to find a suitable statement in the literature, hence we provide a complete and elementary proof. In Section 5.2, we use this criterion to show that when $(3, q-1) = 1$ and $p = 2$, the lattice Γ_0 is a cocompact lattice in $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$ of minimal covolume. We also show that when $(3, q-1) = 1$ and $p = 3$, Γ_0' is a maximal lattice in $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$, and that when $(3, q-1) = 1$ and $p \neq 3$, Γ_0 is a maximal lattice in $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$. We conclude the discussion of covolumes with a conjecture about the cocompact lattice of minimal covolume in $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$ when $(3, q-1) = 1$ and p is odd.

Finally, in Section 6, we discuss how our results answer some open questions from the work of Essert [**11**]. For example, Theorem 2 implies that for all q such that $(3, q-1) = 1$, the group $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$ contains a lattice which acts simply transitively on the set of panels of each type in Δ.

To obtain the lattices Γ_0' and Γ_0 in Theorem 1, we use a construction of Cartwright and Steger from [**7**], which generalises work of [**6**]. This construction gives cocompact lattices $\Gamma < \widetilde{\Gamma}$ in the automorphism group $\mathrm{Aut}(\widetilde{\mathcal{A}})$ of a certain algebra $\widetilde{\mathcal{A}}$, such that $\mathrm{Aut}(\widetilde{\mathcal{A}})$ is isomorphic to $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$. The lattice Γ acts simply transitively on the vertex set of Δ, and $\widetilde{\Gamma} = H\Gamma$ where H is a finite group which is the stabiliser in $\widetilde{\Gamma}$ of a vertex of Δ. We review and slightly extend this construction in Section 3, assuming no background in cyclic algebras from the reader. Our treatment applies to any cyclic Galois extension rather than just the extension of finite fields $\mathbb{F}_{q^d} \supseteq \mathbb{F}_q$. In Section 3.3 we choose an explicit isomorphism $\mathrm{Aut}(\widetilde{\mathcal{A}}) \to \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$ and so move our discussion explicitly into $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$. We also show that H is isomorphic to the normaliser of a Singer cycle S in $\mathrm{PGL}_d(q)$.

We then define Γ_0' and Γ_0 to be the subgroups of $\widetilde{\Gamma}$ generated by suitable $\widetilde{\Gamma}$–conjugates of S or H, respectively. Since $\widetilde{\Gamma}$ is a discrete subgroup of $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$, it is immediate that Γ_0' and Γ_0 are discrete. Using geometric arguments, we show that Γ_0' and Γ_0 act cocompactly on Δ, hence are cocompact lattices. The main additional ingredient in the proof of Theorem 2 is our determination in Section 3 of the intersection of H with $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. This intersection is also used to show that, for certain values of d and q, in fact $\Gamma_0 = \widetilde{\Gamma} \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ or $\Gamma_0' = \widetilde{\Gamma} \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. In fact, an anonymous referee pointed out to us the following result, which is proved in Section 4.3:

THEOREM 3. *The lattice Γ_0' is the type-preserving subgroup of $S\widetilde{\Gamma}$, and the lattice Γ_0 is the type-preserving subgroup of $\widetilde{\Gamma}$. Hence in particular, Γ_0' and Γ_0 are commensurable with $\widetilde{\Gamma}$.*

2. Preliminaries.

We briefly recall some definitions and results, and fix notation.

2.1. Singer cycles and projective spaces.

The following definitions and results are taken from [**10**]. Let q be a power of a prime p and let V be the vector space $\mathbb{F}_q^d$, for $d \geq 2$. A cyclic subgroup S of $\mathrm{GL}_d(q)$ that acts simply transitively on the set of non-zero vectors of V is called a *Singer cycle of* $\mathrm{GL}_d(q)$. Its generator s is an element of $\mathrm{GL}_d(q)$ of order $(q^d - 1)$ and so $|S| = q^d - 1$. The image of a Singer cycle of $\mathrm{GL}_d(q)$ in $\mathrm{PGL}_d(q)$ under the canonical epimorphism is called a *Singer cycle of* $\mathrm{PGL}_d(q)$. The intersection of a Singer cycle S of $\mathrm{GL}_d(q)$ with $\mathrm{SL}_d(q)$, that is, $S \cap \mathrm{SL}_d(q)$, is called a *Singer cycle of* $\mathrm{SL}_d(q)$. Its image under the canonical epimorphism from $\mathrm{SL}_d(q)$ onto $\mathrm{PSL}_d(q)$ is called a *Singer cycle of* $\mathrm{PSL}_d(q)$. A Singer cycle of $\mathrm{PGL}_d(q)$ or of $\mathrm{SL}_d(q)$ has order $\frac{q^d-1}{q-1}$, and a Singer cycle of $\mathrm{PSL}_d(q)$ has order $\frac{q^d-1}{(q-1)\delta}$ where $\delta = (d, q-1)$.

Note that a Singer cycle of $\mathrm{PGL}_d(q)$ acts simply transitively on the set of 1–dimensional subspaces of V, and hence acts simply transitively on the set of $(d-1)$–dimensional subspaces of V as well.

We denote by $\mathrm{PG}(n, q)$ the projective space of dimension $n = d - 1$ over the finite field $\mathbb{F}_q$. Recall that the set of *points* of $\mathrm{PG}(n, q)$ is the set of 1–dimensional subspaces of V, and the set of *lines* is the set of 2–dimensional subspaces of V.

Thus in particular, a Singer cycle of $\mathrm{PGL}_3(q)$ acts simply transitively on both the set of points and the set of lines of the projective plane $\mathrm{PG}(2, q)$. If $(3, q-1) = 1$, the order of a Singer cycle of $\mathrm{PSL}_3(q)$, $\frac{q^3-1}{q-1}$, coincides with the order of a Singer cycle of $\mathrm{PGL}_3(q)$. It follows immediately that in this case, if we identify $\mathrm{PSL}_3(q)$ with a subgroup of $\mathrm{PGL}_3(q)$, the Singer cycles of $\mathrm{PSL}_3(q)$ and $\mathrm{PGL}_3(q)$ coincide. On the other hand, if 3 divides $q-1$ (that is, $(3, q-1) = 3 \neq 1$), the order of a Singer cycle of $\mathrm{PSL}_3(q)$ is $\frac{q^3-1}{3(q-1)}$ and so this subgroup cannot act transitively on the q^2+q+1 points of the projective plane $\mathrm{PG}(2, q)$. In fact, a simple application of the Orbit-Stabiliser Theorem shows that even the normaliser of a Singer cycle of $\mathrm{PSL}_3(q)$ cannot act transitively on the points of $\mathrm{PG}(3, q)$. Moreover, for large enough q, the only p'–subgroups of $\mathrm{PSL}_3(q)$ that act transitively on the points of $\mathrm{PG}(2, q)$ are Singer cycles and their normalisers and only when $(3, q-1) = 1$. This follows immediately from an inspection of the maximal subgroups of $\mathrm{PSL}_3(q)$ that are provided by a result of Hartley and Mitchell (Theorem 6.5.3 of [**14**]). Hence for large enough q, if 3 divides $(q-1)$ there are no p'–subgroups of $\mathrm{SL}_3(q)$ that act transitively on the set of points of $\mathrm{PG}(2, q)$.

2.2. Buildings of type $\widetilde{A}_n$. We assume basic knowledge of buildings, and extract from [**6**] and [**7**] the facts that we will need. A reference for this theory is [**25**]. We also recall the Levi decomposition of a vertex stabiliser in $\mathrm{SL}_d(\mathbb{F}_q(\!(t)\!))$ or $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$.

Let Δ be the building $\tilde{A}_n(K, \nu)$ on which $\mathcal{G}(K)$ acts, where $K = \mathbb{F}_q(\!(t)\!)$, as in the introduction. Let $\mathcal{O} := \{a \in K : \nu(a) \geq 0\} = \mathbb{F}_q[[t]]$. A *lattice in* K^d is a free $\mathcal{O}$–submodule of K^d of rank d, and two lattices L and L' are said to be *equivalent* if $L' = La$ for some $a \in K^\times$. The vertices of Δ are the equivalence classes of lattices in K^d. The group $G = \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$ acts transitively on the vertex set of Δ, so that the stabiliser of the equivalence class represented by $\mathcal{O}^d$ is $P_0 := \mathrm{PGL}_d(\mathbb{F}_q[[t]])$. Thus we may identify the vertex set of Δ with the set of cosets G/P_0. For $g \in \mathrm{GL}_d(\mathbb{F}_q(\!(t)\!))$, we denote the image of g in $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$ by $\overline{g}$. The *type* of the vertex $\overline{g}P_0$ is $\nu(\det(g)) \pmod d$.

Let v_0 be the vertex of Δ identified with the trivial coset of P_0. Then v_0 is the vertex of type 0 in the standard chamber of Δ. For $i = 1, \ldots, d-1$, the vertex v_i of type i in the standard chamber is a coset of the form $\overline{g_i}P_0$ where $g_i \in \mathrm{GL}_d(\mathbb{F}_q(\!(t)\!))$ has entries in $\mathcal{O}$, and $\nu(\det(g_i)) = i$. The set of all vertices adjacent to v_0 corresponds to the elements of the projective space $\mathrm{PG}(n, q)$, and moreover we may choose the types so that for each $i = 1, \ldots, d-1$, the vertices neighbouring v_0 of type i correspond to the i–dimensional subspaces of $V = \mathbb{F}_q^d$.

The action of each $\overline{g} \in \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$ on Δ induces a permutation of the set of types of the form $i \mapsto i + c \pmod d$, where $c = \nu(\det(g))$. Any automorphism of Δ which induces a permutation of types of the form $i \mapsto i + c \pmod d$, for some c, is said to be *type-rotating*. In particular, a type-rotating automorphism fixes either no type or all types.

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We will need the following decomposition of vertex stabilisers, which is a special case of a result for topological Kac–Moody groups in [**8**].

PROPOSITION 4 (Levi decomposition). *Let $G = \mathcal{G}(\mathbb{F}_q((t)))$ where $\mathcal{G}$ is SL_d or PSL_d, $d \geq 2$, and q is a power of a prime p. Let v be a vertex of the building Δ associated to G. Then the stabiliser of v in G has Levi decomposition*

$$L_v \ltimes U_v$$

where L_v is isomorphic to the finite group $\mathcal{G}(\mathbb{F}_q)$, and U_v is pro–p.

3. Generalisation of the Cartwright–Steger construction. We first in Section 3.1 describe the basics of cyclic algebras, following Pierce [**23**]. We then in Section 3.2 extend the construction of [**6**] and [**7**] to general cyclic extensions, using invariant language. For brevity, we will refer to the construction in [**6**] and [**7**] as the Cartwright–Steger construction. Finally in Section 3.3 we restrict to the case of finite fields and recall or prove facts that will be useful for our constructions of lattices in Section 4 below. We note that our constructions of lattices require only the finite fields case.

3.1. Basic definitions and properties. Let $\mathbb{E} \supseteq \mathbb{K}$ be a cyclic Galois extension of degree d, $\sigma \in \mathrm{Gal}(\mathbb{E}/\mathbb{K})$ a generator and $a \in \mathbb{K}^\times$ an element. The cyclic algebra $(\mathbb{E}, \sigma, a)$ is generated as a ring by $\mathbb{E}$ and an extra element t, with $\mathbb{E}$ a subring so that the ring operations of $\mathbb{E}$ are retained in $(\mathbb{E}, \sigma, a)$. The relations involving t are

$$t^d = a, \quad tb = \sigma(b)t \text{ for all } b \in \mathbb{E}.$$

The following are well-known properties of the cyclic algebras:

(1) $(\mathbb{E}, \sigma, a)$ is a central simple algebra over $\mathbb{K}$ of dimension d^2;
(2) $\mathbb{E}$ is a maximal subfield of $(\mathbb{E}, \sigma, a)$; and
(3) the elements $1, t, t^2, \ldots, t^{d-1}$ form a basis of $(\mathbb{E}, \sigma, a)$ over $\mathbb{E}$.

In particular, each cyclic algebra defines an element $[(\mathbb{E}, \sigma, a)]$ in the relative Brauer group $\mathrm{Br}(\mathbb{E}/\mathbb{K})$. Recall the definitions of the trace and the norm $T, N : \mathbb{E} \to \mathbb{K}$:

$$T(a) = \sum_{k=0}^{d-1} \sigma^k(a), \quad N(a) = \prod_{k=0}^{d-1} \sigma^k(a).$$

The norm image $N(\mathbb{E}^\times)$ is a subgroup of $\mathbb{K}^\times$. We also need the following properties [**23**]:

(4) $(\mathbb{E}, \sigma, a) \cong M_d(\mathbb{K})$ if and only if $a \in N(\mathbb{E}^\times)$; and
(5) if $a \in \mathbb{K}^\times$ and the order of $aN(\mathbb{E}^\times) \in \mathbb{K}^\times/N(\mathbb{E}^\times)$ is d then $(\mathbb{E}, \sigma, a)$ is a division algebra.

The cyclic extension $\mathbb{E} \supseteq \mathbb{K}$ gives rise to two further cyclic Galois extensions: the fields of rational functions $\mathbb{E}(Y) \supseteq \mathbb{K}(Y)$ and the fields of Laurent series $\mathbb{E}((Y)) \supseteq \mathbb{K}((Y))$. One can think of them as Galois extensions with the same Galois group, so that σ acts on the coefficients while $\sigma(Y) = Y$.

3.2. The construction for general cyclic extensions, using invariant language. The first cyclic algebra of interest to us is

$$\mathcal{A} := (\mathbb{E}(Y), \sigma, 1 + Y).$$

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It is a division algebra, by property (5) [**15**, p.84]: the equation

$$N\left(\frac{a_0 + \cdots + a_m Y^m}{b_0 + \cdots + b_k Y^k}\right) = (1+Y)^n$$

with $a_m \neq 0 \neq b_k$ gets rewritten as

$$N(a_m)Y^{md} + \mathcal{O}(Y^{md-1}) = (N(b_k)Y^{kd} + \mathcal{O}(Y^{kd-1}))(Y^n + \mathcal{O}(Y^{n-1})).$$

Comparing the highest terms, $md = kd + n$. Hence n must be divisible by d, for $(1+Y)^n$ to be a norm of some element. Since $N(1+Y) = (1+Y)^d$, the order of $(1+Y)N(\mathbb{E}^\times)$ is exactly d. By (5), $\mathcal{A}$ is a division algebra.

The second cyclic algebra of interest is

$$\widetilde{\mathcal{A}} := (\mathbb{E}((Y)), \sigma, 1+Y) \cong \mathbb{K}((Y)) \otimes_{\mathbb{K}(Y)} \mathcal{A}.$$

It is isomorphic to the matrix algebra $M_d(\mathbb{K}((Y)))$ by (4). To observe this, let us note that the trace $T : \mathbb{E}((Y)) \to \mathbb{K}((Y))$ is surjective. Indeed, pick any $x \in \mathbb{E}((Y))$ with nonzero trace $T(x) = \beta \in \mathbb{K}((Y))$, then for every $\alpha \in \mathbb{K}((Y))$ we have $T(\alpha\beta^{-1}x) = \alpha$. This allows to solve the equation

$$N(1 + x_1 Y + x_2 Y^2 + \cdots) = 1 + Y$$

recursively: x_1 is a solution of $T(x_1) = 1$, and each consecutive term x_n will be a solution of $T(x_n) = f_n(x_1, \ldots, x_{n-1})$ for a certain function f_n of all the previously found terms.

We would like to write an explicit isomorphism Ψ from $\widetilde{\mathcal{A}}$ to a matrix algebra. Observe that in $\widetilde{\mathcal{A}}$ for any $a, b \in \mathbb{E}((Y))$

$$(at)b = \sigma(b)at \quad \text{and} \quad (at)^d = a\sigma(a)t^2(at)^{d-2} = \cdots = N(a)t^d = N(a)(1+Y).$$

Hence, if $X \in \mathbb{E}((Y))$ is a solution of $N(X) = 1 + Y$ then

$$\sum_j a_j t^j \mapsto \sum_j a_j X^j \hat{t}^j$$

is an isomorphism from $\widetilde{\mathcal{A}}$ to $(\mathbb{E}((Y)), \sigma, 1)$. The latter is known as the *skew group algebra* and admits an explicit isomorphism to the matrix algebra $\mathrm{End}_{\mathbb{K}((Y))}(\mathbb{E}((Y))$ given by $a\hat{t}^j : b \mapsto a\sigma^j(b)$. Composing these isomorphisms, we arrive at an explicit isomorphism

$$\Psi : \widetilde{\mathcal{A}} \to \mathrm{End}_{\mathbb{K}((Y))}(\mathbb{E}((Y)))$$

given by

$$\Psi\left(\sum_j a_j t^j\right) : b \mapsto \sum_j a_j X^j \sigma^j(b), \quad b \in \mathbb{E}((Y)). \tag{1}$$

We will abuse notation by denoting various restrictions of Ψ, for instance to $\mathcal{A}$, by the same letter. On the level of multiplicative groups we have an injective homomorphism

$$\Psi : \mathcal{A}^\times \to \mathrm{GL}_{\mathbb{K}((Y))}(\mathbb{E}((Y))).$$

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By the Skolem–Noether Theorem, every $\mathbb{K}(Y)$–linear automorphism of $\mathcal{A}$ is inner, so we have another injective group homomorphism

$$\overline{\Psi} : \mathrm{Aut}(\mathcal{A}) \cong \mathcal{A}^\times / Z(\mathcal{A}^\times) \to \mathrm{PGL}_{\mathbb{K}((Y))}(\mathbb{E}((Y))).$$

Now we are ready to introduce the *Cartwright–Steger groups* [**6**, **7**]. Let $\mathcal{A}_0$ be the $\mathbb{E}[Y^{-1}]$–span of the elements t^m, $m < d$ in $\mathcal{A}$. Notice that it is not a subring: $t^d = 1 + Y \notin \mathcal{A}_0$. The "big" Cartwright–Steger group $\widetilde{\Gamma}$ is defined as

$$\widetilde{\Gamma} := \{\gamma \in \mathrm{Aut}(\mathcal{A}) \mid \gamma(\mathcal{A}_0) \subseteq \mathcal{A}_0\}.$$

Why is $\widetilde{\Gamma}$ a subgroup? To show this we choose a $\mathbb{K}(Y)$–basis $\mathcal{B}$ of $\mathcal{A}$ consisting of the elements at^m, $m < d$, $a \in \mathbb{E}$. The basis $\mathcal{B}$ is also a $\mathbb{K}((Y))$–basis of $\widetilde{\mathcal{A}}$. Writing automorphisms in this basis gives an injective homomorphism

$$\Phi : \mathrm{Aut}(\mathcal{A}) \to \mathrm{GL}_{\mathbb{K}(Y)}(\mathcal{A}) \to \mathrm{GL}_{\mathbb{K}((Y))}(\widetilde{\mathcal{A}}) \cong \mathrm{GL}_{d^2}(\mathbb{K}((Y))).$$

Moreover, each $\Phi(\gamma)$ is an automorphism of $\widetilde{\mathcal{A}}$. By the Skolem–Noether Theorem, $\Phi(\gamma)(x) = y_\gamma x {y_\gamma}^{-1}$ for a certain $y_\gamma \in \widetilde{\mathcal{A}} \cong M_d(\mathbb{K}((Y)))$. It follows that $\det(\Phi(\gamma)) = \det(y_\gamma)^d \det(y_\gamma)^{-d} = 1$ [7, p.129]. Thus, we can restrict the image of Φ to the special linear group:

$$\Phi : \mathrm{Aut}(\mathcal{A}) \to \mathrm{SL}_{d^2}(\mathbb{K}((Y))).$$

Clearly, $\gamma \in \widetilde{\Gamma}$ if and only if the coefficients of $\Phi(\gamma)$ lie in $\mathbb{K}[Y^{-1}]$. Thus,

$$\widetilde{\Gamma} = \Phi^{-1}(\mathrm{SL}_{d^2}(\mathbb{K}[Y^{-1}]))$$

is a subgroup. Since $\gamma \in \widetilde{\Gamma}$ is $\mathbb{K}(Y)$–linear, we have $\gamma(Y^{-1}\mathcal{A}_0) \subseteq Y^{-1}\mathcal{A}_0$ for any $\gamma \in \widetilde{\Gamma}$. Thus γ defines a linear map $\Theta(\gamma) \in \mathrm{End}_{\mathbb{K}}(\mathcal{A}_+)$ where $\mathcal{A}_+ = \mathcal{A}_0 / Y^{-1}\mathcal{A}_0$. The map Θ is a semigroup homomorphism from a group, so its image consists of invertible elements:

$$\Theta : \widetilde{\Gamma} \to \mathrm{GL}_{\mathbb{K}}(\mathcal{A}_+) \cong \mathrm{GL}_{d^2}(\mathbb{K}).$$

In essence, Θ is the Y–degree zero term of Φ: the basis $\mathcal{B}$ defined above gives an $\mathbb{K}$–basis of $\mathcal{A}_+$. The basis $\mathcal{B}$ has a partial order coming from the degree of t in $[at^j] = at^j + Y^{-1}\mathcal{A}_0$. Let T be the group of "unitriangular" transformations in this basis, that is,

$$T = \{\pi \in \mathrm{GL}_{\mathbb{K}}(\mathcal{A}_+) \mid \forall a \in \mathbb{E}, j < d \;\; \pi([at^j]) = [at^j] + \sum_{i=0}^{j-1} [a_i t^i],\ a_i \in \mathbb{E}\}.$$

Finally, the "small" Cartwright–Steger group is

$$\Gamma := \Theta^{-1}(T) \leq \widetilde{\Gamma}.$$

(Since not all of T may be in the image of Θ, we should perhaps write that $\Gamma = \Theta^{-1}(T) \cap \mathrm{Im}(\Theta)$.)

LEMMA 5. *If* $\gamma \in \Gamma$ *then*

$$\gamma(t) = t + \mathcal{O}(Y^{-1}) \quad \textit{and} \quad \gamma(t^{d-1}) = t^{d-1} + \mathcal{O}(Y^{-1})$$

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where $\mathcal{O}(Y^{-1})$ denotes a polynomial in negative degrees of Y with coefficients in $\mathcal{A}_0$.

Proof. By definition of Γ,

$$\gamma(t) = t + a + \mathcal{O}(Y^{-1}) \text{ and } \gamma(t^{d-1}) = t^{d-1} + b_{d-2}t^{d-2} + \cdots + b_1 t + b_0 + \mathcal{O}(Y^{-1})$$

for some $a, b_i \in \mathbb{E}^\times$. Let us analyse the key equation

$$1 + Y = \gamma(1 + Y) = \gamma(t^d) = \gamma(t)\gamma(t^{d-1}).$$

Since $t^d = 1 + Y$ we get the equation

$$(a + \sigma(b_{d-2}))t^{d-1} + (ab_{d-2} + \sigma(b_{d-3}))t^{d-2} + \cdots + (ab_1 + \sigma(b_0))t^1 + ab_0 + \mathcal{O}(Y^{-1}) = 0.$$

If $a = 0$ then we immediately conclude that all $\sigma(b_i) = 0$. Hence all $b_i = 0$ and we are done. If $a \neq 0$ then we conclude that all $b_0 = 0$. Then $b_1 = 0$. Recursively, all $b_i = 0$ and we are done. □

To contemplate the difference between Γ and $\widetilde{\Gamma}$, let us introduce another group H: as a set H consists of $\gamma \in \mathrm{Aut}(\mathcal{A})$ that are conjugations by at^j, where $a \in \mathbb{E}$ and $j < d$.

PROPOSITION 6.
(1) *H is a subgroup of $\widetilde{\Gamma}$.*
(2) *$H \cap \Gamma = \{1\}$.*
(3) *$H\Gamma$ is a subgroup of $\widetilde{\Gamma}$ and Γ is normal in $H\Gamma$.*

As recalled in Section 3.3 below, in the case of finite fields $H\Gamma = \widetilde{\Gamma}$, which may or may not hold over arbitrary fields. This is an interesting question.

Proof. Let us calculate in $\mathcal{A}$, writing $x \sim y$ when x and y give the same conjugation in $\mathrm{Aut}(\mathcal{A})$. Since $1 + Y \sim 1$,

$$(at^j)^{-1} = t^{d-j}a^{-1}(1 + Y)^{-1} \sim \sigma^{d-j}(a^{-1})t^{d-j} \quad \text{and}$$

$$(at^j)(bt^i) = a\sigma^j(b)t^{i+j} \sim a\sigma^j(b)t^{i+j-d},$$

showing that H is a subgroup of $\mathrm{Aut}(\mathcal{A})$. If $\gamma \in H$ is a conjugation by at^j, where $a \in \mathbb{E}$ and $j < d$, then

$$\begin{aligned}\gamma(bt^i) &= at^j bt^i t^{d-j} a^{-1}(1 + Y)^{-1}\\ &= a\sigma^j(b)t^{d+i}a^{-1}(1 + Y)^{-1}\\ &= a\sigma^i(a^{-1})\sigma^j(b)t^i.\end{aligned}$$

Thus H is a subgroup of $\widetilde{\Gamma}$. Moreover, $\gamma \in \Gamma$ if and only if $b = a\sigma^i(a^{-1})\sigma^j(b)$ for all b and i if and only if $a \in \mathbb{K}$ and $j = 0$ if and only if $\gamma = 1$. This proves (2).

Finally, it suffices to check that $\gamma\Gamma\gamma^{-1} \subseteq \Gamma$ where γ is a conjugation by x, and x is either t or $a \in \mathbb{E}^\times$. If $\beta \in \Gamma$, then

$$\gamma\beta\gamma^{-1}(y) = x\beta(x^{-1})\beta(y)(x\beta(x^{-1}))^{-1}.$$

Note that elements of Γ are characterised by the fact that

$$\beta(bt^i) = bt^i + \mathcal{O}(t^{i-1}) + \mathcal{O}(Y^{-1})$$

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for all $b \in \mathbb{E}$ and $i \in \{0, 1, \dots, d-1\}$, where $\mathcal{O}(t^{i-1})$ denotes a polynomial in $1, t, \dots, t^{i-1}$ with coefficients in $\mathbb{E}$ and $\mathcal{O}(Y^{-1})$ denotes a polynomial in negative degrees of Y with coefficients in $\mathcal{A}_0$.

If $x = a$ then

$$\beta(a) = a + \mathcal{O}(Y^{-1}), \quad \beta(a^{-1}) = a^{-1} + \mathcal{O}(Y^{-1})$$

by the definition of Γ and

$$x\beta(x^{-1}) = 1 + \mathcal{O}(Y^{-1}), \quad (x\beta(x^{-1}))^{-1} = \beta(x)x^{-1} = 1 + \mathcal{O}(Y^{-1}).$$

Finally,

$$\begin{aligned}\gamma\beta\gamma^{-1}(y) &= (1 + \mathcal{O}(Y^{-1}))(bt^i + \mathcal{O}(t^{i-1}) + \mathcal{O}(Y^{-1}))(1 + \mathcal{O}(Y^{-1})) \\ &= bt^i + \mathcal{O}(t^{i-1}) + \mathcal{O}(Y^{-1})\end{aligned}$$

because there would not be enough powers of t to cancel all of the Y^{-j} using $t^d = 1 + Y$ and produce at least an i-th power of t.

Similarly, if $x = t$ then

$$\beta(t) = t + \mathcal{O}(Y^{-1}), \quad \beta(t^{-1}) = (1 + Y)^{-1}(t^{d-1} + \mathcal{O}(Y^{-1}))$$

by Lemma 5. Since $(1 + Y)^{-1} = Y^{-1} - Y^{-2} + Y^{-3} - \cdots$,

$$x\beta(x^{-1}) = 1 + \mathcal{O}(Y^{-1}), \quad (x\beta(x^{-1}))^{-1} = \beta(x)x^{-1} = 1 + \mathcal{O}(Y^{-1}).$$

Finally,

$$\gamma\beta\gamma^{-1}(y) = bt^i + \mathcal{O}(t^{i-1}) + \mathcal{O}(Y^{-1})$$

as in the case of $x = a$. □

It would be useful for us to know how the image $\overline{\Psi}(\widetilde{\Gamma})$ intersects with $\mathrm{PSL}_{\mathbb{K}((Y))}(\mathbb{E}((Y)))$. We can understand this for the image of H. By $(\mathbb{K}^\times)^k$ we denote the subgroup of the multiplicative group $\mathbb{K}^\times$ consisting of k-th powers. Let $\gamma : \mathcal{A}^\times \to \mathrm{Aut}(\mathcal{A})$ be the homomorphism assigning the conjugation by x to each $x \in \mathcal{A}^\times$.

PROPOSITION 7. *Let p be the characteristic of $\mathbb{K}$. Denote by $\mathrm{Ord}_p(m)$ the largest power of p that divides an integer m (or 1 if $p = 0$). Then*

$$\begin{aligned}&\overline{\Psi}(H) \cap \mathrm{PSL}_{\mathbb{K}((Y))}(\mathbb{E}((Y))) \\ &\quad = \{\Psi(\gamma(at^k)) \mid a \in \mathbb{E}^\times, N(a) \in (\mathbb{K}^\times)^d, \mathrm{Ord}_p(k) \geq \mathrm{Ord}_p(d)\}.\end{aligned}$$

Proof. The element $\overline{\Psi}(\gamma(at^k))$ is in $\mathrm{PSL}_{\mathbb{K}((Y))}(\mathbb{E}((Y)))$ if and only if one can multiply $\Psi(at^k)$ by a scalar matrix zI_d, $z \in \mathbb{K}((Y))$, so that the determinant of the product is 1. Now the product

$$z\Psi(at^k) : b \mapsto zaX^k\sigma^k(b), \quad \forall b \in \mathbb{E}((Y))$$

is a composition of four linear maps

$$(b \mapsto zb) \circ (b \mapsto ab) \circ (b \mapsto X^k b) \circ \sigma^k$$

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so its determinant is the product of four determinants:

$$\det(z\Psi(at^k)) = z^d \cdot N(a) \cdot (1+Y)^k \cdot (-1)^{(d-1)k}.$$

Here we use the fact that the determinant of the multiplication $(b \mapsto ab)$ is the norm $N(a)$. In particular, we see three norms, including $N(z) = z^d$ and $N(X^k) = (1+Y)^k$. From Galois theory, we know that the action of σ on $\mathbb{E}((Y))$ is conjugate to the permutation matrix of a cycle of length d that gives the last determinant.

Thus, we just need a d-th root of $(-1)^k N(a)(1+Y)^k$ in $\mathbb{K}((Y))$. The free term of such a root is a d-th root of $N((-1)^k a)$. Therefore it is necessary and sufficient to have d-th roots of both $N((-1)^k a)$ and $(1+Y)^k$. The existence of the former is equivalent to $N((-1)^k a) \in (\mathbb{K}^\times)^d$, while the existence of the latter is equivalent to $\mathrm{Ord}_p(k) \geq \mathrm{Ord}_p(d)$.

The last statement needs an explanation. Write $d = \mathrm{Ord}_p(d)d'$. Extracting a d'-th root of $(1+Y)^k$ can be done because d' is invertible in $\mathbb{K}$: the equation

$$(1 + x_1 Y + x_2 Y^2 + \cdots)^{d'} = (1+Y)^k$$

can be solved recursively: x_1 is a solution of $d'x_1 = k$, and each consecutive term x_n will be a solution of $d'x_n = f_n(x_1, \ldots, x_{n-1})$ for a certain function f_n of all the previously found terms. It remains to contemplate extracting of the p-th root in characteristic p: since

$$(1 + x_1 Y + x_2 Y^2 + \cdots)^p = 1 + x_1^p Y^p + x_2^p Y^{2p} + \cdots$$

this can be done if and only if $(1+Y)^k$ is already a p-th power, that is, if and only if p divides k.

Finally, since $\Psi(\gamma((-1)^k at^k)) = \Psi(\gamma(at^k))$ we can replace $(-1)^k a$ with a. □

3.3. Application to the case of finite fields, and summary of useful results. While the algebraic properties of the construction in Section 3.2 above are upheld in any cyclic extension, we would like to move to its topological and metric properties. For this, from now on we assume that the extension $\mathbb{E} \supseteq \mathbb{K}$ is a finite field extension $\mathbb{F}_{q^d} \supseteq \mathbb{F}_q$ with $q = p^a$, p a prime.

PROPOSITION 8. *Let $\mathbb{E} = \mathbb{F}_{q^d}$ and $\mathbb{K} = \mathbb{F}_q$. Then*

$$\left|\overline{\Psi}(H) : \left(\overline{\Psi}(H) \cap \mathrm{PSL}_{\mathbb{K}((Y))}(\mathbb{E}((Y)))\right)\right| = \delta \cdot \mathrm{Ord}_p(d)$$

where δ is the greatest common divisor of d and $(q-1)$ (note that δ is a divisor of $(q^d-1)/(q-1)$).

Proof. Clearly $at^k \sim bt^m$ (with $k, m < d$) if and only if $ab^{-1} \in \mathbb{K}$ and $k = m$. Thus, we can compute the contributions to the index from a and from t separately. The powers of t of degrees $\mathrm{Ord}_p(d)$, $2\,\mathrm{Ord}_p(d)$, $\ldots$, $d - \mathrm{Ord}_p(d)$ are exactly those that produce elements of the subgroup. So, $\mathrm{Ord}_p(d)$ is the contribution from t. The contribution from a is the index

$$\left|\mathbb{E}^\times : \mathbb{K}^\times N^{-1}((\mathbb{K}^\times)^d)\right| = \left|\mathbb{E}^\times : N^{-1}((\mathbb{K}^\times)^d)\right| = \left|\mathbb{K}^\times : (\mathbb{K}^\times)^d\right| = n.$$

The first equality holds because $\mathbb{K}^\times \subseteq N^{-1}((\mathbb{K}^\times)^d)$. Indeed, $N(a) = a^d \in (\mathbb{K}^\times)^d$ for all $a \in \mathbb{K}^\times$. The second equality holds since N is surjective and $(\mathbb{K}^\times)^d$ has index n in $\mathbb{K}^\times$. □

Using the explicit expression for Ψ at (1) above, one can construct an explicit image of H in the locally compact, totally disconnected group $G = \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$ under $\overline{\Psi}$. Interestingly enough, if $(p, d) = 1$, one can see that $\overline{\Psi}(H)$ can be realised as a subgroup of $\mathrm{PGL}_d(q)$ naturally embedded in $\mathrm{PGL}_d(\mathbb{F}_q[[t]])$. However, if $p \mid d$, this is not possible and $\overline{\Psi}(H) \cap \mathrm{PGL}_d(q)$ is a subgroup of index $\mathrm{Ord}_p(d)$ in $\overline{\Psi}(H)$. This difference comes from the fact that in the former case X (a solution of $N(X) = 1 + Y$) can be realised over $\mathbb{F}_q$, while in the latter case this is not possible.

So far we have been working in $\mathrm{Aut}(\widetilde{\mathcal{A}})$. However, it will now be convenient to switch our discussion explicitly into $G = \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$. To avoid excessive notations, we identify $\widetilde{\Gamma}$ with its image $\overline{\Psi}(\widetilde{\Gamma})$ in G. From now on we call this image $\widetilde{\Gamma}$. Likewise, we call $\widetilde{\Gamma}_v$, now in G, again by H (instead of using $\overline{\Psi}(H)$).

We now recall the facts about $\widetilde{\Gamma}$ that will be useful for us. Most of them can be derived from Section 3.2 but, as they already appear in [**7**], we just restate them. We have:

(1) $\widetilde{\Gamma}$ is a cocompact lattice of $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$;
(2) Γ acts simply transitively on the set of vertices of the building Δ associated to $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$;
(3) $H = \widetilde{\Gamma}_v$ for a vertex v of Δ;
(4) $|H| = \frac{q^d-1}{q-1}d$; and
(5) $\widetilde{\Gamma} = H\Gamma$.

We will now discuss the structure of H and some of its properties.

LEMMA 9. *Let $H = \widetilde{\Gamma}_v$ for a vertex v of Δ the building associated to $G = \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$. Then the following conditions hold:*

(1) *H is a subgroup of $G_v \cong \mathrm{PGL}_d(\mathbb{F}_q[[t]])$;*
(2) *H contains a normal cyclic subgroup S of order $\frac{q^d-1}{q-1}$ where S is a Singer cycle of $\mathrm{PGL}_d(q)$;*
(3) *$H \cong N_{\mathrm{PGL}_d(q)}(S)$; and*
(4) *if we identify $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ with a subgroup of G, then*

$$|H \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))| = \frac{d}{\mathrm{Ord}_p(d)} \cdot \frac{q^d-1}{(q-1)(d, q-1)}.$$

Proof. Part (1) follows immediately from the fact that $H = \widetilde{\Gamma}_v$, hence $H \leq G_v$, and the fact that $G_v \cong \mathrm{PGL}_d(\mathbb{F}_q[[t]])$, as discussed in Section 2.2.

For (2), using the notation of Proposition 6, let S be the image of $mE^{\times}$ in $\mathrm{PGL}_d(q)$. Obviously, S is a cyclic subgroup of H of order $\frac{q^d-1}{q-1}$. Now from the proof of (1) of Proposition 6, it follows that S indeed is normal in H. Moreover, as S is an abelian subgroup of $\mathrm{PGL}_d(q)$ of order $\frac{q^d-1}{q-1}$, Proposition 2.2 of [**10**] implies that S is a Singer cycle of $\mathrm{PGL}_d(q)$.

To prove (3), we have $H \leq G_v \cong \mathrm{PGL}_d(\mathbb{F}_q[[t]]) \cong U_v \rtimes \mathrm{PGL}_d(q)$ where U_v is a pro–p group. If $(p, d) = 1$, then $(|H|, p) = 1$ and so $H \cap U_v = 1$. Suppose that $p \mid d$. Assume that $H \cap U_v \neq 1$. Then there exists $1 \neq h \in H \cap U_v$, an element of order p. It follows that $[h, S] \leq U_v \cap S = 1$ since on the one hand $h \in U_v \lhd G_v$ and $S \leq G_v$, while on the other, h normalises S and $(p, |S|) = 1$. Thus h centralises S. Using calculations from the proof of Proposition 6(1) we observe that S is self-centralising in H. We have reached a contradiction that proves that $H \cap U_v = 1$. It follows immediately that $H \cong \overline{H} \leq \overline{G}_v := G_v/U_v \cong \mathrm{PGL}_d(q)$.

Now $\overline{H}$ contains a normal subgroup $\overline{S} \cong S$ which is a Singer cycle of $\overline{G}_v$, by Proposition 2.2 of [**10**]. Moreover, $|\overline{H}| = |N_{\mathrm{PGL}_d(q)}(\overline{S})|$. Therefore (3) holds.

Finally using (1), (2) and (3) together with Proposition 8, we conclude that (4) holds. □

4. The lattices Γ_0 and Γ_0'. In this section we prove our main results, Theorems 1 and 2, for all $d \geq 3$. We construct and establish the properties of the lattices $\Gamma_0' \leq \Gamma_0$ in $\mathrm{PGL}_d(\mathbb{F}_q((t)))$ in Section 4.1, and investigate the intersections $\Lambda_0' := \Gamma_0' \cap \mathrm{PSL}_d(\mathbb{F}_q((t)))$ and $\Lambda_0 := \Gamma_0 \cap \mathrm{PSL}_d(\mathbb{F}_q((t)))$ in Section 4.2. We then discuss the relationship between Γ_0 and Γ_0' and the Cartwright–Steger lattices in Section 4.3.

4.1. Lattices in $\mathrm{PGL}_d(\mathbb{F}_q((t)))$. Recall the construction of the cocompact lattice $\widetilde{\Gamma} \leq \mathrm{PGL}_d(\mathbb{F}_q((t)))$ described in Section 3 above. This construction appears in [**7**]. As noted in Section 3.3(5) above, the lattice $\widetilde{\Gamma}$ is a product of a vertex stabiliser H of order $d\frac{q^d-1}{q-1}$ and a vertex-regular lattice Γ. Denote by $\widetilde{\Gamma}'$ the subgroup of $\widetilde{\Gamma}$ which is the product of Γ with the Singer cycle $S < H$ guaranteed by Lemma 9 above. Then by construction, S is a vertex stabiliser in $\widetilde{\Gamma}'$. (Since $\Gamma \leq \widetilde{\Gamma}' \leq \widetilde{\Gamma}$, the group $\widetilde{\Gamma}'$ is also a cocompact lattice in $\mathrm{PGL}_d(\mathbb{F}_q((t)))$.)

For $i = 0, \dots, d-1$ let v_i be the vertex of type i in the standard chamber, as in Section 2.2 above. Let N_i be the stabiliser of v_i in $\widetilde{\Gamma}$, and let S_i be the stabiliser of v_i in $\widetilde{\Gamma}'$. Then each $N_i \cong H$ and each $S_i \cong S$. We now define

$$\Gamma_0' := \langle S_0, \dots, S_{d-1} \rangle \leq \widetilde{\Gamma}'$$

to be the subgroup of $\widetilde{\Gamma}'$ generated by $S_0, \dots, S_{d-1}$, and

$$\Gamma_0 := \langle N_0, \dots, N_{d-1} \rangle \leq \widetilde{\Gamma}$$

to be the subgroup of $\widetilde{\Gamma}$ generated by $N_0, \dots, N_{d-1}$. Clearly $\Gamma_0' \leq \Gamma_0$.

We claim that Γ_0' and Γ_0 are cocompact lattices in $\mathrm{PGL}_d(\mathbb{F}_q((t)))$. Recall from the introduction that $\Gamma < G$ is a cocompact lattice in G if it is a discrete subgroup of G which acts cocompactly on Δ. Hence it suffices to show that Γ_0 is a discrete subgroup of $\mathrm{PGL}_d(\mathbb{F}_q((t)))$ and that Γ_0' acts cocompactly on Δ. The following lemma is immediate, since by construction Γ_0 is a subgroup of the discrete group $\widetilde{\Gamma} \leq \mathrm{PGL}_d(\mathbb{F}_q((t)))$.

LEMMA 10. *Γ_0 is a discrete subgroup of $\mathrm{PGL}_d(\mathbb{F}_q((t)))$.*

To show that Γ_0' acts cocompactly on Δ, we first consider the action of the groups S_i which generate Γ_0'.

LEMMA 11. *For $i = 0, \dots, d-1$ and $j = i-1, i+1 \pmod d$, the group S_i acts simply transitively on the vertices neighbouring v_i of type j.*

Proof. From the discussion of Singer cycles in Section 2.1 and types in Section 2.2, the group S_0 acts simply transitively on the vertices neighbouring v_0 of type j, for $j = -1, 1 \pmod d$. Now $\widetilde{\Gamma}'$ consists of type-rotating automorphisms, since the Cartwright–Steger lattice $\widetilde{\Gamma}$, which contains $\widetilde{\Gamma}'$, consists of type-rotating automorphisms. By construction and the definition of type-rotating, for $i = 1, \dots, d-1$ the group S_i is the image of S_0 under conjugation by an element of $\widetilde{\Gamma}'$ which adds i

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(mod d) to each type. Thus for $i = 1, \ldots, d-1$, the group S_i acts simply transitively on the vertices neighbouring v_i of type $j = i-1, i+1 \pmod{d}$. □

PROPOSITION 12. *For $i = 0, \ldots, d-1$, the group Γ_0' acts transitively on the vertices of type i in Δ.*

Proof. We will show that Γ_0' acts transitively on the vertices of type 0 in Δ. The same argument will apply for types $i = 1, \ldots, d-1$. It suffices to show that for each vertex w_0 of type 0, there is an element of Γ_0' which takes w_0 to v_0. We prove this by induction on the distance $\delta(w_0, v_0) \in 2\mathbb{N}$.

If $\delta(w_0, v_0) = 2$ we consider the following cases.

(1) w_0 is adjacent to v_1. By Lemma 11 above, S_1 acts transitively on the type 0 neighbours of v_1, and so the claim follows in this case.
(2) w_0 is adjacent to some vertex $s_0 v_1$ with $s_0 \in S_0$. Then $s_0^{-1} w_0$ is adjacent to v_1, and we apply the argument from Case (1).
(3) w_0 is adjacent to v_i where $i \in \{2, \ldots, d-1\}$. Then there is a vertex v_{i-1}' of type $(i-1)$ so that v_i, w_0 and v_{i-1}' are mutually adjacent. Since S_i acts transitively on the type $(i-1)$ neighbours of v_i, we have that $s_i v_{i-1}' = v_{i-1}$ for some $s_i \in S_i$. Thus $s_i w_0$ is adjacent to v_{i-1}. By repeating this argument, we obtain after finitely many steps that for some $\gamma \in \Gamma_0$ we have γw_0 adjacent to v_1, and we may then apply the argument from Case (1).
(4) w_0 is adjacent to a vertex $v_i' \neq v_i$ of type $i \in \{2, \ldots, d-1\}$, with $\delta(v_0, v_i') = \delta(v_i', w_0) = 1$. Choose a vertex v_1' of type 1 so that v_0, v_1' and v_i' are mutually adjacent. Then there is an $s_0 \in S_0$ such that $s_0 v_1' = v_1$, and hence $s_0 v_i'$ is a neighbour of v_1 of type i. Now choose a vertex v_2' of type 2 so that v_1, v_2' and $s_0 v_i'$ are mutually adjacent. Then there is an $s_1 \in S_1$ such that $s_1 v_2' = v_2$, and hence $s_1 s_0 v_i'$ is a neighbour of v_2 of type i. By repeating this argument, we obtain that $\gamma v_i'$ is a neighbour of v_{i-1} of type i, for some $\gamma \in \Gamma_0'$. Then there is an $s_{i-1} \in S_{i-1}$ such that $s_{i-1}\gamma v_i' = v_i$. Thus $s_{i-1}\gamma w_0$ is a neighbour of v_i, and so we may apply the argument from Case (3).

Now suppose that $\delta(w_0, v_0) = 2k$. Then there is a vertex w_0' of Δ of type 0 such that $\delta(w_0, w_0') = 2(k-1)$ and $\delta(w_0', v_0) = 2$. By the base case of the induction there is an element $\gamma \in \hat{\Gamma}_0$ such that $\gamma w_0' = v_0$. But then $\delta(\gamma w_0, v_0) = \delta(\gamma w_0, \gamma w_0') = \delta(w_0, w_0') = 2(k-1)$ so by inductive assumption there is a $\gamma' \in \hat{\Gamma}_0$ such that $\gamma'\gamma w_0 = v_0$, as required. □

COROLLARY 13. *Γ_0' acts cocompactly on Δ.*

Proof. By Proposition 12 above, Γ_0' has finitely many (at most d) orbits of vertices on Δ. Since Δ is locally finite, this implies that Γ_0' acts cocompactly. □

We have established the claim that Γ_0' and Γ_0 are cocompact lattices in $\mathrm{PGL}_d(\mathbb{F}_q((t)))$. To finish the proof of Theorem 1, we further describe the actions of Γ_0' and Γ_0 on Δ.

COROLLARY 14. *The action of Γ_0' and of Γ_0 is type-preserving and transitive on each type of vertex in Δ. For $i = 0, \ldots, d-1$, the stabiliser of v_i in Γ_0' is the group S_i, and in Γ_0 is the group N_i.*

Proof. Each N_i is a subgroup of the type-rotating group $\widetilde{\Gamma}$ and stabilises a vertex of type i, hence each N_i fixes all types. It follows that Γ_0 and thus Γ_0' is type-preserving. By Proposition 12, the action of Γ_0' and thus of Γ_0 is transitive on each type of vertex of Δ. For $i = 0, \dots, d-1$, the stabiliser of v_i in Γ_0' is S_i since by construction

$$S_i \leq \mathrm{Stab}_{\Gamma_0'}(v_i) \leq \mathrm{Stab}_{\widetilde{\Gamma}'}(v_i) = S_i.$$

Similarly, the stabiliser of v_i in Γ_0 is N_i. □

4.2. Lattices in $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. We will first prove that $\Lambda_0 := \Gamma_0 \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ is a cocompact lattice in $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. The proof that $\Lambda_0' := \Gamma_0' \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ is a cocompact lattice in $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ is similar.

Since Γ_0 is discrete, it is immediate that Λ_0 is a discrete subgroup of $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. Now Γ_0 acts cocompactly on Δ, so to show that Λ_0 act cocompactly on Δ it suffices to show that Λ_0 is of finite index in Γ_0.

Consider the determinant homomorphism $\det : \mathrm{GL}_d(\mathbb{F}_q(\!(t)\!)) \to \mathbb{F}_q(\!(t)\!)^\times$, with kernel $\mathrm{SL}_d(\mathbb{F}_q(\!(t)\!))$. This homomorphism induces a well-defined homomorphism

$$\overline{\det} : \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!)) \to \mathbb{F}_q(\!(t)\!)^\times / (\mathbb{F}_q(\!(t)\!)^\times)^d$$

where $(\mathbb{F}_q(\!(t)\!)^\times)^d$ is the subgroup of $\mathbb{F}_q(\!(t)\!)^\times$ consisting of dth powers of invertible elements of $\mathbb{F}_q(\!(t)\!)$. The kernel of $\overline{\det}$ is $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$.

The group Γ_0 is finitely generated by torsion elements, since each N_i is finite. Hence the restriction of $\overline{\det}$ to Γ_0 has finite image. But the kernel of this restriction is $\Gamma_0 \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!)) = \Lambda_0$. Thus Λ_0 has finite index in Γ_0, as required. We conclude that Λ_0 is a cocompact lattice in $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$.

We now describe these intersections Λ_0 and Λ_0'. We list the outcomes for various pairs of d and q in the next statement, which follows from Proposition 8 and Lemma 9 above. Recall that S_i is a Singer cycle of $\mathrm{PGL}_d(q)$, hence $S_i \cong C_{\frac{q^d-1}{q-1}}$, and that $N_i \cong C_{\frac{q^d-1}{q-1}} \rtimes C_d$.

LEMMA 15. *Let $q = p^a$, $a \in \mathbb{N}$, $d \geq 3$, and $i \in \{0, \dots, d-1\}$.*

(1) *Suppose that $(d, q-1) = 1$.*

(a) *If p does not divide d, then*

$$N_i \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!)) \cong C_{\frac{q^d-1}{q-1}} \rtimes C_d$$

is equal to N_i. Hence $\Lambda_0' = \Gamma_0'$ and $\Lambda_0 = \Gamma_0$.

(b) *If p divides d, then*

$$N_i \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!)) \cong C_{\frac{q^d-1}{q-1}} \rtimes C_{\frac{d}{\mathrm{Ord}_p(d)}}$$

is a proper subgroup of N_i. Moreover, $S_i \leq N_i \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. Hence $\Gamma_0' = \Lambda_0'$ and Λ_0 is a proper subgroup of Γ_0.

(2) *Suppose that $(d, q-1) \neq 1$.*

(a) *If p does not divide d, then*

$$N_i \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!)) \cong C_{\frac{q^d-1}{(q-1)(d,q-1)}} \rtimes C_d$$

is a proper subgroup of N_i. Moreover, S_i is not contained in $N_i \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. Hence Γ_0' is a proper subgroup of Λ_0' and Λ_0 is a proper subgroup of Γ_0.

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(b) *If p divides d, then*

$$N_i \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!)) \cong C_{\frac{q^d-1}{(q-1)(d,q-1)}} \rtimes C_{\frac{d}{\mathrm{Ord}_p(d)}}$$

is a proper subgroup of N_i. Moreover, S_i is not contained in $N_i \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. Hence Γ_0' is a proper subgroup of Λ_0' and Λ_0 is a proper subgroup of Γ_0.

In each case in which $\Lambda_0' = \Gamma_0'$ (respectively, $\Lambda_0 = \Gamma_0$), the same arguments as in Section 4.1 above show that Γ_0' (respectively, Γ_0) is a cocompact lattice in $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$, with action as described in Corollary 14 above. This completes the proof of Theorem 2.

On the other hand, if Λ_0' is a proper subgroup of Γ_0' (respectively, Λ_0 is a proper subgroup of Γ_0), then all that we can say about the action is that, since Λ_0' (respectively, Λ_0) is a type-preserving cocompact lattice, it has finitely many orbits of vertices of each type. In particular, we do not know whether Λ_0' (respectively, Λ_0) acts transitively on the set of vertices of Δ of each type. For instance, if $d = 3$ and $(d, q-1) \neq 1$, then by Lemma 9(4) above, $H \cap \mathrm{PSL}_3(\mathbb{F}_q(\!(t)\!))$ has order $(q^2 + q + 1)$. Moreover, as $H \cap \mathrm{PSL}_3(\mathbb{F}_q(\!(t)\!)) = H \cap \mathrm{PSL}_3(\mathbb{F}_q[[t]])$, H is a normaliser of a Singer cycle of $\mathrm{PSL}_3(q)$. Thus as discussed in Section 2.1, $H \cap \mathrm{PSL}_3(\mathbb{F}_q(\!(t)\!))$ cannot act transitively on the set of points and the set of lines of the projective plane over $\mathbb{F}_q$. Hence the arguments used to prove Proposition 12 above cannot be applied in this case.

4.3. Relationships between Γ_0 and Γ_0' and the Cartwright–Steger lattice. In this section we establish some relationships between the lattices Γ_0 and Γ_0' that we constructed in Section 4.1 above, and the Cartwright–Steger lattice $\widetilde{\Gamma}$.

We first prove Theorem 3 of the introduction. Recall from Section 3.3 above that $\widetilde{\Gamma} = H\Gamma$. In Section 4.1 above, we denoted by $\widetilde{\Gamma}'$ the product $S\Gamma$. Since $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ are type-rotating, they have finite index (normal) subgroups consisting of all type-preserving elements. Thus the following result establishes Theorem 3.

Proposition 16. *The lattice Γ_0 is the type-preserving subgroup of $\widetilde{\Gamma}$, and the lattice Γ_0' is the type-preserving subgroup of $\widetilde{\Gamma}'$.*

Proof. Denote by $\widetilde{\Gamma}_0$ the type-preserving subgroup of $\widetilde{\Gamma}$. Since $\Gamma_0 \leq \widetilde{\Gamma}$ and Γ_0 is type-preserving, we have that $\Gamma_0 \leq \widetilde{\Gamma}_0$. By Corollary 14 above, Γ_0 acts transitively on the vertices of each type in Δ. Hence $\widetilde{\Gamma}_0$ acts transitively on the vertices of each type in Δ. Now let v be any vertex of Δ and let $x \in \widetilde{\Gamma}_0$. Then there is a $y \in \Gamma_0$ so that $xy^{-1} \cdot v = v$. In particular, xy^{-1} is an element of $\mathrm{Stab}_{\widetilde{\Gamma}_0}(v)$. But $\mathrm{Stab}_{\Gamma_0}(v) \leq \mathrm{Stab}_{\widetilde{\Gamma}_0}(v) \leq \mathrm{Stab}_{\widetilde{\Gamma}(v)}$ and as $\mathrm{Stab}_{\Gamma_0}(v) = \mathrm{Stab}_{\widetilde{\Gamma}}(v) = N_i$ for some i, we have that $\mathrm{Stab}_{\Gamma_0}(v) = \mathrm{Stab}_{\widetilde{\Gamma}_0}(v)$. Thus $xy^{-1} \in \mathrm{Stab}_{\Gamma_0}(v)$ and so $x \in \mathrm{Stab}_{\Gamma_0}(v)y \subset \Gamma_0$. Therefore $\widetilde{\Gamma}_0 \leq \Gamma_0$. Thus $\Gamma_0 = \widetilde{\Gamma}_0$, the type-preserving subgroup of $\widetilde{\Gamma}$, as required.

The proof for Γ_0' is similar. □

In the case that $(d, q-1) = 1$, we can also specify the relationship between Γ_0 and Γ_0' and the Cartwright–Steger lattice $\widetilde{\Gamma}$ as follows.

Lemma 17. *Assume that $(d, q-1) = 1$. If p does not divide d, then $\Gamma_0 = \widetilde{\Gamma} \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$, while if p divides d, then $\Gamma_0' = \widetilde{\Gamma} \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$.*

Proof. If p does not divide d, then by Lemma 15(1)(a) above, the lattice Γ_0 is contained in $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. Since we constructed Γ_0 as a subgroup of $\widetilde{\Gamma}$, it follows that

$\Gamma_0 \leq \widetilde{\Gamma} \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. Now let $g \in \widetilde{\Gamma} \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$. Then as the action of $\mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ on Δ is type-preserving, the vertex gv_0 has type 0. Since Γ_0 acts transitively on vertices of type 0, there is a $g_0 \in \Gamma_0$ such that $g_0^{-1} g v_0 = v_0$. Thus as $\Gamma_0 \leq \widetilde{\Gamma}$, the element $h := g_0^{-1} g$ is in $\mathrm{Stab}_{\widetilde{\Gamma}}(v_0)$. But $\mathrm{Stab}_{\Gamma_0}(v_0) = \mathrm{Stab}_{\widetilde{\Gamma}}(v_0) = N_0$, and thus $g = g_0 h \in \Gamma_0$. Hence $\Gamma_0 = \widetilde{\Gamma} \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ as required.

The proof that if p divides d then $\Gamma_0' = \widetilde{\Gamma} \cap \mathrm{PSL}_d(\mathbb{F}_q(\!(t)\!))$ is similar. □

5. Minimality of covolumes. In Section 5.1 we discuss whether cocompact lattices in the matrix groups we have been considering can contain p–elements. We then in Section 5.2 discuss minimality of covolumes of cocompact lattices in $G = \mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$.

5.1. Cocompact lattices, do they contain p–elements? We begin by establishing an analogue for $G = \mathrm{SL}_d(\mathbb{F}_q(\!(t)\!))$ of Godement's Cocompactness Criterion (or the Kazhdan-Margulis Theorem) [**4**, **17**, **22**]. We will use the general result contained in Proposition 18 below. A similar statement can be found in, for example, [**13**, page 10]. The proof in [**13**] requires a compact fundamental domain, that cannot be assured in our case. Hence, for the sake of completeness, we exhibit a variation of their argument here. The existence of a discrete cocompact subgroup will make the group G locally compact, but we still formulate the result for a topological group because local compactness is not used in the proof.

PROPOSITION 18. *Let G be a topological group and Γ a discrete cocompact subgroup of G. If $u \in \Gamma$, then*

$$u^G := \{gug^{-1} \mid g \in G\}$$

is a closed subset of G.

Proof. Let $g_i u g_i^{-1}$, $g_i \in G$, be a net converging to $v \in G$. Since Γ is cocompact, the set $\{g_i\Gamma\}$ admits a convergent subnet, so without loss of generality, $g_i\Gamma \to g\Gamma$. Thus, there exist such $x_i \in \Gamma$ that $g_i x_i \to g$. Since $g_i u g_i^{-1} = (g_i x_i)(x_i^{-1} u x_i)(g_i x_i)^{-1}$, the net $x_i^{-1} u x_i$ converges to $g^{-1} v g$. Since all $x_i^{-1} u x_i$ are elements of the discrete subgroup Γ, the net must stabilise, hence, $x_j^{-1} u x_j = g^{-1} v g$ for some j, and so we arrive at $v \in u^G$. □

It is an interesting question whether cocompact lattices in groups defined over a field of characteristic p contain p–elements. In [**18**] Lubotzky uses Proposition 18 above to show that cocompact lattices in $\mathrm{SL}_2(\mathbb{F}_q(\!(t)\!))$, where $q = p^a$, contain no p–elements. In fact, this statement can be generalised in the following way.

PROPOSITION 19. *Let $G = \mathrm{SL}_d(\mathbb{F}_q(\!(t)\!))$ where $q = p^a$ with p prime and $d \geq 2$. Let Γ be a lattice in G. Then Γ is cocompact if and only if Γ does not contain any elements of order p.*

Proof. First suppose that Γ is non-cocompact and let A be a set of vertices of the building for G which represent the orbits of Γ. Then by the remarks in the introduction, A is infinite and the series $\mu(\Gamma\backslash G) = \sum_{a \in A} |\mathrm{Stab}_\Gamma(a)|^{-1}$ converges, hence Γ contains vertex stabilisers of arbitrarily large order. The Levi decomposition (Proposition 4 above) then implies that Γ must have elements of order p.

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For the converse, by Proposition 18 above, it is enough to show that if $u \in G$ is a p–element then there is $g \in G$ such that $g^k u g^{-k} \to I$ as $k \to \infty$, where I is the identity matrix in G.

So let $u \in G$ be such that $u^p = I \neq u$. Since we are working over a field of characteristic p, it follows that $(u - I)^p = 0$ and thus u is a unipotent element of $G = \mathrm{SL}_d(\mathbb{F}_q(\!(t)\!))$ (recall that by definition, unipotent elements are those with all eigenvalues equal to 1). Thus u is conjugate in G to a matrix with all 1s on the diagonal and all below-diagonal elements 0. Without loss of generality we may assume that u itself has all 1s on the diagonal and all below-diagonal elements 0. It is then not hard to construct a suitable diagonal matrix $g \in G$ such that $g^k u g^{-k}$ converges to I. For example, for $d = 3$, g can be taken to be the following matrix:

$$\begin{pmatrix} t^2 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-3} \end{pmatrix}.$$

□

The proof of Proposition 19 makes essential use of the fact that in $\mathrm{SL}_d(\mathbb{F}_q(\!(t)\!))$, an element of order p is a good unipotent element (cf. [**26**]). However, *one needs to be careful about cocompact lattices in other matrix groups*!

Let us look again at the Cartwright–Steger lattice $\widetilde{\Gamma}$ in $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$. As we saw, $\widetilde{\Gamma} = \Gamma H$ where H is a finite subgroup of $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$ of order $d\frac{(q^d-1)}{(q-1)}$. Suppose that p divides d (for example, if $p = 3 = d$). Then obviously H, and thus $\widetilde{\Gamma}$, contains an element $\widetilde{h} \in H$ of order p. On the other hand, $\widetilde{\Gamma}$ is a cocompact lattice in $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$. What is going on? The answer comes from the fact that under the natural map $\mathrm{GL}_d(\mathbb{F}_q(\!(t)\!)) \to \mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$, $\widetilde{h}$ is the image of an element $h \in \mathrm{GL}_d(\mathbb{F}_q(\!(t)\!))$ of infinite order. In particular, $\widetilde{h}$ is not a good unipotent element and the proof of Proposition 19 above does not work. In fact the conjugacy class of $\widetilde{h}$ in $\mathrm{PGL}_d(\mathbb{F}_q(\!(t)\!))$ is closed, so there is no contradiction with Proposition 18 above.

5.2. Minimality of covolumes. As discussed in the introduction, our original motivation was to find cocompact lattices of minimal covolume in $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$, and this led us to considering vertex stabilisers which are Singer cycles or normalisers of Singer cycles. We now consider covolumes of cocompact lattices in the special case that $G = \mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$ and $(3, q-1) = 1$. Notice that in particular, $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!)) = \mathrm{PSL}_3(\mathbb{F}_q(\!(t)\!))$.

By Theorem 2 and the remarks in the introduction, we have that Γ_0' is a cocompact lattice in G of covolume

$$\mu(\Gamma_0' \backslash G) = \sum_{i=0}^{2} \frac{1}{|\operatorname{Stab}_{\Gamma_0'}(v_i)|} = \sum_{i=0}^{2} \frac{1}{|S_i|} = \frac{3}{q^2+q+1}.$$

Also, if $p \neq 3$, then Γ_0 is a cocompact lattice in G of covolume

$$\mu(\Gamma_0 \backslash G) = \sum_{i=0}^{2} \frac{1}{|\operatorname{Stab}_{\Gamma_0}(v_i)|} = \sum_{i=0}^{2} \frac{1}{|N_i|} = \frac{3}{3(q^2+q+1)} = \frac{1}{q^2+q+1}.$$

Now let Γ be any cocompact lattice in $G = \mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$. Then by Proposition 19 above, each vertex stabiliser in Γ is a finite p'–subgroup of a vertex stabiliser in G.

The Levi decomposition (Proposition 4 above) then implies that each vertex stabiliser in Γ is isomorphic to a p'–subgroup of $\mathrm{SL}_3(q) = \mathrm{PSL}_3(q)$. We thus consider maximal p'–subgroups of $\mathrm{PSL}_3(q)$, in Lemma 20 below.

Note that since Γ is type-preserving, Γ has at least one orbit of vertices of each type $i = 0, 1, 2$. It follows that if $|\,\mathrm{Stab}_\Gamma(v_i)| \leq q^2$ for each i, then $\mu(\Gamma\backslash G) > \mu(\Gamma_0'\backslash G)$ and so Γ is not a cocompact lattice of minimal covolume. Similarly, if $p \neq 3$ and $|\,\mathrm{Stab}_\Gamma(v_i)| < 3(q^2+q+1)$ for each i, then $\mu(\Gamma\backslash G) > \mu(\Gamma_0\backslash G)$ and so again Γ is not a cocompact lattice of minimal covolume. Hence in the next statement we consider only maximal p'–subgroups H of $\mathrm{PSL}_3(q)$ with $|H| > q^2$ in the case $p = 3$, and $|H| \geq 3(q^2+q+1)$ otherwise.

Lemma 20. *Let $K = \mathrm{PSL}_3(q)$, where $q = p^a > 72$ with p prime and $a \in \mathbb{N}$. Assume that $(3, q-1) = 1$ and $q > 72$. Let H be a maximal p'–subgroup of K.*

(1) *If $p = 2$ and $|H| \geq 3(q^2+q+1)$, then H is the normaliser of a Singer cycle of K and $|H| = 3(q^2+q+1)$.*
(2) *If $p = 3$ and $|H| > q^2$ then one of the following holds:*
 (a) *H is a subgroup of the normaliser of a maximal split torus of K and $|H| = 2(q-1)^2$;*
 (b) *H is the normaliser of a Singer cycle of K and $|H| = (q^2+q+1)$; or*
 (c) *H is a subgroup of a Levi complement of a maximal parabolic subgroup of K and $|H| = 2(q^2-1)$.*
(3) *If $p \geq 5$ and $|H| \geq 3(q^2+q+1)$, then one of the following holds:*
 (a) *H is the normaliser of a maximal split torus of K and $|H| = 6(q-1)^2$; or*
 (b) *H is the normaliser of a Singer cycle of K and $|H| = 3(q^2+q+1)$.*

Proof. The result follows immediately from the theorem of Hartley and Mitchell (cf. Theorem 6.5.3 of [**14**]). □

From this, the following minimality result in characteristic 2 is immediate:

Proposition 21. *Suppose that $(3, q-1) = 1$ and that $p = 2$. Then for q large enough, the lattice Γ_0 is a cocompact lattice of minimal covolume in $G = \mathrm{SL}_3(\mathbb{F}_q((t)))$.*

Proof. Let Γ be any cocompact lattice in $\mathrm{SL}_3(\mathbb{F}_q((t)))$ and assume that $q > 72$. By Lemma 20 and the discussion preceding it, for $i = 0, 1, 2$, we have $|\,\mathrm{Stab}_\Gamma(v_i)| \leq |\,\mathrm{Stab}_{\Gamma_0}(v_i)| = 3(q^2+q+1)$ and so $\mu(\Gamma\backslash G) \geq \mu(\Gamma_0\backslash G)$ as required. □

It would be nice either to prove or to disprove Proposition 21 in an arbitrary characteristic p. At the moment of writing, we cannot do it, for reasons we now explain.

A lattice $\Gamma' \leq G = \mathrm{SL}_3(\mathbb{F}_q((t)))$ is said to be *maximal* if for every lattice $\Gamma \leq G$ such that $\Gamma' \leq \Gamma$, in fact $\Gamma' = \Gamma$. It is clear that a cocompact lattice of minimal covolume must be a maximal lattice. In fact, the following is true.

Proposition 22. *Suppose that $(3, q-1) = 1$. Then for q large enough, if $p = 3$, the lattice Γ_0' is a maximal lattice in $G = \mathrm{SL}_3(\mathbb{F}_q((t)))$ and if $p \geq 5$, the lattice Γ_0 is a maximal lattice in $G = \mathrm{SL}_3(\mathbb{F}_q((t)))$.*

Proof. We give the proof for $p \geq 5$. The proof for $p = 3$ is similar. Suppose that Γ is a lattice in G such that $\Gamma_0 \leq \Gamma$. Then Γ is cocompact, since Γ_0 is cocompact. Since Γ is type-preserving and Γ_0 is transitive on each type of vertex, Γ is transitive on each type of vertex. By Lemma 20, the vertex stabilisers in Γ_0 are maximal p'–subgroups of $\mathrm{PSL}_3(q)$. It follows that for $i = 0, 1, 2$ we have $\mathrm{Stab}_\Gamma(v_i) = \mathrm{Stab}_{\Gamma_0}(v_i)$ and hence $\mu(\Gamma\backslash G) = \mu(\Gamma_0\backslash G)$. Thus $\Gamma = \Gamma_0$ as required. □

For $p \geq 5$, we have found a candidate besides Γ_0 for the cocompact lattice of minimal covolume. Let H_1 be the normaliser of a maximal split torus of $\mathrm{PSL}_3(q)$. Using complexes of groups (see [**5**]), for p odd and $(3, q-1) = 1$ we are able to construct a group Γ_1 which acts transitively on the set of vertices of each type in *some* building of type $\tilde{A}_2$ (possibly exotic), so that each vertex stabiliser in Γ_1 is isomorphic to H_1. However, for $p \geq 5$ we do not know whether Γ_1 embeds in $G = \mathrm{SL}_3(\mathbb{F}_q((t)))$ as a cocompact lattice acting transitively on the set of vertices of each type in the building for G, with $\mathrm{Stab}_{\Gamma_1}(v_i) \cong H_1$ for $i = 0, 1, 2$. (For $p = 3$, the whole group H_1 cannot be a vertex stabiliser, since it contains an element of order 3.) If there is such an embedding of Γ_1, then by the same arguments as for Proposition 22, Γ_1 is a maximal lattice in G, and it will have a smaller covolume than Γ_0:

$$\mu(\Gamma_1\backslash G) = \sum_{i=0}^{2} \frac{1}{|\,\mathrm{Stab}_{\Gamma_1}(v_i)|} = \sum_{i=0}^{2} \frac{1}{|6(q-1)^2|}$$
$$= \frac{3}{6(q-1)^2} = \frac{1}{2(q-1)^2} < \frac{1}{q^2+q+1}.$$

Hence, we would like to finish this section with the following question and conjecture.

Question. Does $G = \mathrm{SL}_3(\mathbb{F}_q((t)))$ admit a lattice Γ_1 as described above?

Conjecture. Let $(p, 3) = 1 = (3, q-1)$ and $G = \mathrm{SL}_3(\mathbb{F}_q((t)))$. Then either Γ_0 is a cocompact lattice of minimal covolume, or G admits a cocompact lattice Γ_1 as described above, and Γ_1 is a cocompact lattice of minimal covolume.

6. Relationship with the work of Essert. Recall from the introduction that Essert [**11**] constructed cocompact lattices which act simply transitively on the set of panels of the same type in some $\tilde{A}_2$–building, possibly exotic. Such lattices are said to be *panel-regular*. We now conclude by addressing some open questions from [**11**].

Let Δ be the building $\tilde{A}_2(K, \nu)$, for some field K with discrete valuation ν, and let $G = \mathcal{G}(K)$ where $\mathcal{G}$ is in the set $\{\mathrm{PGL}_3, \mathrm{SL}_3, \mathrm{PSL}_3\}$. With the exception of one lattice which is realised explicitly in the group $\mathrm{SL}_3(\mathbb{F}_2((t)))$ (see the Remark in [**11**, Section 5.3]), it is an open question in [**11**] whether the panel-regular lattices constructed there are lattices in the full automorphism group $\mathrm{Aut}(\Delta)$ of any building Δ of the form $\tilde{A}_2(K, \nu)$, and whether they can be embedded in any $G = \mathcal{G}(K)$ (see the Introduction to [**11**]). We consider these questions in the case that $K = \mathbb{F}_q((t))$. Note that since $G/Z(G)$ is cocompact in $\mathrm{Aut}(\Delta)$, if Γ is a panel-regular lattice in G, then Γ will be a panel-regular lattice in $\mathrm{Aut}(\Delta)$.

Proposition 23. *For all q, the group $\mathrm{PGL}_3(\mathbb{F}_q((t)))$ admits a panel-regular lattice, hence the full automorphism group of the building $\Delta = \tilde{A}_2(\mathbb{F}_q((t)), \nu)$ admits a panel-regular lattice.*

Proof. Consider the lattice $\Gamma_0' \leq \mathrm{PGL}_3(\mathbb{F}_q((t)))$ constructed in Section 4.1 above, in the case $d = 3$. Since the vertex stabilisers of Γ_0' are Singer cycles of $\mathrm{PGL}_3(q)$, and Γ_0' acts transitively on the set of vertices of each type in Δ, it follows that the lattice Γ_0' acts simply transitively on the set of panels of each type in Δ. Hence Γ_0' is a panel-regular lattice in $\mathrm{PGL}_3(\mathbb{F}_q(\!(t)\!))$ and thus in $\mathrm{Aut}(\Delta)$. □

COROLLARY 24. *Assume* $(3, q-1) = 1$. *Then* $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$ *admits a panel-regular lattice.*

Proof. We showed in Section 4.2 above that in this case, the lattice Γ_0' is also contained in $\mathrm{PSL}_3(\mathbb{F}_q(\!(t)\!)) = \mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$. By the proof of Proposition 23 above, the lattice Γ_0' is panel-regular. Hence for all q such that $(3, q-1) = 1$, there is a panel-regular lattice in $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$. □

On the other hand:

PROPOSITION 25. *If* $3|(q-1)$ *and q is large enough,* $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$ *does not admit a panel-regular lattice.*

Proof. Suppose that $3 \mid (q-1)$. From the Levi decomposition (Proposition 4 above) and Proposition 19 above, if Γ is a cocompact lattice in $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$, then the vertex stabilisers in Γ are isomorphic to p'–subgroups of $\mathrm{SL}_3(q)$. However, when q is large enough and $3 \mid (q-1)$, there is no p'–subgroup of $\mathrm{SL}_3(q)$ which acts transitively on the points of the projective plane (see Section 2.1). Hence no vertex stabiliser in Γ can act transitively on the set of adjacent panels of the same type. Thus if q is large enough and $3 \mid (q-1)$, there is no lattice $\Gamma < \mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$ which acts (simply) transitively on the set of panels of the same type. Thus in this case, $\mathrm{SL}_3(\mathbb{F}_q(\!(t)\!))$ does not admit a panel-regular lattice. □

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