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DEFINING RELATIONS IN ORTHOGONAL GROUPS OF CHARACTERISTIC TWO

GEORG GÜNTHER AND WOLFGANG NOLTE

Introduction. It is well-known that a group is uniquely determined by a system of generators, and a set of defining relations on those generators. Clearly it is of interest to find relations that are as simple as possible. In this paper, this question is dealt with for certain orthogonal groups of characteristic 2, which are generated by involutions.

Let V be a vector space over a field K of characteristic 2 (we always exclude the prime field $K = GF(2)$). Let Q be a quadratic form over V, and let S be the set of orthogonal transformations of (V, Q) whose path is 1-dimensional and not contained in the radical of V. Letting O^* be the group generated by S, we shall show that every relation among generators in S is a consequence of relations of length 2, 3, or 4. Similar results for char $K \neq 2$ were proved in the regular case by Becken [**4**] and by Ahrens, Dress and Wolff [**1**] in the general case. Furthermore, K. Meyer [**17**] solved this problem for orthogonal groups of characteristic 2 for dim $V < \infty$, $|K| \geqq 4$-dim V, again in the regular case. In this paper, we solve the problem for the most general case for char $K = 2$ (Theorem 6.1).

In the last part of this paper, we treat a similar question for orthogonal groups which are generated by a subset T of S: we let $\bar{V}$ be a subspace of V, and let T be the set of those isometries in S whose paths belong to $\bar{V}$. Then we can prove a similar theorem (Theorem 8.3) for the group G which is generated by T. A similar investigation in the case that char $K \neq 2$ was made by Nolte [**18**].

We remark in conclusion that similar results are also known for unitary and symplectic groups. We refer here to papers by Becken [**4**], Ellers [**11**], Götzky [**12**], and Spengler [**23**].

In § 1, we gather some known results about isometries in metric vector spaces. Using these results, we prove a number of preliminary lemmas about the behaviour of certain products of simple isometries. In § 3, we find four types of fundamental relations between simple isometries, and with their aid set up an equivalence relation on the set of relations which we examine in detail in § 4. All of § 5 is devoted to proving the main lemma of this paper (Lemma 5.9); we then use this lemma in § 6 to prove Theorem 6.1, which states that every relation in the group O^* between simple isometries is in fact a consequence of the elementary relations introduced in § 3.

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1. Isometries in metric vector spaces. Let (V, Q) be a metric vector space over a field K, where the metric in V is given by the quadratic form Q and its associated bilinear form f. Throughout this paper, we exclude the case that $K = GF(2)$. The isometries of V are those transformations in $GL(V)$ that preserve Q, and hence also preserve f. With each isometry π, we have two associated subspaces

$$B(\pi) : = \{\pi(x) - x \mid x \in V\}, \text{ called the } path \text{ of } \pi, \quad \text{and}$$
$$F(\pi) : = \{x \in V \mid \pi(x) = x\}, \text{ called the } fix \text{ of } \pi.$$

An isometry σ is *simple* if $\dim B(\sigma) = 1$; in particular, every non-singular vector p not in rad V has associated with it the unique involutory simple isometry σ_p for which $B(\sigma_p) = \langle p \rangle$ and $F(\sigma_p) = p^\perp$. Wherever we write the symbol σ_p, we shall be referring to such a simple isometry. We let $S : = \{\sigma | \dim B(\sigma) = 1, \text{ and } B(\sigma) \subsetneq \text{rad } V, \text{ and } B(\sigma) \text{ non-singular}\}$. Let $O^* = O^*(V, Q)$ be the group of isometries of V generated by S.

We admit the possibility that V might be infinite-dimensional. In this context, we state the following.

LEMMA 1.1. *Let (V, Q) be a metric vector space.*

(a) *If U is a finite-dimensional subspace of V, then there exists a finite-dimensional subspace T containing U such that $V = T + T^\perp$.*

(b) *Suppose $V = T + T^\perp$, and we have simple isometries σ_{a_i}, $i = 1, \ldots, k$ with $a_i \in T$. Then $\sigma_{a_1} \ldots \sigma_{a_k} = 1$ if and only if*

$$(\sigma_{a_1} \ldots \sigma_{a_k})|_T = 1|_T.$$

The proof of (a) goes as given in [18] for the case of general characteristic; the proof of (b) is immediate. This lemma allows us to reduce the general case to the case of finite dimension. For this reason, unless explicitly stated to the contrary, we restrict ourselves to the case that V is finite-dimensional.

For easy reference, we gather some well-known results in the next lemma. For proofs, we refer to [**1**], [**2**], [**6**], [**10**], [**11**] and [**14**].

LEMMA 1.2. *Let $\sigma_a, \sigma_b \in S$, and let $\alpha, \beta \in O^*$.*

(a) $\sigma_a = \sigma_b$ *if and only if* $\langle a \rangle = \langle b \rangle$.

(b) $\sigma_b \sigma_a \sigma_b = \sigma_{\sigma_b(a)}$.

(c) *If $\langle a \rangle \neq \langle b \rangle$, then $a \perp b$ if and only if $\sigma_a \sigma_b = \sigma_b \sigma_a$.*

(d) *$B(\alpha\beta) \subseteq B(\alpha) + B(\beta)$. In particular, if $\alpha = \sigma_{a_1} \ldots \sigma_{a_k}$, then*

$$B(\alpha) \subseteq \langle a_1, \ldots, a_k \rangle.$$

(e) $F(\alpha) \subseteq B(\alpha)^\perp$.

(f) *If A is a subspace of V, then*

$$\dim A^\perp = \dim V - \dim A + \dim (A \cap \text{rad } V).$$

(g) *If* char $K = 2$, *then* $\dim (A/\text{rad } A)$ *is even for all subspaces A.*

In the special case that V is regular, we can refine (d) and (e) of Lemma 1.2 to obtain.

LEMMA 1.3. *Let* (V, Q) *be a regular metric vector space, and let* $\alpha \in O^*$. *Then*
(a) $F(\alpha) = B(\alpha)^{\perp}$ *and* $B(\alpha) = F(\alpha)^{\perp}$.
(b) *Given* $\sigma \in S$, *we have* $B(\alpha\sigma) = B(\alpha) + B(\sigma)$ *if and only if* $B(\sigma) \subsetneq B(\alpha)$, *and* $\dim B(\alpha\sigma) = \dim B(\alpha) - 1$ *if and only if* $B(\sigma) \subseteq B(\alpha)$.

The next lemma will turn out to be of considerable importance in our development. Define

$$Q \text{ rad } V := \{x \in \text{rad } V \mid Q(x) = 0\}.$$

LEMMA 1.4. *Let* $\pi = \sigma_{a_1} \ldots \sigma_{a_k}$. *Let* $A = \langle a_1, \ldots, a_k \rangle$ *and suppose* $\dim A = k$. *Then*
(a) $F(\pi) = A^{\perp}$.
(b) $\dim B(\pi) = \dim A - \dim (A \cap \text{rad } V)$.
(c) *If* $\dim (A \cap Q \text{ rad } V) = k - 1$, *then* $B(\pi) \subseteq Q \text{ rad } V$ *if and only if* k *is even.*

Proof. (a) The proof given in 1.4 of [**13**] goes through if we replace η_i by $(Q(q_i))^{-1}$.
(b) This follows easily from (a) and 1.2 f.
(c) If $\dim (A \cap Q \text{ rad } V) = k - 1 = \dim A - 1$, then by (b), $\dim B(\pi) = 1$, and so π is simple. When $X = Y \oplus Z$ and $Y \perp Z$ we shall write $X = Y \textcircled{\perp} Z$. Then $A = \langle a \rangle \textcircled{\perp} (A \cap Q \text{ rad } V)$, so that $\langle a_i \rangle = \langle a + q_i \rangle$ for suitable $q_i \in A \cap Q \text{ rad } V$. A simple computation shows that

$$\pi(x) = x - \frac{f(x, a)}{Q(a)} (\alpha a + q_1 - q_2 + q_3 \ldots \pm q_k),$$

where $\alpha = 1$ or $\alpha = 0$ as k is odd or even. This proves (c).

We finally state a lemma which gives some information about the occurence of singular vectors in a subspace.

LEMMA 1.5. *Let* A *be a non-singular subspace of* V. *Suppose* A *contains a singular subspace* B *with* $\dim B = \dim A - 1$. *Then one of the following holds:*
(a) $B \subseteq \text{rad } A$ *and every singular vector of* A *lies in* B.
(b) $B \subsetneq \text{rad } A$. *Then* A *contains a second singular subspace* C *with* $\dim C = \dim A - 1$, *and* $\text{rad } A = B \cap C$. *All singular vectors in* A *lie in* $B \cup C$, *and* $\sigma_p(B) = C$ *for any* $p \in A$.

Proof. We refer to [**14**], where this lemma is proved in the case that V is regular, and remark that the proof is identical in case $\text{rad } V \neq 0$.

This lemma is of interest if char $K = 2$, for then (a) implies that A is isotropic (by (g) of Lemma 1.2), and (b) implies that A is not isotropic.

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2. Preliminary lemmas. For the remainder of this paper, we shall assume that K is some field of characteristic 2, excluding always the prime field $GF(2)$.

(2.1) (a) *Suppose A is a 2-dimensional, non-singular subspace, and suppose A contains a singular, non-zero vector. If A is isotropic, then it contains exactly one singular,* 1*-dimensional subspace. If A is non-isotropic, it contains two distinct singular* 1*-dimensional subspaces.*

(b) *Suppose A is a* 3*-dimensional non-singular subspace containing a* 2*-dimensional singular subspace B. If A is isotropic, then all singular vectors in A are contained in B; if A is non-isotropic, it contains a second* 2*-dimensional singular subspace C, such that all singular vectors in A are contained in $B \cup C$. In this case, $B \cap C = \text{rad } A$.*

It is a simple thing to compute products $\sigma_a\sigma_b\sigma_c$. If $c = \alpha a + \beta b$ for some $\alpha, \beta \in K$, than the vector

$$d := (\beta Q(b) + \alpha f(a, b))a + \alpha Q(b)b$$

is easily seen to be non-singular ($Q(d) = Q(a)Q(b)Q(c)$). In fact, we have

$$(*) \quad \sigma_a\sigma_b\sigma_c(x) = x + \frac{f(x, d)}{Q(d)} d \quad \text{for all } x \in V.$$

If $d \in \text{rad } V$, then $f(x, d) = 0$ and $\sigma_a\sigma_b\sigma_c = 1$; if d $\notin \text{rad } V$, we obtain the familiar formula $\sigma_a\sigma_b\sigma_c = \sigma_d$.

We state next the following.

LEMMA 2.2. *Let π be an isometry. Then $B(\pi) \cap \text{rad } V \subseteq Q \text{ rad } V$. In particular, if $B(\pi) = \langle p \rangle$, then either $p \notin \text{rad } V$ and $Q(p) \neq 0$ (implying that π is the simple isometry σ_p) or $p \in Q \text{ rad } V$.*

For the proof of this, we refer to Lemma 3 in [**10**].

In the rest of this section, we make some statements about products of two, three, or four simple isometries. First we look at products $\sigma_a\sigma_b$, where $\langle a, b \rangle \cap \text{rad } V = \langle r \rangle$. By 1.4 b and c, and by 2.2, we deduce that $B(\sigma_a\sigma_b) = \langle r \rangle$ if and only if $r \in Q \text{ rad } V$.

If r is non-singular, we obtain

LEMMA 2.3. *Let $\sigma_a\sigma_b$ be an isometry such that $\langle a, b \rangle \cap \text{rad } V = \langle r \rangle$ is non-singular. Then $\sigma_a\sigma_b = \sigma_c$ for some $c \in \langle a, b \rangle$. Conversely, if $\sigma_a\sigma_b = \sigma_c$, then the vectors a, b, c are linearly dependent, but pairwise independent, and $\langle a, b, c \rangle \cap \text{rad } V$ is non-singular. In particular, if $\langle a, b \rangle \cap \text{rad } V = \langle r \rangle$ with $Q(r) \neq 0$, then we can express b uniquely as $b = \alpha a + \beta r$, and $\sigma_a\sigma_b = \sigma_c$, where $c = Q(b)a + \alpha Q(a)b$.*

Proof. For the first part we use 1.4 and 2.2 to deduce that $B(\sigma_a\sigma_b) \neq \langle r \rangle$, and hence $\sigma_a\sigma_b = \sigma_c$ for some $c \in \langle a, b \rangle$. Conversely, if $\sigma_a\sigma_b = \sigma_c$, then 1.2 d and 1.4 b imply that $\dim \langle a, b \rangle \cap \text{rad } V = 1$, and the result follows from 1.4 c and 2.2. The formula for c follows from simple computation.

If r is singular, we obtain

LEMMA 2.4. *Let $\sigma_a\sigma_b$ be an isometry such that $\langle a, b\rangle \cap Q$ rad $V = \langle r\rangle$. Then $B(\sigma_a\sigma_b) = \langle r\rangle$, and if $b = \alpha a + \beta r$, then*

$$\sigma_a\sigma_b(x) = x + \frac{\beta}{\alpha}\cdot\frac{f(x, a)}{Q(a)}\, r.$$

Proof. This is an immediate consequence of Lemmas 1.5 and 2.2. The last part of the lemma follows from a simple computation.

We calculate easily that the following is true.

(2.5) *If a, b are vectors such that $f(a, b) = 0$, if $r \in Q$ rad V, and $s = a + b$ is singular, $s \notin Q$ rad V, then*

$$\sigma_a\sigma_{a+r}\sigma_b\sigma_{b+r}(x) = x + \frac{f(x, s)}{Q(a)}\, r.$$

We now introduce some projective notation. We say that four subspaces $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$ and $\langle d\rangle$ contained in a 3-dimensional subspace form a *quadrangle* if any three of the vectors a, b, c, d are independent. The 2-dimensional subspace L containing $\langle u\rangle = \langle a, b\rangle \cap \langle c, d\rangle$, $\langle v\rangle = \langle a, c\rangle \cap \langle b, d\rangle$ and $\langle w\rangle = \langle a, d\rangle \cap \langle b, c\rangle$ shall be called the *diagonal* of the quadrangle. If L is singular, we say that the quadrangle is an *sd-quadrangle*. In this case, the vectors u, v, w are singular, and we can normalize to obtain $u = a + b = c + d$, and $v = a + c = b + d$.

We now state

LEMMA 2.6. *Suppose $\pi = \sigma_a\sigma_b\sigma_c\sigma_d$, and suppose that $A = \langle a, b, c, d\rangle$ is 3-dimensional. Suppose $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ form an sd-quadrangle in A whose singular diagonal is L. Suppose further, as above, that we have normalized so that $a + b = c + d = u \in L$ and $a + c = b + d = v \in L$. Then we have the following:*

(a) *If A is isotropic, then $Q(a) = Q(b) = Q(c) = Q(d)$, and*

$$\pi(x) = x + Q(a)^{-1}\,(f(x, u)v + f(x, v)u) \text{ for all } x \in V.$$

(b) *If A is not isotropic, and if* rad $A = \langle u\rangle$, *then*

$$\pi(x) = x + \lambda^{-1}(f(x, u)s + f(x, s)u) \text{ for all } x \in V,$$

where $\lambda = Q(a)Q(c)$ and $s = Q(c)a + Q(a)c$. In this case s is singular and $B(\pi)$ is singular. If dim $B(\pi) = 2$, *then $B(\pi) \neq L$.*

Proof. The formulas given for $\pi(x)$ are easily verified. All that remains to be proved is the last part of (b). But in this case, $B(\pi) = \langle u, s\rangle$, where $s = Q(c)a + Q(a)c$. Since $\langle u\rangle =$ rad A and $u \in \langle a, b\rangle$, we know that $a^\perp \cap A = \langle a, b\rangle$, and hence $f(a, c) \neq 0$. But $v = a + c$, and $Q(v) = 0$, and so $Q(a) +$

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$Q(c) = f(a, c) \neq 0$ implying $Q(a) \neq Q(c)$. Now

$$f(s, v) = f(Q(a)c + Q(c)a, a + c) = Q(a)f(a, c) + Q(c)f(a, c)$$
$$= (Q(a) + Q(c))f(a, c) = f(a, c)^2 \neq 0$$

and thus $s \notin L$.

LEMMA 2.7. *Let* $\pi = \sigma_a\sigma_b\sigma_c$, *and let* $A = \langle a, b, c\rangle$ *be 3-dimensional with the property that* $A \cap \text{rad } V = \langle d\rangle$ *is non-singular. Suppose that* $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ *form an sd-quadrangle with diagonal* L. *Then we have:*

(a) A *is isotropic.*

(b) $B(\pi) = L$.

(c) *There exist* σ_f, σ_g, σ_h *with* $f, g, h \in A$ *so that* $\pi = \sigma_a\sigma_f\sigma_g\sigma_h$, *and* $\langle a\rangle$, $\langle f\rangle$, $\langle g\rangle$, $\langle h\rangle$ *is an sd-quadrangle with diagonal* L.

Proof. Since d is non-singular, we know $d \notin L$, and so $A = \langle d\rangle + L$, implying (as $d \in \text{rad } V$) that A is isotropic. We can again normalize to obtain $u = a + b = c + d \in L$, so that $a + b + c + d = 0$, and $b + c \in L$. Then clearly $Q(a) = Q(b) = Q(c) = Q(d) := \lambda \neq 0$. Then we have

$$\pi(x) = x + \lambda^{-1}(f(x, a)a + f(x, b)b + f(x, c)c),$$

and so

$$f(\pi(x), x) = \lambda^{-1}(f(x, a)^2 + f(x, b)^2 + f(x, c)^2) = \lambda^{-1}(f(x, a + b + c))^2$$
$$= \lambda^{-1}f(x, d)^2 = 0$$

for all $x \in V$. But $Q(\pi(x) - x) = f(\pi(x), x) = 0$. Hence $B(\pi)$ is singular, and by Lemma 1.4 (b), $\dim B(\pi) = 2$. Thus, by Lemma 1.5, we know that $B(\pi) = L$. If we let $g = \alpha c + d$ and $h = c + \alpha d$ for some $\alpha \neq 0, 1$, then by Lemma 2.3 $\sigma_g\sigma_h = \sigma_c$, and so we have $\pi = \sigma_a\sigma_b\sigma_g\sigma_h$. Also,

$$\langle a, b\rangle \cap \langle g, h\rangle = \langle a, b\rangle \cap \langle \alpha c + d, c + \alpha d\rangle = \langle a, b\rangle \cap \langle c, d\rangle$$
$$= \langle u\rangle \subseteq L \quad \text{and}$$

$$\langle a, g\rangle \cap \langle b, h\rangle = \langle a, \alpha c + d\rangle \cap \langle a + u, c + \alpha d\rangle$$
$$= \langle (1 + \alpha)(a + c) + u\rangle \subseteq L.$$

The next lemma is a converse to 2.5.

LEMMA 2.8. *Suppose* $\pi(x) = x + \lambda^{-1}f(x, s)u$, *with* $u \in Q$ rad V, *and* $s \notin$ rad V *but* $Q(s) = 0$. *For every isometry* σ_a *we can then find* σ_b, σ_c, σ_d *with* $b,c,d \in \langle a, s, u\rangle$, *such that* $\pi = \sigma_a\sigma_b\sigma_c\sigma_d$.

Proof. Choose any $\sigma_a \in S$. Suppose first that $f(a, s) \neq 0$. We may assume $Q(a) \neq 1$, as otherwise we replace a by αa for some $\alpha \neq 1$. Let $f(a, s) = \mu$, and define $s' = \mu^{-1}s$, and $u' = \mu\lambda^{-1}u$. Then of course, $\pi(x) = x + f(x, s')u'$. Now we put $v = Q(a)^{-1}s' + Q(a)^{-2}a$, and define $c := a + v$. An immediate check shows that $f(a, s') = 1$, $Q(v) = 0$, $\langle v\rangle \neq \langle s'\rangle$ and $s' = Q(c)a + Q(a)c$. Also, $f(a, v) = Q(a)^{-1} = f(a, c)$, and $Q(c) = Q(a) + Q(a)^{-1} \neq 0$, since

$Q(a) \neq 1$. Now we put $b := a + u'$ and $d = c + u'$, and then by (2.5) we see that $\sigma_a\sigma_b\sigma_c\sigma_d = \pi$. If $f(a, s) = 0$, then $\langle a, s, u \rangle$ is isotropic. We now put $s' = Q(a)\lambda^{-1}s$, $b := a + u$, $c := a + s'$ and $d = a + u + s'$, and we again obtain $\pi = \sigma_a\sigma_b\sigma_c\sigma_d$ if c, d both do not lie in rad V. It could conceivably happen that both c and d lie in rad V, so that σ_c, σ_d do not exist. If both lie in rad V, then

$$\sigma_a\sigma_b(x) = x + f(x, a)Q(a)^{-1}a.$$

But since by assumption, $a + s' \in$ rad V, this implies that $f(x, a) = f(x, s')$, so that

$$\sigma_a\sigma_b(x) = x + Q(a)^{-1}f(x, s')u = x + \lambda^{-1}f(x, s)u = \pi(x).$$

Thus we choose any σ_g with $g \in \langle a, s, u \rangle$, and see that $\sigma_a\sigma_b\sigma_g\sigma_g = \pi$, as claimed.

In 2.6, we saw that certain products of four simple isometries yield a singular isometry π with 2-dimensional path L for which $L \cap$ rad $V = 0$. The next lemma shows that we have considerable freedom in expressing such an isometry as a product of simple isometries.

LEMMA 2.9. *Let $\pi(x) = x + f(x, u)v + f(x, v)u$, where $\langle u, v \rangle = L$ is singular and 2-dimensional, and $L \cap$ rad $V = 0$. Then, for all $\sigma_a \in S$, we can find σ_b, σ_c, $\sigma_d \in S$ with $b, c, d \in L + \langle a \rangle$ such that $\pi = \sigma_a\sigma_b\sigma_c\sigma_d$.*

Proof. Let $A = \langle a \rangle + L$.

(i) Assume first that A is non-isotropic. Then $A \cap$ rad $V = 0$. We may clearly assume that $f(a, v) \neq 0$, and indeed, $f(a, v) \neq 1$. Let rad $A = \langle u' \rangle$, then $u' \in L$, and we can write $u' = u + v$, so that

$$\pi(x) = x + f(x, u')v + f(x, v)u'.$$

Now let $c' := Q(a)^{-1}a + v$. Since $f(a, v) \neq 1$ we see that $Q(c') \neq 0$. We now define $c = Q(c')^{-1}c'$, and see that $Q(c) = Q(c')^{-1}$, and hence $c' = Q(c)^{-1}c$. Thus we may write $v = Q(a)^{-1}a + Q(c)^{-1}c$. We now let $r = a + c$, and see that

$$Q(r) = Q(a + c) = Q(v) = 0.$$

We define $b := a + u'$, and $d := c + u'$, and use Lemma 2.6 (b) to deduce that $\sigma_a\sigma_b\sigma_c\sigma_d = \pi$.

(ii) If A is isotropic, then we define $b := a + u$, $c := a + v$ and $d := a + u + v$. If none of b, c, d lies in rad V, then we calculate easily that $\sigma_a\sigma_b\sigma_c\sigma_d = \pi$. Also, at most one of b, c, d can lie in rad V, as otherwise $L \cap$ rad $V \neq 0$. But if one of b, c, d lies in rad V, it is an easy computation to show that the product of σ_a with the remaining two simple isometries is equal to π, and then we may use Lemma 2.7 to express this product of 3 simple isometries as a product $\sigma_a\sigma_f\sigma_g\sigma_h$.

Lemmas 2.6, 2.8, and 2.9 can be combined to yield

LEMMA 2.10. *Let* $\pi = \sigma_a\sigma_b\sigma_c\sigma_d$, *and let* $A = \langle a, b, c, d\rangle$ *be 3-dimensional. If* $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ *form an sd-quadrangle in* A, *then for any* $\sigma_{a'} \in S$, *we can find* $\sigma_{b'}, \sigma_{c'}, \sigma_{d'} \in S$ *so that* $\pi = \sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'}$.

3. Relations between simple isometries. Our aim in this paper is to find a presentation for the group O^* generated by the set S. For $n = 1, 2, \ldots$, we let $S^n = \{(\sigma_1, \ldots, \sigma_n) \mid \sigma_i \in S\}$, and $S^0 = \{\emptyset\}$. An element $(\sigma_1, \ldots, \sigma_n) \in S^n$ is a *relation of length n* if $\sigma_1 \ldots \sigma_n = 1$. If in addition, $\dim (B(\sigma_1) + \ldots + B(\sigma_n)) = k$, then we talk of a *k-dimensional* relation of length n. The trivial relation $\emptyset$ is 0-dimensional and of length 0.

In this section, we discuss four types of fundamental relations. We shall use these to define the equivalence relation $\sim$ on the words of the free group $W := \bigcup_{n\in\mathbf{N}\cup\{0\}} S^n$ generated by S: two words $(\sigma_{a_1}, \ldots, \sigma_{a_n})$ and $(\sigma_{b_1}, \ldots, \sigma_{b_m})$ are equivalent if one can be obtained from the other by successive insertion or deletion of these fundamental relations.

We say that $B(\sigma_1) + \ldots + B(\sigma_k)$ is the subspace associated with the word $(\sigma_1, \ldots, \sigma_k) \in W$. Given two equivalent words $a = (\sigma_{a_1}, \ldots, \sigma_{a_m})$ and $b = (\sigma_{b_1}, \ldots, \sigma_{b_n})$ with associated subspaces A and B respectively, we say that b is *elementary-equivalent to* a and write $a \gtrsim b$, if B is contained in A and if further all the fundamental relations whose insertion or deletion yields b from a also have associated subspaces which all lie in A. The relation $\gtrsim$ is not an equivalence relation; it is reflexive and transitive, but not symmetric. We do not require this concept till later in this paper; however, as proof of equivalence in most cases is also proof of elementary equivalence, we introduce the concept at this point.

For the sake of brevity, we write $\sigma_{a_1} \ldots \sigma_{a_k} \sim \sigma_{b_1} \ldots \sigma_{b_m}$ instead of the formally correct $(\sigma_{a_1}, \ldots, \sigma_{a_k}) \sim (\sigma_{b_1}, \ldots, \sigma_{b_m})$, and we shall use the same abbreviated notation with the symbol $\gtrsim$.

The following is a listing of the four types of fundamental relations:

(a) 1-dimensional relations of length 2. These are of the form (σ, σ) for every $\sigma \in S$.

(b) 2-dimensional relations of length 3. These are of the form $(\sigma_a, \sigma_b, \sigma_c)$, where a, b, c are dependent but pairwise independent, and $\langle a, b\rangle$ contains a non-singular vector of rad V.

(c) 2-dimensional relations of length 4. These are of the form $(\sigma_a, \sigma_b, \sigma_c, \sigma_d)$, where a, b, c are linearly dependent, and d is the vector given in (*) of § 2.

(d) 3-dimensional relations of length 4. These are of the form $(\sigma_a, \sigma_b, \sigma_c, \sigma_d)$ where $\dim \langle a, b, c\rangle = 3$, and $\dim \langle a, b, c\rangle \cap Q \operatorname{rad} V = 2$.

The relations (a) express the fact that the generators are involutions. Lemma 2.3 characterizes the two types of 2-dimensional relations (b) and (c). The relations of type (b) are unique to the characteristic 2 case, as rad $V = Q$ rad V if char $V \neq 2$. Finally, in view of 1.4, we see that all relations of length

4 are either consequences of types (a), (b) or (c), or are described by type (d). Thus the list above describes all the relations of length at most 4.

4. Equivalence of n-tuples. In this section we prove a sequence of results related to the concepts of equivalence and elementary-equivalence defined in § 3. The first of these arises immediately out of the considerations of § 3.

(4.1) *If* $\sigma_a\sigma_b\sigma_c\sigma_d = 1$, *then* $\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \emptyset$.

Repeated application of formula (*) in § 2 gives

(4.2) *Let* $\pi = \sigma_{a_1} \ldots \sigma_{a_m}$ *with* $\dim \langle a_1, \ldots, a_m \rangle \leqq 2$. *Then either*

$$\sigma_{a_1} \ldots \sigma_{a_m} \gtrsim \emptyset \quad \text{or} \quad \sigma_{a_1} \ldots \sigma_{a_m} \gtrsim \sigma_b \quad \text{or} \quad \sigma_{a_1} \ldots \sigma_{a_m} \gtrsim \sigma_{b_1}\sigma_{b_2}.$$

In a similar vein, we prove

(4.3) *Let* $\pi = \sigma_{a_1} \ldots \sigma_{a_5}$ *with* $\dim \langle a_1, \ldots, a_5 \rangle = 3$. *Then*

$$\sigma_{a_1} \ldots \sigma_{a_5} \gtrsim \sigma_{b_1} \ldots \sigma_{b_k} \quad \text{with} \quad 0 \leqq k \leqq 3.$$

Proof. If either $\{a_1, a_2, a_3\}$ or $\{a_4, a_5\}$ are dependent sets, then (4.3) follows at once from the considerations of § 3. If both $\{a_1, a_2, a_3\}$ and $\{a_4, a_5\}$ are independent, then $\langle a_4, a_5 \rangle$ intersects one of $\langle a_1, a_2 \rangle$, $\langle a_1, a_3 \rangle$ or $\langle a_2, a_3 \rangle$ in some non-singular $\langle r \rangle$. If $\langle r \rangle = \langle a_1, a_2 \rangle \cap \langle a_4, a_5 \rangle$, and $r \notin \operatorname{rad} V$ then

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_2}\sigma_{a_4}\sigma_{a_5} \gtrsim \sigma_{a_1}\sigma_{a_2}\sigma_r\sigma_r\sigma_{a_3}\sigma_r\sigma_r\sigma_{a_4}\sigma_{a_5},$$

and (4.3) follows from § 3. If $r \in \operatorname{rad} V$, then

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_{a_4}\sigma_{a_5} \gtrsim \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}$$

by § 3. A similar argument yields the result if $r \in \langle a_1, a_3 \rangle$ or $r \in \langle a_2, a_3 \rangle$.

It is clear that if we conjugate any σ_a by a σ_b, then $\sigma_b\sigma_a\sigma_b \gtrsim \sigma_c$. Doing this repeatedly gives rise to

(4.4) *Let* $\pi = \sigma_{a_1} \ldots \sigma_{a_k}$, *and suppose* $1 \leqq i_1 \leqq \ldots \leqq i_r \leqq k$. *Then we can find* $\sigma_{b_{r+1}}, \ldots, \sigma_{b_k}$ *and* $\sigma_{c_{r+1}}, \ldots, \sigma_{c_k}$ *such that*

$$\sigma_{a_1} \ldots \sigma_{a_k} \gtrsim \sigma_{a_{i_1}} \ldots \sigma_{a_{i_r}}\sigma_{b_{r+1}} \ldots \sigma_{b_k} \quad \text{and}$$
$$\sigma_{a_1} \ldots \sigma_{a_k} \gtrsim \sigma_{c_{r+1}} \ldots \sigma_{c_k}\sigma_{a_{i_r}} \ldots \sigma_{a_{i_1}}.$$

Next we show

(4.5) *Suppose* $\pi = \sigma_a\sigma_b\sigma_c$ *and* $\dim \langle a, b, c \rangle = 3$. *Suppose that* $\langle a, b, c \rangle \cap \operatorname{rad} V = \langle s \rangle$ *is non-singular, and that* $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle s \rangle$ *form a quadrangle whose diagonal is not singular. Then there exist* σ_u, σ_v *so that* $\sigma_a\sigma_b\sigma_c \gtrsim \sigma_u\sigma_v$.

Proof. By Lemma 1.5 (b) and 2.2, we know that $\dim B(\pi) = 2$, and that $s \notin B(\pi)$. Since the diagonal of $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle s \rangle$ is not singular, we may assume that $\langle p \rangle = \langle a, b \rangle \cap \langle c, s \rangle$ is not singular. Also $\langle p \rangle \neq \langle s \rangle$ and

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rad $V \cap \langle a, b, c \rangle = \langle s \rangle$, and hence $p \notin$ rad V. Thus σ_p exists, and so we have

$$\sigma_a\sigma_b\sigma_c \gtrsim \sigma_a\sigma_b\sigma_p\sigma_p\sigma_c \gtrsim \sigma_u\sigma_v,$$

by § 3.

(4.6) *Suppose* $\pi = \sigma_a\sigma_b\sigma_c$, *and* dim $\langle a, b, c \rangle = 3$. *Suppose that* $\langle a, b, c \rangle \cap$ rad V *contains a non-singular* $\langle d \rangle$ *for which* $\langle a \rangle, \langle b \rangle, \langle c \rangle, \langle d \rangle$ *is an sd-quadrangle whose diagonal is* L. *Then we have:*

(a) *If* dim $\langle a, b, c \rangle \cap$ rad $V = 2$, *there exist* σ_x, σ_y *such that* $\sigma_a\sigma_b\sigma_c \gtrsim \sigma_x\sigma_y$.

(b) *If* dim $\langle a, b, c \rangle \cap$ rad $V = 1$, *then there exist* $\sigma_f, \sigma_g, \sigma_h$ *such that* $\sigma_a\sigma_b\sigma_c \gtrsim \sigma_a\sigma_f\sigma_g\sigma_h$. *In this case,* $\langle a \rangle, \langle f \rangle, \langle g \rangle, \langle h \rangle$ *is an sd-quadrangle with diagonal* L, *and* $B(\pi)$ *is singular, and in fact,* $B(\pi) = L$.

Proof. (a) Let $\langle a, b, c \rangle \cap$ rad $V = B$. Since B contains the non-singular vector d, we know that B is non-singular, and hence either $\langle a, b \rangle \cap B$ or $\langle b, c \rangle \cap B$ is non-singular. In either case, Lemma 2.3 yields the result.

(b) This is just a restatement of Lemma 2.7.

We now prove the important

LEMMA 4.7. *If* $\sigma_a\sigma_b\sigma_c\sigma_{c'}\sigma_{b'}\sigma_{a'} = 1$, *then* $\sigma_a\sigma_b\sigma_c\sigma_{c'}\sigma_{b'}\sigma_{a'} \gtrsim \emptyset$.

Proof. If either of $\{a, b, c\}$ or $\{a', b', c'\}$ is a dependent set, then the result follows at once from § 3 and (4.1). We let

$$\pi = \sigma_a\sigma_b\sigma_c = \sigma_{a'}\sigma_{b'}\sigma_{c'},$$

and we may assume that both the associated subspaces $A = \langle a, b, c \rangle$ and $A' = \langle a', b', c' \rangle$ are 3-dimensional. If $A = A'$, the lemma follows immediately from (4.3) and (4.1). We may therefore assume that $A \neq A'$. This of course implies by Lemma 1.4 (b) that $B(\pi) \neq A$ and hence that

$$\dim (A \cap \text{rad } V) = \dim (A' \cap \text{rad } V) \neq 0.$$

We consider separately four cases. In this case distinction, let

$$A \cap \text{rad } V = Z \quad \text{and} \quad A' \cap \text{rad } V = Z'.$$

(a) Z is 2-dimensional and singular. Then by § 3, $\sigma_a\sigma_b\sigma_c \gtrsim \sigma_{a''}$, and hence (4.1) yields the result.

(b) Z is 2-dimensional and non-singular. Now dim $B(\pi) = 1$, implying that dim $Z' = 2$. By (a), we may assume that Z' is also non-singular. Then the methods used in (4.6), (a) show that $\sigma_a\sigma_b\sigma_c \gtrsim \sigma_x\sigma_y$ and that $\sigma_{a'}\sigma_{b'}\sigma_{c'} \gtrsim \sigma_{x'}\sigma_{y'}$. Now (4.1) yields the result.

(c) Z is 1-dimensional and singular. Let $Z = \langle z \rangle$. By Lemma 1.5, we know that dim $B(\pi) = 2$, and hence we can deduce that Z' is 1-dimensional. Also, $B(\pi) \subseteq A \cap A'$, and hence dim $(A + A') = 4$ and $B(\pi) = A \cap A'$. Now let $\langle a, b \rangle \cap \langle c, z \rangle = \langle t \rangle$. If $\langle t \rangle \neq \langle z \rangle$, we can write

$$\sigma_a\sigma_b\sigma_c \gtrsim \sigma_a\sigma_b\sigma_t\sigma_t\sigma_c \gtrsim \sigma_{a''}\sigma_t\sigma_c \quad \text{with} \quad z \in \langle t, c \rangle.$$

Thus we may assume that either $z \in \langle a, b \rangle$ or $z \in \langle b, c \rangle$. Either situation implies that $z \in B(\pi)$, as an easy computation will show, and therefore $z \in A'$, implying that $Z = Z'$. Indeed, we may clearly assume that $\langle z \rangle = \langle b, c \rangle \cap \langle b', c' \rangle$, and so $B(\pi) = \langle a, z \rangle = \langle a', z \rangle$, and hence a, a', z are linearly dependent.

If $\langle a \rangle = \langle a' \rangle$, then

$$\sigma_a\sigma_b\sigma_c\sigma_{c'}\sigma_{b'}\sigma_{a'} \gtrsim \sigma_{\bar{b}}\sigma_{\bar{c}}\sigma_{\bar{c}'}\sigma_{\bar{b}'}$$

for suitable $\bar{b}, \bar{c}, \bar{c}', \bar{b}'$ and the result follows from (4.1). Thus we may assume that a, a' are independent, and hence we can normalize them to obtain $a' = a + z$. Similarly we can write $c = b + z$ and $c' = b' + z$.

Now $a, a', b, c \in A$. If the quadrangle $\langle a \rangle, \langle a' \rangle, \langle b \rangle, \langle c \rangle$ is not an *sd*-quadrangle, then we know that one of $\langle a, b \rangle \cap \langle a', c \rangle = \langle p \rangle$ or $\langle a, c \rangle \cap \langle a', b \rangle = \langle q \rangle$ is non-singular, and in either case, $\sigma_{a'}\sigma_a\sigma_b\sigma_c \gtrsim \sigma_f\sigma_g$ for suitable f, g, and we are done. Thus we may assume that $\langle a \rangle, \langle a' \rangle, \langle b \rangle, \langle c \rangle$ is an *sd*-quadrangle. The same argument permits us to restrict ourselves to the case that both $\langle a' \rangle, \langle b' \rangle, \langle c' \rangle, \langle a \rangle$ and $\langle b \rangle, \langle c \rangle, \langle b' \rangle, \langle c' \rangle$ are *sd*-quadrangles. In this case, both $\langle a, c' \rangle \cap \langle a', b' \rangle$ and $\langle b', c \rangle \cap \langle b, c' \rangle$ are singular. But this implies that the 2-dimensional subspace

$$G := \langle a, b, b' \rangle \cap \langle a', c, c' \rangle$$

is singular as it contains the distinct 1-dimensional subpaces

$$\langle a, b \rangle \cap \langle a', c \rangle, \langle a, b' \rangle \cap \langle a', c' \rangle \quad \text{and} \quad \langle b, b' \rangle \cap \langle c, c' \rangle,$$

which are all singular. As z does not lie in $\langle a, b, b' \rangle$ (as otherwise $b' \in A$, implying $A = A'$), we know that $z \notin G$, and hence $H := \langle z \rangle + G$ is a singular, 3-dimensional subspace of $A + A'$. Now from our normalization, we know that $a + a' = b + c = b' + c' = z$; also $a' + b$ is singular, since $\langle a' + b \rangle$ is contained in the diagonal of the *sd*-quadrangle $\langle a \rangle, \langle a' \rangle, \langle b \rangle, \langle c \rangle$. Now we have

$$\sigma_{a'}\sigma_a\sigma_b\sigma_c = \sigma_{b'}\sigma_{c'}.$$

Repeated application of Lemma 2.4 yields:

$$\sigma_{a'}\sigma_a\sigma_b\sigma_c(x) = x + f(x, Q(a)^{-1}a + Q(b)^{-1}b)z \text{ and}$$
$$\sigma_{b'}\sigma_{c'}(x) = x + f(x, Q(b')^{-1}b')z.$$

Hence we see that

$$Q(a)^{-1}a + Q(b)^{-1}b + Q(b')^{-1}b' = s$$

lies in rad V. Since a, b, b' are independent, we know $s \neq 0$; since $z \notin \langle a, b, b' \rangle$, we know $\langle s \rangle \neq \langle z \rangle$.

We now make two case-distinctions:

(i) $Q(s) = 0$. Since $\langle s \rangle \neq \langle z \rangle$, we know that

$$(A + A') \cap \text{rad } V = \langle s, z \rangle = L$$

is singular. Now $F := \langle b', c', z, s \rangle$ is 3-dimensional and isotropic and hence, by Lemma 1.5, all singular vectors of F belong to L. Now $F \cap A$ contains z, but since $A \cap \text{rad } V = \langle z \rangle$, we know that $F \cap A$ is non-singular, and 2-dimensional. Thus we can find b'', $c'' \in F \cap A$ so that $\sigma_{b'}\sigma_{c'}\sigma_{c''}\sigma_{b''} = 1$. But then

$$\sigma_a\sigma_b\sigma_c\sigma_{c'}\sigma_{b'}\sigma_{a'} \gtrsim \sigma_a\sigma_b\sigma_c\sigma_{c''}\sigma_{b''}\sigma_{a'},$$

where a, b, c, a', b'', $c'' \in A$. Now the result follows from (4.3) and (4.1).

We are left with

(ii) $Q(s) \neq 0$. But then $s \notin H$, and so we see that $A + A' = H \oplus \langle s \rangle$, where H is singular. Hence in this case, $A + A'$ is isotropic. Now since $|K| \geqq 4$, we know that every 2-dimensional subspace M contains at least five 1-dimensional subspaces, at most one of which is singular if M is non-singular. In particular, $\langle a', b' \rangle$ contains at least two other non-singular $\langle p_1 \rangle$, $\langle p_2 \rangle$ (beside $\langle a' \rangle$, $\langle b' \rangle$). Let $\langle q_i \rangle = \langle p_i, c' \rangle \cap \langle a, a' \rangle$. Both $\langle q_1 \rangle$ and $\langle q_2 \rangle$ are non-singular. Let $\langle r_i \rangle = \langle q_i, c \rangle \cap \langle a, b \rangle$. At least one of the $\langle r_i \rangle$ is non-singular. Hence we can find $p \in \langle a', b' \rangle$ so that $\langle p \rangle$, $\langle q \rangle = \langle p, c' \rangle \cap \langle a, a' \rangle$ and $\langle r \rangle = \langle q, c \rangle \cap \langle a, b \rangle$ are all non-singular. But then

$$\sigma_a\sigma_b\sigma_c\sigma_{c'}\sigma_{b'}\sigma_{c'} \gtrsim \sigma_a\sigma_b\sigma_r\sigma_r\sigma_c\sigma_q\sigma_q\sigma_{c'}\sigma_p\sigma_p\sigma_{b'}\sigma_{a'} \gtrsim \sigma_{a''}\sigma_{b''}\sigma_{c''}\sigma_{d''}$$

by repeated application of the 3-reflection theorem. Now (4.1) yields the result.

We are left only with

(d) Z is 1-dimensional and non-singular. Let $Z = \langle z \rangle$. If

$$z \in \langle a, b \rangle \cup \langle b, c \rangle \cup \langle a, c \rangle,$$

then by § 3, $\sigma_a\sigma_b\sigma_c \gtrsim \sigma_u\sigma_v$. Also, dim $B(\pi) = 2$, and so $B(\pi) = \langle u, v \rangle \subseteq A'$. Thus (4.3) and (4.1) yield the result. The same reasoning holds if $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle z \rangle$ is not an *sd*-quadrangle. Thus we may assume that $\langle a \rangle$, $\langle b \rangle$, $\langle c \rangle$, $\langle z \rangle$ is an *sd*-quadrangle with diagonal L. By (4.6) (b), $B(\pi) = L$, and $\sigma_a\sigma_b\sigma_c \gtrsim \sigma_a\sigma_f\sigma_g\sigma_h$ for suitable f, g, $h \in A$. Hence we see that $L = B(\pi) = A \cap A'$, and $L \cap \text{rad } V = 0$. But then $A' \cap \text{rad } V = Z' = \langle z' \rangle$ is non-singular, and hence

$$A + A' = L \oplus \langle z \rangle \oplus \langle z' \rangle,$$

implying that $A + A'$ is isotropic. Also,

$$(A + A') \cap \text{rad } V = \langle z, z' \rangle.$$

The singular vectors of A, respectively A', all lie on L. Let $\langle a, z \rangle \cap L = \langle u \rangle$. If $u \in \langle c', z' \rangle$, we interchange the roles of a' and c'. (We can do this without fear, as A is isotropic, and so $\sigma_{a'}\sigma_{b'}\sigma_{c'} \gtrsim \sigma_{c'}\sigma_{b'}\sigma_{a'}$). If $u \notin \langle c', z' \rangle$, then $u \notin \langle a', b' \rangle$, and so $\langle u, z' \rangle \cap \langle a', b' \rangle = \langle a'' \rangle$ is non-singular, and we can replace

$\sigma_{a'}\sigma_{b'}$ by $\sigma_{a''}\sigma_{a''}\sigma_{a'}\sigma_{b'}$. Thus we see that we may assume that

$$\langle a, z\rangle \cap L = \langle a', z'\rangle \cap L = \langle b, c\rangle \cap L = \langle b', c'\rangle \cap L = \langle u\rangle.$$

Now, $C = \langle a, a', z, z'\rangle$ and $C' = \langle b, c, b', c'\rangle$ are two 3-dimensional subspaces of $A + A'$ where $C \cap C' = M$ and M is 2-dimensional, with $u \in M$. Now let $\langle r\rangle = \langle a, a'\rangle \cap \langle z, z'\rangle$. If r is non-singular, then

$$\sigma_a\sigma_{a'} \gtrsim \sigma_{a''} \quad \text{and} \quad \sigma_{a''} = \sigma_b\sigma_c\sigma_{b'}\sigma_{c'},$$

implying that $a'' \in C'$, and then (4.3) and (4.1) yield the result. If r is singular, then $B(\sigma_a\sigma_{a'}) = \langle r\rangle \subseteq C'$. But now from

$$\sigma_a\sigma_b\sigma_c\sigma_{c'}\sigma_{b'}\sigma_{a'} = 1,$$

we obtain (since A is isotropic) that

$$\sigma_b\sigma_c\sigma_{b'} = \sigma_a\sigma_{a'}\sigma_{c'},$$

where

$$\langle b, c, b'\rangle \cap \text{rad } V = C' \cap \text{rad } V = \langle r\rangle$$

which is singular. This puts us back into case (c), and thus the lemma is proved.

Now we require a technical

LEMMA 4.8. *Suppose* $\pi = \sigma_a\sigma_b\sigma_c\sigma_d$, *and* $\pi \neq 1$. *Suppose that* $A = \langle a, b, c, d\rangle$ *is* 3-*dimensional, and that* $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ *is an sd-quadrangle in* A *with diagonal* L. *Then*

(a) *For any* $u \in \text{rad } A$, $u \neq 0$, *we can find* $a', b', c', d' \in A$ *such that*

$$\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'}, \quad \textit{and} \quad \langle u\rangle = \langle a', b'\rangle \cap \langle c', d'\rangle.$$

(b) *If* $\langle a, b\rangle \cap \langle c, d\rangle = \langle u\rangle$ *and* A *is isotropic, then for all non-singular* $b' \in \langle a, b\rangle$ *with* $b' \notin \text{rad } V$, *and* $\langle b'\rangle \neq \langle a\rangle$, *we can find* $c', d' \in A$ *such that*

$$\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \sigma_a\sigma_{b'}\sigma_{c'}\sigma_{d'}.$$

Proof. (a) Since $u \in \text{rad } A$, we know $u \in L$. If $\langle u\rangle = \langle a, b\rangle \cap \langle c, d\rangle$, we are done. So suppose $b, c, d \notin \langle u, a\rangle$. Observe that if $\langle x\rangle \neq \langle u\rangle$, $x \neq 0$ and $x \in \langle u, a\rangle$, then x is non-singular, as $\langle u, a\rangle$ is istropic. Now at least one of $\langle b, c\rangle \cap \langle u, a\rangle$, $\langle b, d\rangle \cap \langle u, a\rangle$ or $\langle c, d\rangle \cap \langle u, a\rangle$ is non-singular, and not in rad V. By (4.4), we can assume that $\langle b'\rangle = \langle b, c\rangle \cap \langle u, a\rangle$ is this non-singular subspace. But then

$$\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \sigma_a\sigma_{b'}\sigma_{b'}\sigma_b\sigma_c\sigma_d \gtrsim \sigma_a\sigma_{b'}\sigma_{c'}\sigma_d.$$

(b) Now suppose $\langle a, b\rangle \cap \langle c, d\rangle = \langle u\rangle$ and A isotropic, implying that both $\langle a, b\rangle$ and $\langle c, d\rangle$ are isotropic. Choose any non-singular $b' \in \langle a, b\rangle$ with $\langle b'\rangle \neq \langle a\rangle$ and $b' \notin \text{rad } V$. Now $\langle b, c\rangle \cap \langle a, d\rangle = \langle v\rangle$ is singular. Since $\langle b'\rangle \neq \langle a\rangle$, we know that $\langle b, c\rangle \cap \langle b', d\rangle \neq \langle v\rangle$, and hence $\langle b, c\rangle \cap$

$\langle b', d\rangle = \langle p\rangle$ is non-singular. If $p \notin \text{rad } V$, then

$$\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \sigma_a\sigma_b\sigma_c\sigma_p\sigma_p\sigma_d\sigma_{b'}\sigma_{b'} \gtrsim \sigma_a\sigma_{c''}\sigma_{d''}\sigma_{b'} \gtrsim \sigma_a\sigma_{b'}\sigma_{c'}\sigma_{d'}$$

for suitable c'', d'', c', d'. If $p \in \text{rad } V$, then

$$\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \sigma_a\sigma_b\sigma_c\sigma_d\sigma_{b'}\sigma_{b'} \gtrsim \sigma_a\sigma_e\sigma_f\sigma_{b'} \gtrsim \sigma_a\sigma_{b'}\sigma_{c'}\sigma_{d'}$$

for suitable e, f, c', d'.

This lemma is useful in the following important

LEMMA 4.9. *Let* $\pi = \sigma_a\sigma_b\sigma_c\sigma_d \neq 1$, *and suppose* $A = \langle a, b, c, d\rangle$ *is 3-dimensional, such that* $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$ *is an sd-quadrangle in* A. *Then for any* $\sigma_{a'} \in S$, *we can find* $\sigma_{b'}$, $\sigma_{c'}$, $\sigma_{d'}$ *such that*

$$\sigma_a\sigma_b\sigma_c\sigma_d \sim \sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'}.$$

Proof. We first prove

(*) Let D be 3-dimensional, and M, N be two 2-dimensional subspaces of D such that M is isotropic, N is not isotropic, and $M \cap N$ is singular. Then

$$\sigma_{m'}\sigma_m\sigma_{n'}\sigma_n \gtrsim \sigma_x\sigma_y$$

for all $m', m \in M$ and all $n', n \in N$.

To prove (*), it is sufficient to observe that rad D is non-singular, as rad $D \subseteq M$, but rad $D \neq M \cap N$. Hence D contains no singular 2-dimensional subspaces, implying that $\langle m'\rangle$, $\langle m\rangle$, $\langle n'\rangle$, $\langle n\rangle$ is not an *sd*-quadrangle, and so $\langle m, n'\rangle \cap \langle m', n\rangle$ may be assumed non-singular. This immediately yields (*).

Now for the proof of Lemma 4.9: Let L be the diagonal of $\langle a\rangle$, $\langle b\rangle$, $\langle c\rangle$, $\langle d\rangle$. If A is isotropic, let $Z = L$. If A is not isotropic, we know by (2.1) (b) that A contains a second singular 2-dimensional subspace $Z \neq L$. A combination of (2.6), (2.8), (2.9) tells us that for a given $\sigma_{a'} \in S$, there exist $\sigma_{b'}$, $\sigma_{c'}$, $\sigma_{d'}$ with $b', c', d' \in \langle a'\rangle + Z$ for which

$$\sigma_a\sigma_b\sigma_c\sigma_d = \sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'}.$$

We must prove the equivalence of these two expressions for π.

If $a' \in A$, the result follows from (4.3) and Lemma 4.7. Hence we are left with the case that $a' \notin A$. We let $C := \langle a'\rangle + A$, and $A' := \langle a'\rangle + Z$. We now distinguish three cases.

(a) A is isotropic. If $a' \notin Z^\perp$, then there exists $u \in Z$ with $f(a', u) \neq 0$. By Lemma 4.8, there exist $b^*, c^*, d^* \in A$ such that

$$\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \sigma_a\sigma_{b^*}\sigma_{c^*}\sigma_{d^*} \text{ with } \langle u\rangle = \langle a, b^*\rangle \cap \langle c^*, d^*\rangle.$$

Also, the same argument as was used in the proof of Lemma 4.8 (a) shows that there exists $b'' \in \langle a', u\rangle$ such that

$$\sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'} \gtrsim \sigma_{a'}\sigma_{b''}\sigma_{c''}\sigma_{d''}$$

for suitable c'', d''. Now let $D = \langle a, b^*, a', b'' \rangle$, let $M = \langle a, b^* \rangle$, $N = \langle a', b'' \rangle$. Then apply $(*)$ to conclude that

$$\sigma_{b''}\sigma_{a'}\sigma_a\sigma_{b^*} \sim \sigma_x\sigma_y.$$

But

$$\sigma_{d''}\sigma_{c''}\sigma_{b''}\sigma_{a'}\sigma_a\sigma_{b^*}\sigma_{c^*}\sigma_{d^*} \gtrsim \sigma_{d''}\sigma_{c''}\sigma_x\sigma_y\sigma_{c''}\sigma_{d''}$$

and so now the result follows from Lemma 4.7.

If $a' \in Z^\perp$, then of course, $Z \subseteq \text{rad } C$, and so in particular, A' is isotropic also. As above, we write

$$\sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'} \gtrsim \sigma_{a'}\sigma_{b''}\sigma_{c''}\sigma_{d''}$$

where $\langle u \rangle = \langle a, b \rangle \cap \langle c, d \rangle = \langle a', b'' \rangle \cap \langle c'', d'' \rangle$. Clearly

$$\dim \langle a, b, a', b'' \rangle = \dim \langle a, b, c'', d'' \rangle = 3.$$

If $u \notin \text{rad } V$, but if

$$\langle a', b'' \rangle \cap \text{rad } V = \langle s \rangle \neq \langle 0 \rangle \quad \text{and} \quad \langle c'', d'' \rangle \cap \text{rad } V = \langle t \rangle \neq \langle 0 \rangle,$$

then both s, t are non-singular, implying that

$$\sigma_{a'}\sigma_{b''}\sigma_{c''}\sigma_{d''} \gtrsim \sigma_x\sigma_y,$$

and then Lemma 4.7 yields the result.

So we now assume that $\langle a', b'' \rangle \cap \text{rad } V \subseteq \langle u \rangle$. But then it is clear that $\langle a', b'' \rangle$ contains at least one non-singular vector $\bar{b}$ for which $\langle p \rangle = \langle a, \bar{b} \rangle \cap \langle a', b \rangle$ is non-singular with $\langle \bar{b} \rangle \neq \langle a' \rangle$. But by Lemma 4.8 (b), we know that

$$\sigma_{a'}\sigma_{b''}\sigma_{c''}\sigma_{d''} \gtrsim \sigma_{a'}\sigma_{\bar{b}}\sigma_{\bar{c}}\sigma_{\bar{d}}$$

for suitable $\bar{c}$, $\bar{d} \in A'$. Also

$$\sigma_{\bar{b}}\sigma_{a'}\sigma_a\sigma_b \gtrsim \sigma_x\sigma_y$$

for suitable σ_x, σ_y by using the non-singular $\langle p \rangle$, and so Lemma 4.7 again yields the result.

Similarly, we obtain the result in the case

(b) A' is isotropic.

We thus are left to consider

(c) Both A, A' are not isotropic. Now let $\langle u \rangle = \text{rad } A$. By Lemma 4.8, we know that

$$\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \sigma_a\sigma_{b^*}\sigma_{c^*}\sigma_{d^*},$$

with $\langle u \rangle = \langle a, b^* \rangle \cap \langle c^*, d^* \rangle$, and as in case (a), we know that

$$\sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'} \gtrsim \sigma_{a'}\sigma_{b''}\sigma_{c''}\sigma_{d''}$$

with $u \in \langle a', b'' \rangle$. If $a' \notin u^\perp$, we argue as in (a), again using $(*)$ to obtain the

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desired result. If $a' \in u^{\perp}$, but $u \notin \text{rad } V$, we know that there exists $\sigma_w \in S$ with $f(w, u) \neq 0$; by Corollary 4.10,

$$\sigma_a\sigma_b\sigma_c\sigma_d = \sigma_w\sigma_x\sigma_y\sigma_z$$

for suitable x, y, z. Now $w \notin u^{\perp}$ implies that the methods above yield

$$\sigma_a\sigma_b\sigma_c\sigma_d \sim \sigma_w\sigma_x\sigma_y\sigma_z \sim \sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'},$$

and we are done.

We are left with the case that $u \in \text{rad } V$. In this case, $C = A + A'$ has a non-trivial radical. Also, since A is not isotropic, we know that C is not isotropic, and hence rad $C = W$ is 2-dimensional, and $u \in W$. Observe that $A \cap W = \langle u \rangle = A' \cap W$. Now let

$$H_1 : = \langle a', b'' \rangle + W \quad \text{and} \quad H_2 : = \langle c'', d'' \rangle + W.$$

We know dim $H_1 =$ dim $H_2 = 3$, and H_1, H_2 are both isotropic. Let $F_1 : = H_1 \cap A$ and $F_2 : = H_2 \cap A$. Now it is easy to see that F_1, F_2, Z are three mutually distinct 2-dimensional isotropic subspaces of A containing $\langle u \rangle$. By (2.1), at most two of these are singular. Since Z is singular, we may therefore conclude that F_1 is non-singular. But then there exists $\sigma_{\bar{a}}$, with $\bar{a} \in F_1$, such that

$$\sigma_a\sigma_b\sigma_c\sigma_d \sim \sigma_{\bar{a}}\sigma_{\bar{b}}\sigma_{\bar{c}}\sigma_{\bar{d}}$$

for suitable $\bar{b}$, $\bar{c}$, $\bar{d}$, and by Lemma 4.8, we can choose $\bar{b}$ in such a way that $u \in \langle \bar{a}, \bar{b} \rangle$. But now we have

$$\sigma_a\sigma_b\sigma_c\sigma_d \sim \sigma_{\bar{a}}\sigma_{\bar{b}}\sigma_{\bar{c}}\sigma_{\bar{d}};\ \sigma_{\bar{a}}\sigma_{\bar{b}}\sigma_{\bar{c}}\sigma_{\bar{d}} = \sigma_{a'}\sigma_{b''}\sigma_{c''}\sigma_{d''};\ \sigma_{a'}\sigma_{b''}\sigma_{c''}\sigma_{d''} \sim \sigma_{a'}\sigma_{b'}\sigma_{c'}\sigma_{d'}.$$

Hence

$$\sigma_{b''}\sigma_{a'}\sigma_{\bar{a}}\sigma_{\bar{b}} = \sigma_{c''}\sigma_{d''}\sigma_{\bar{d}}\sigma_{\bar{c}}.$$

Now let $B = \langle b'', a', \bar{a}, \bar{b} \rangle$ and $B' = \langle c'', d'', \bar{d}, \bar{c} \rangle$. Since $W \subseteq B$, we know that B is isotropic. Thus we have reduced this case to case (a). This completes the proof of the lemma.

Out of the proof of (a) in Lemma 4.9, we pull the following

COROLLARY 4.10. *Under the assumptions of Lemma 4.9, we have: If A is isotropic, then* $\sigma_{a'}\sigma_a\sigma_b\sigma_c\sigma_d \gtrsim \sigma_{b'}\sigma_{c'}\sigma_{d'}$.

We complete this section with

LEMMA 4.11. *Suppose* $\pi = \sigma_{a_1} \ldots \sigma_{a_6}$ *and suppose* dim $\langle a_1, \ldots, a_6 \rangle = 4$. *Then*

$$\sigma_{a_1} \ldots \sigma_{a_6} \sim \sigma_{b_1} \ldots \sigma_{b_k}$$

with $k \leqq 5$.

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Proof. In view of (4.4), we may assume that $a_1, \ldots, a_4$ are linearly independent. If $\langle a_5 \rangle = \langle a_6 \rangle$, we are done at once. So we may now assume also that a_5, a_6 are independent, and of course, $\langle a_5, a_6 \rangle \cap \text{rad } V \subseteq Q \text{ rad } V$. Let $A_1 := \langle a_2, a_3, a_4 \rangle$, $A_2 := \langle a_1, a_3, a_4 \rangle$, $A_3 := \langle a_1, a_2, a_4 \rangle$, $A_4 := \langle a_1, a_2, a_3 \rangle$. If $\langle a_5, a_6 \rangle$ lies in one of A_1, A_2, A_3, A_4, then (4.4) and (4.3) yield the result. If $\langle a_5, a_6 \rangle$ is not contained in any of the A_i, then let $\langle p_i \rangle = A_i \cap \langle a_5, a_6 \rangle$ for $i = 1, 2, 3, 4$. At least one of the $\langle p_i \rangle$ is non-singular. Then $p_i \notin \text{rad } V$, and in view of (4.4), we may assume that $\langle a_2, a_3, a_4 \rangle \cap \langle a_5, a_6 \rangle = \langle p_i \rangle$ is non-singular, and so

$$\sigma_{a_2}\sigma_{a_3}\sigma_{a_4}\sigma_{a_5}\sigma_{a_6} \sim \sigma_{a_2}\sigma_{a_3}\sigma_{a_4}\sigma_{p_i}\sigma_q \quad \text{where} \quad \sigma_q = \sigma_{p_i}\sigma_{a_5}\sigma_{a_6}.$$

Either

$$\sigma_{a_2}\sigma_{a_3}\sigma_{a_4}\sigma_{p_i} \sim \sigma_x\sigma_y,$$

or $\langle a_2 \rangle, \langle a_3 \rangle, \langle a_4 \rangle, \langle p_i \rangle$ is an *sd*-quadrangle. But by Lemma 4.9, we know there exist $\sigma_u, \sigma_v, \sigma_w$ such that

$$\sigma_{a_2}\sigma_{a_3}\sigma_{a_4}\sigma_{p_i} \sim \sigma_{a_1}\sigma_u\sigma_v\sigma_w, \quad \text{and so}$$

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_{a_4}\sigma_{a_5}\sigma_{a_6} \sim \sigma_{a_1}\sigma_{a_1}\sigma_u\sigma_v\sigma_w,$$

which proves the lemma.

5. The main lemma. We start this section with the following

Definition. Let H be a subspace, and let $\sigma_{a_1} \ldots \sigma_{a_k}$ be some word in W. We say that $\sigma_{a_1} \ldots \sigma_{a_k}$ is *H-equivalent* if there exist $b_1, \ldots, b_m \in H \cup H^\perp$ such that $\sigma_{a_1} \ldots \sigma_{a_k} \sim \sigma_{b_1} \ldots \sigma_{b_m}$. We say that $\sigma_{a_1} \ldots \sigma_{a_k}$ is *H-sufficient* if $a_1, \ldots, a_k \in H \cup H^\perp$.

Thus $\sigma_{a_1} \ldots \sigma_{a_k}$ is H-equivalent if it is equivalent to some H-sufficient word $\sigma_{a_1} \ldots \sigma_{b_m}$.

In the following, H shall be some fixed, regular, 2-dimensional subspace.

LEMMA 5.1. *Let H' be isometric to H such that $H' \subsetneq H^\perp$. Then there exists $\sigma_1, \sigma_2 \in S$ such that $\sigma_1\sigma_2(H) = H'$ or $\sigma_1(H) = H'$.*

Proof. Suppose first that $H \cap H' = \langle a \rangle$, where $Q(a) \neq 0$. Then there exist $b \in H$, $c \in H'$ such that $H = \langle a, b \rangle$, $H' = \langle a, c \rangle$ and $Q(b) = Q(c)$, $f(a, b) = f(a, c)$. Further $f(b, c) \neq 0$, as otherwise replace b by $b' = \sigma_a(b)$. This choice of b and c means that $\sigma_{b+c}(a) = a$ and $\sigma_{b+c}(b) = c$ so that $\sigma_{b+c}(H) = H'$.

Now, suppose $H \cap H'$ is singular. Let $q \in H$ be non-singular, such that $q \notin H'^\perp$, and let $U = \{y \in H' \mid Q(y) = Q(q)\}$. Since H' and H are isometric, we know that U contains at least two elements. Now, for $y \in U$, we know that $Q(y + q) = f(y, q)$. Our choice of $q \notin H'^\perp$ guarantees the existence of at least one $p \in U$ so that $Q(p + q) = f(p, q) \neq 0$. Hence σ_{p+q} exists, and $\sigma_{p+q}(q) = p$

with $Q(p) \neq 0$. Thus $\sigma_{p+q}(H) = H''$, with $H \cap H'' = \langle p \rangle$ where $Q(p) \neq 0$. As we saw above, we can now map H'' to H' by some $\sigma \in S$.

LEMMA 5.2. *Let $\pi \in 0^*$ such that $\pi(H) = H$, but $H \subsetneq F(\pi)$. Then $H \cap B(\pi)$ is non-singular. If $H \cap B(\pi) = \langle h \rangle$, then $\pi|_H = \sigma_h|_H$, or equivalently, $H \subseteq F(\sigma_h\pi)$.*

Proof. Since $H \subsetneq F(\pi)$, there exists a non-singular $h' \in H$ with $\pi(h') \neq h'$. Hence $f(\pi(h'), h') \neq 0$ (as H is regular) and so $Q(\pi(h') + h') \neq 0$. Thus

$$h := h' + \pi(h') \in H \cap B(\pi)$$

and h is non-singular. Observe that

$$\sigma_h(h') = h' + f(h'h)Q(h)^{-1}h = h' + h = \pi(h'),$$

and hence $\sigma_h\pi(h') = h'$. If $H \cap B(\pi) = \langle h \rangle$, then $\pi(h) = h$ and so in this case, $H \subseteq F(\sigma_h\pi)$.

We next prove a converse to Lemma 5.1.

LEMMA 5.3. *Let $\pi = \sigma_p\sigma_q$, and suppose $H' = \pi(H)$. Then $H' \subsetneq H^\perp$.*

Proof. If $H' = H$, there is nothing to prove. So suppose $H' \neq H$. Let $H'' = \sigma_q(H)$, so that $H' = \sigma_p(H'')$. Now $H'' \subseteq \langle H, q \rangle$ and $H' \subseteq \langle H'', p \rangle$, and so $H'' \cap H \supseteq \langle t \rangle$ and $H' \cap H'' \supseteq \langle s \rangle$, where $s, t \neq 0$. If $\langle s \rangle = \langle t \rangle$, then $H' \cap H = \langle t \rangle$, and $t \notin H^\perp$, so $H' \subsetneq H^\perp$. If $\langle s \rangle \neq \langle t \rangle$, then $H'' = \langle s, t \rangle$. But H'' is regular, implying $f(s, t) \neq 0$. Since $s \in H'$ and $t \in H$, we again conclude $H' \subsetneq H^\perp$.

LEMMA 5.4. *Suppose $\sigma_{a_1} \ldots \sigma_{a_k}(H) = H$, for $k \geqq 3$. Then there exists i, with $1 \leqq i \leqq k - 2$ so that*

$$\sigma_{a_i}\sigma_{a_{k-1}}\sigma_{a_k}(H) \subsetneq H^\perp, \quad \textit{and}$$

$$\sigma_{a_1} \ldots \sigma_{a_k} \sim \sigma_{b_1} \ldots \sigma_{b_{k-3}}\sigma_{a_i}\sigma_{a_{k-1}}\sigma_{a_k}.$$

Proof. Let $\tau = \sigma_{a_{k-1}}\sigma_{a_k}$, and let $H'' = \tau(H)$. By Lemma 5.3, $H'' \subsetneq H^\perp$. If $a_i \in H^\perp$ for some i, then $\sigma_{a_i}(H^\perp) = H^\perp$, and hence $\sigma_{a_i}(H'') \subsetneq H^\perp$. So we must see what happens if none of the a_i for $1 \leqq i \leqq k - 2$ lie in $H^\perp$. Suppose that $\sigma_{a_i}\tau(H) \subseteq H^\perp$ for all $i \leqq k - 2$. This means that

$$f(\sigma_{a_i}\tau(h), h') = 0 \quad \text{for all } h, h' \in H,$$

or equivalently,

$$f(\tau(h), h') + Q(a_i)^{-1}f(\tau(h), a_i)f(h', a_i) = 0$$

for all $h, h' \in H$ and all $i \leqq k - 2$. Since we now assume $a_i \notin H$, we know that for each i, there exists a unique $\langle h(i) \rangle \subseteq H$, $h(i) \neq 0$, so that $f(a_i, h(i)) = 0$. Hence, for each i, there exists $h(i) \in H\backslash\{0\}$ such that $f(\tau(h), h(i)) = 0$ for all $h \in H$, and so $\tau(h) \in h(i)^\perp$ for all $h \in H$. Since $\tau(H) \subsetneq H^\perp$, we conclude that

$h(i)$, $h(j)$ are dependent for all $i, j \leqq k - 2$. So we see that $\sigma_i \tau(H)$ is orthogonal to H for all $i \leqq k - 2$ if and only if there exists some $h \in H \backslash \{0\}$ with $\tau(H) \subseteq h^{\perp}$, and $a_i \in h^{\perp}$ for all $i \leqq k - 2$. This implies that $\sigma_{a_i}(h^{\perp}) = h^{\perp}$, and hence $\sigma_{a_i}(\tau(H)) \subseteq h^{\perp}$ for all i. But then

$$\sigma_{a_1} \ldots \sigma_{a_{k-2}}(\tau(H)) = H \subseteq h^{\perp} \text{ for } h \in H,$$

implying that H is not regular, a contradiction.

The last part of the lemma follows at once from (4.4).

LEMMA 5.5. *Let* $\pi = \sigma_{a_1} \ldots \sigma_{a_k}$, *and suppose the associated subspace* $A = \langle a_1, \ldots, a_k \rangle$ *is k-dimensional. Suppose further that* $\pi(H) = H$. *Let* $Z = A \cap H^{\perp}$. *If* dim $B(\pi) \cap H \leqq 1$, *then* dim $Z \geqq k - 1$. *If* $H \subseteq B(\pi)$, *then* dim $Z = k - 2$. *If* dim $Z = k - 1$, *and Z is singular, then A is isotropic.*

Proof. If $H \subsetneq F(\pi)$, then by Lemma 5.2, $H \cap B(\pi)$ is non-singular. Thus, if $H \subsetneq B(\pi)$, either $H \cap B(\pi) = \langle h \rangle \neq \langle 0 \rangle$, or $H \subseteq F(\pi)$. But $H \subseteq F(\pi)$ implies by Lemma 1.5 that $H \subseteq F(\pi) \subseteq A^{\perp}$, and hence $A \subseteq A^{\perp\perp} \subseteq H^{\perp}$, so that $Z = A$. If $H \cap B(\pi) = \langle h \rangle$, then also by Lemma 5.2, we see that $\pi(h) = h$, so $h \in F(\pi) = A^{\perp}$. But then $A \subseteq h^{\perp}$, and $A \cap H^{\perp} = A \cap (h_1^{\perp} \cap h)$ for some $h_1 \in H^{\perp}$, implying that

$$\dim A \cap H^{\perp} = \dim A \cap h_1^{\perp} = k - 1.$$

Certainly, since dim $H^{\perp} = n - 2$, and dim $A = k$, we know that dim $Z \geqq k - 2$. If $H \subseteq B(\pi)$, then $H \subseteq A$. Also, dim $Z \geqq k - 2$, and so in fact, $A = H \textcircled{\perp} Z$. Finally, if dim $Z = k - 1$ and Z is singular, then $A = Z \textcircled{\perp} \langle h \rangle$ is clearly isotropic.

In a similar spirit, we prove

LEMMA 5.6. *Suppose* $\pi = \sigma_{a_1} \ldots \sigma_{a_k}$, *and* $A = \langle a_1, \ldots, a_k \rangle$. *If* dim $A = k - 1$, *and if* $H \subseteq F(\pi)$, *then* dim $Z \geqq$ dim $A - 1$ *and* $B(\pi) \subseteq Z$, *where* $Z = A \cap H^{\perp}$.

Proof. We can assume that $a_2, \ldots, a_k$ are linearly independent. Since $H \subseteq F(\pi)$, we know that

$$\sigma_{a_2} \ldots \sigma_{a_k}(h) = \sigma_{a_1}(h)$$

for all $h \in H$. But dim $a_1^{\perp} \cap H \geqq 1$, and so there exists $h_0 \in H \backslash \{0\}$ such that $\sigma_{a_1}(h_0) = h_0$, implying $h_0 \in F(\sigma_{a_2}, \ldots, \sigma_{a_k})$, and hence $h_0 \in A^{\perp}$, or $A \subseteq h_0^{\perp}$. Thus again

$$\dim (A \cap H^{\perp}) \geqq \dim A - 1.$$

From Lemma 1.2 (e), we know that $F(\pi) \subseteq B(\pi)^{-1}$, and hence $B(\pi) \subseteq B(\pi)^{\perp\perp} \subseteq F(\pi)^{\perp}$. Also, $H \subseteq F(\pi)$ implies $F(\pi)^{\perp} \subseteq H^{\perp}$. Hence $B(\pi) \subseteq H^{\perp}$, and trivially, $B(\pi) \subseteq A$ implies $B(\pi) \subseteq Z$.

Now we prove

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LEMMA 5.7. *Let* $\pi = \sigma_{a_1} \ldots \sigma_{a_k}$ *with* $k \leqq 3$, *and suppose* $\pi(H) = H$. *Then* $\sigma_{a_1} \ldots \sigma_{a_k}$ *is H-equivalent.*

Proof. (a) If $k = 1$, then $\pi = \sigma_{a_1}$, and $\sigma_{a_1}(H) = H$ if and only if $a_1 \in H \cup H^\perp$.

(b) If $k = 2$, then $\pi = \sigma_{a_1}\sigma_{a_2}$. If $\langle a_1 \rangle = \langle a_2 \rangle$, then $\pi = 1$, hence $\pi \sim \sigma_h \sigma_h$ for $h \in H$. If $\langle a_1 \rangle \neq \langle a_2 \rangle$, then either $H \subseteq F(\pi)$, and so $H \subseteq \langle a_1, a_2 \rangle^\perp$, implying $\langle a_1, a_2 \rangle \subseteq H^\perp$, in which case the lemma holds, or there exists non-singular $h \in H \cap B(\pi) \subseteq H \cap \langle a_1, a_2 \rangle$, $h \neq 0$. But then

$$\sigma_{a_1}\sigma_{a_2} \sim \sigma_h \sigma_h \sigma_{a_1} \sigma_{a_2},$$

where $\sigma_h \sigma_{a_1} \sigma_{a_2} \sim \emptyset$, or $\sigma_h \sigma_{a_1} \sigma_{a_2} \sim \sigma_d$. Then (a) yields the result.

(c) If $k = 3$, then $\pi = \sigma_{a_1}\sigma_{a_2}\sigma_{a_3}$. If a_1, a_2, a_3 are dependent, then either

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \emptyset \quad \text{or} \quad \sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \sigma_d,$$

and (a) yields the result.

So assume a_1, a_2, a_3 are independent. Let $A = \langle a_1, a_2, a_3 \rangle$. If $H \subseteq F(\pi)$, then $H \subseteq A^\perp$, and hence $A \subseteq H^\perp$. If $H \subsetneq F(\pi)$, let $Z := A \cap H^\perp$. If $H \subseteq B(\pi)$, then $H \subseteq A$, and so one of $\langle a_1, a_2 \rangle$, $\langle a_1, a_3 \rangle$ or $\langle a_2, a_3 \rangle$ intersects H in a non-singular 1-dimensional subspace $\langle h \rangle$. But then

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_h \sim \sigma_x \sigma_y$$

and so

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_h\sigma_h \sim \sigma_x\sigma_y\sigma_h,$$

and we use (b) to conclude the result.

If $H \subsetneq B(\pi)$, then by Lemma 5.2, $H \cap B(\pi) = \langle h \rangle$, where $\langle h \rangle$ is non-singular, and $\dim Z = 2$. If Z is non-singular, we repeat the argument immediately above with Z instead of H to deduce the result. If Z is singular, then by Lemma 5.5, A is isotropic.

Consider the 1-dimensional subspaces $\langle a_1 \rangle$, $\langle a_2 \rangle$, $\langle a_3 \rangle$, $\langle h \rangle$. If this is not an *sd*-quadrangle, then

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_h \sim \sigma_x\sigma_y$$

and we conclude as above. If it is an *sd*-quadrangle, and if $Z \subseteq Q$ rad V, then

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \sigma_u$$

and we are done by (a). If $Z \subsetneq Q$ rad V, then there exists σ_{b_1} with $b_1 \in H^\perp$. By Lemma 4.9, there exists $\sigma_{b_2}, \sigma_{b_3}, \sigma_{b_4}$ with $b_2, b_3, b_4 \in \langle b_1 \rangle + Z \subseteq H^\perp$ such that

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_h \sim \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}\sigma_{b_4},$$

and hence we deduce

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_h\sigma_h \sim \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}\sigma_{b_4}\sigma_h,$$

with $b_i \in H^\perp$ for $i = 1, 2, 3, 4$.

In a similar vein, we prove

LEMMA 5.8. *Let* $\pi = \sigma_{a_1} \ldots \sigma_{a_k}$, *with* $3 \leqq k \leqq 6$ *such that* $H \subseteq F(\pi)$. *Then* $\sigma_{a_1} \ldots \sigma_{a_k}$ *is H-equivalent.*

Proof. If $k = 3$, the lemma holds by Lemma 5.7. So suppose $4 \leqq k \leqq 6$, and suppose the lemma is true for $k - 1$. We let $A = \langle a_1, \ldots, a_k \rangle$, and consider the following cases:

(a) dim $A = k$. But then $A \subseteq H^\perp$, and so the claim is true.

(b) dim $A = k - 1$. By Lemma 5.6, if $Z := A \cap H^\perp$, we know that dim $Z \geqq k - 2$. If dim $Z = k - 1$, then $Z = A$, and so $A \subseteq H^\perp$, and we are done. So now suppose dim $Z = k - 2$. Since $k \geqq 4$, we know that dim $A \geqq 3$, and we may assume that a_1, a_2, a_3 are linearly independent. Let

$$L := \langle a_1, a_2, a_3 \rangle \cap H^\perp = \langle a_1, a_2, a_3 \rangle \cap Z.$$

Then dim $L \geqq 2$. If dim $L = 3$, we are done at once, as then $a_1, a_2, a_3 \in H^\perp$. Now suppose dim $L = 2$. If L is non-singular, then either

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \sigma_{c_1}\sigma_{c_2}, \quad \text{or} \quad \sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \sigma_{d_1}\sigma_{d_2}\sigma_{d_3}$$

with $d_1 \in L \subseteq H^\perp$, and in both cases the lemma follows by the induction hypothesis. Now suppose L is singular. If $L \subseteq Q$ rad V, then

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \sigma_d,$$

and we are again finished. So assume $L \subsetneq Q$ rad V. Let

$$u \in Q \text{ rad } \langle a_1, a_2, a_3 \rangle, \ u \neq 0.$$

Then $u \in L$, and as in the proof of Lemma 4.8, there exist a_2', a_3' such that

$$\sigma_{a_1}\sigma_{a_2}\sigma_{a_3} \sim \sigma_{a_1}\sigma_{a_2'}\sigma_{a_3'}, \quad \text{and} \quad u \in \langle a_1, a_2' \rangle.$$

Now choose $c \in \langle a_1, a_2, a_3 \rangle$ such that $\langle a_1 \rangle$, $\langle a_2' \rangle$, $\langle a_3' \rangle$, $\langle c \rangle$ is an *sd*-quadrangle, with diagonal D. Here $D = L$ if $\langle a_1, a_2, a_3 \rangle$ is isotropic, and D is the second singular 2-dimensional subspace of $\langle a_1, a_2, a_3 \rangle$ if $\langle a_1, a_2, a_3 \rangle$ is not isotropic. Then c is clearly non-singular. If $c \in$ rad V, then $B(\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}) = L$; if $c \notin$ rad V, then $B(\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_c) \subseteq L$. Since $L \subsetneq Q$ rad V, we know that $H^\perp$ is not isotropic, and hence there exists $\sigma_d \in S$ with $d \in H^\perp$. But now, by Lemma 4.9, either $\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}$ or $\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_c$ is equivalent to $\sigma_d\sigma_v\sigma_w\sigma_x$, with $v, w, x \in \langle d \rangle + L \subseteq H^\perp$. Hence either $\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}$ is H-equivalent, or $\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\sigma_c$ is H-equivalent, and so the induction hypothesis yields the result.

(c) Suppose dim $A = k - 2$. Since $k \leqq 6$, this implies that dim $A \leqq 4$. But then we come back to the induction hypothesis by using (4.3) or Lemma 4.11.

We are now ready to prove the main Lemma.

LEMMA 5.9. *Let* $\pi = \sigma_{a_1} \ldots \sigma_{a_k}$. *If* $\pi(H) = H$, *then* $\sigma_{a_1} \ldots \sigma_{a_k}$ *is H-equivalent.*

Proof. We prove this lemma by induction on k. Lemma 5.7 provides the proof for $k \leqq 3$. So suppose now that $k \geqq 4$, and that the lemma holds for $k - 1$. By Lemma 5.4, we can assume that

$$\sigma_{a_{k-2}}\sigma_{a_{k-1}}\sigma_{a_k}(H) = H' \subsetneqq H^{\perp}.$$

Then either $H' = H$, implying that

$$\sigma_{a_1} \ldots \sigma_{a_{k-3}}(H) = H,$$

and so our induction hypothesis yields the result, or $H' \neq H$, in which case by Lemma 5.1, there exist σ_u, σ_v so that $\sigma_u\sigma_v(H') = H$. But then

$$\sigma_{a_1} \ldots \sigma_{a_k} \sim \sigma_{a_1} \ldots \sigma_{a_{k-3}}\sigma_v\sigma_u\sigma_u\sigma_v\sigma_{a_{k-2}}\sigma_{a_{k-1}}\sigma_{a_k},$$

where

$$\sigma_{a_1} \ldots \sigma_{a_{k-3}}\sigma_v\sigma_u(H) = \sigma_u\sigma_v\sigma_{a_{k-2}}\sigma_{a_{k-1}}\sigma_{a_k}(H) = H.$$

By induction hypothesis, $\sigma_{a_1} \ldots \sigma_{a_{k-3}}\sigma_u\sigma_v$ is H-equivalent. Hence we need only show that

$$\sigma_u\sigma_v\sigma_{a_{k-2}}\sigma_{a_{k-1}}\sigma_{a_k}$$

is H-equivalent.

Let

$$\rho = \sigma_u\sigma_v\sigma_{a_{k-2}}\sigma_{a_{k-1}}\sigma_{a_k}.$$

If $H \subseteq F(\rho)$, then Lemma 5.8 yields the result. If $H \subsetneqq F(\rho)$, we know that $H \cap B(\rho)$ is non-singular. If $\dim H \cap B(\rho) = 1$, say $H \cap B(\rho) = \langle h \rangle$, then by Lemma 5.2, $H \subseteq F(\sigma_h\rho)$, and so, by Lemma 5.8,

$$\sigma_h\sigma_u\sigma_v\sigma_{a_{k-2}}\sigma_{a_{k-1}}\sigma_{a_k}$$

is H-equivalent, implying that

$$\sigma_u\sigma_v\sigma_{a_{k-2}}\sigma_{a_{k-1}}\sigma_{a_k}$$

is H-equivalent.

We are therefore left with the case that

$$H \subseteq B(\rho) \subseteq \langle u, v, a_{k-2}, a_{k-1}, a_k \rangle.$$

Let $A = \langle u, v, a_{k-2}, a_{k-1}, a_k \rangle$. Since $H \subseteq B(\rho)$, we know that there exist $h_1, h_2 \in H$ such that $\rho(h) = \sigma_{h_2}\sigma_{h_1}(h)$ for all $h \in H$, and hence $H \subseteq F(\sigma_{h_1}\sigma_{h_2}\rho)$. If $\dim A \leqq 4$, then we can use (4.3), and lemma 4.11 to deduce that

$$\sigma_{h_1}\sigma_{h_2}\sigma_u\sigma_v\sigma_{a_{k-2}}\sigma_{a_{k-1}}\sigma_{a_k} \sim \sigma_{b_1} \ldots \sigma_{b_e} \quad \text{with } e \leqq 5,$$

and then the lemma follows from Lemma 5.8.

This leaves the case that dim $A = 5$. By Lemma 5.5, we know that $A = H \textcircled{\perp} Z$, where $Z = A \cap H^\perp$, and dim $Z = 3$. Now consider $L := Z \cap \langle a_{k-2}, a_{k-1}, a_k \rangle$. If $L = \langle a_{k-2}, a_{k-1}, a_k \rangle$, then $a_{k-2}, a_{k-1}, a_k \in H^\perp$, and we are done. Hence we may assume that dim $L = 2$. If L is non-singular, then either

$$L \subseteq \text{rad } V, \quad \text{and} \quad \sigma_{ak-2}\sigma_{ak-1}\sigma_{ak} \sim \sigma_s\sigma_t,$$

and we are done, or

$$\sigma_{ak-2}\sigma_{ak-1}\sigma_{ak} \sim \sigma_z\sigma_s\sigma_t \quad \text{with } z \in Z \subseteq H,$$

and we are done.

If L is singular, and $L \subseteq Q$ rad V, then $\sigma_{ak-2}\sigma_{ak-1}\sigma_{ak} \sim \sigma_s$, and we are done. So we must only worry about the case that L is singular, but $L \subsetneq Q$ rad V. But now we proceed precisely as in part (b) of the proof of Lemma 5.8, to conclude that either

$$\sigma_{ak-2}\sigma_{ak-1}\sigma_{ak} \sim \sigma_s\sigma_t\sigma_x\sigma_y, \quad \text{with } s, t, x, y \in H^\perp, \quad \text{or}$$

$$\sigma_{ak-2}\sigma_{ak-1}\sigma_{ak} \sim \sigma_s\sigma_t\sigma_x\sigma_y\sigma_c, \quad \text{with } s, t, x, y \in H^\perp.$$

In either case, we are done.

We conclude this section with

LEMMA 5.10. *Let $\pi = \sigma_{a_1} \ldots \sigma_{a_k}$, and suppose $H \subseteq F(\pi)$ for some regular 2-dimensional subspace H. Then there exist $\sigma_{b_1}, \ldots, \sigma_{b_m}$, with $b_1, \ldots, b_m \in H^\perp$, and*

$$\sigma_{a_1} \ldots \sigma_{a_k} \sim \sigma_{b_1} \ldots \sigma_{b_m}.$$

Proof. By Lemma 5.9, $\sigma_{a_1} \ldots \sigma_{a_k}$ is H-equivalent, and so there exists some H-sufficient word $\sigma_{c_1} \ldots \sigma_{c_n}$ such that

$$\sigma_{a_1} \ldots \sigma_{a_k} \sim \sigma_{c_1} \ldots \sigma_{c_n}.$$

Now the c_i all lie in $H \cup H^\perp$. By Lemma 1.2 (b), we can assume that

$$c_1, \ldots, c_e \in H, \quad \text{and} \quad c_{e+1}, \ldots, c_n \in H^\perp.$$

We have $H \subseteq F(\sigma_{c_1} \ldots \sigma_{c_e})$, and so, by (4.2), $\sigma_{c_1} \ldots \sigma_{c_e} \sim \emptyset$, and we are done.

6. The main theorem. We now state the main theorem of this paper. Observe that in this theorem we make no assumptions about dim V. In fact, we admit the possibility that V is infinite-dimensional.

THEOREM 6.1. *Let (V, Q) be a metric vector space over a field K of characteristic 2, with $K \neq GF(2)$. Let S be the set of simple isometries with non-singular path, and let 0^* be the group of isometries generated by S. Then every relation in 0^* between elements of S is a consequence of the fundamental relations (a), (b), (c), (d) of length 2, 3, or 4, discussed in § 3.*

Proof. Assume first that $\dim V < \infty$. Then $V = H_1 \textcircled{\perp} H_2 \textcircled{\perp} \ldots \textcircled{\perp} H_n \textcircled{\perp} R$ where $R = \operatorname{rad} V$, and $H_1, \ldots, H_n$ are regular, 2-dimensional subspaces. Now suppose $\sigma_{a_1} \ldots \sigma_{a_k} = 1$. By Lemma 5.10, $\sigma_{a_1} \ldots \sigma_{a_k} \sim \sigma_{b_1} \ldots \sigma_{b_e}$, with $b_1, \ldots, b_e \in H_1^\perp = H_2 \textcircled{\perp} \ldots \textcircled{\perp} H_n \textcircled{\perp} R$. Now we apply Lemma 5.10 again to the relation $\sigma_{b_1} \ldots \sigma_{b_e}$ in the subspace $H_2 \textcircled{\perp} \ldots \textcircled{\perp} H_n \textcircled{\perp} R$, and obtain that

$$\sigma_{b_1} \ldots \sigma_{b_e} \sim \sigma_{c_1} \ldots \sigma_{c_m},$$

with

$$c_1, \ldots, c_m \in H_1^\perp \cap H_2^\perp = H_3 \textcircled{\perp} \ldots \textcircled{\perp} H_n \textcircled{\perp} R.$$

After n steps, we have obtained

$$\sigma_{a_1} \ldots \sigma_{a_e} \sim \sigma_{d_1} \ldots \sigma_{d_r},$$

where

$$d_1, \ldots, d_r \in H_1^\perp \cap \ldots \cap H_n^\perp = R,$$

and hence $\sigma_{a_1} \ldots \sigma_{a_k} \sim \emptyset$.

Now suppose $\dim V = \infty$, and suppose $\sigma_{a_1} \ldots \sigma_{a_k} = 1$. Let $U = \langle a_1, \ldots, a_k \rangle$. By Lemma 1.1, there exists a finite-dimensional subspace T with $U \subseteq T$, and $V = T + T^\perp$. Now the theorem is valid in T, and by Lemma 1.2, it is therefore valid in V.

In special cases, we can sharpen the results of Theorem 6.1 to obtain

COROLLARY 6.2. (1) *If* $\operatorname{rad} V = 0$, *then every relation in* O^* *is a consequence of fundamental relations of types* (a) *and* (c).

(2) *If* $Q \operatorname{rad} V = 0$, *then every relation is a consequence of fundamental relations of types* (a), (b) *and* (c).

(3) *If* $\operatorname{rad} V = Q \operatorname{rad} V$, *then every relation is a consequence of fundamental relations of types* (a), (c) *and* (d). *If in addition*, $\dim Q \operatorname{rad} V \leqq 1$, *then we only require types* (a) *and* (c).

7. A class of subgroups of $0^*(V, Q)$. In the following, let $\bar{V}$ be a fixed subspace of V. Define

$$T := \{\sigma_x \in S \mid x \in \bar{V}\}.$$

Let G be the orthogonal group generated by T and call (G, T) the group associated with the pair $V, \bar{V}$. We show that a theorem similar to 6.1 holds for the group (G, T).

Again, we first consider the case that $\dim V < \infty$. For any $\alpha \in G$, we let $\bar{\alpha}$ be the restriction of α to $\bar{V}$, and let $\bar{G} := \{\bar{\alpha} \mid \alpha \in G\}$. With this notation, we have

LEMMA 7.1. *The mapping*

$$\tau : \begin{cases} G \to \bar{G} \\ \alpha \mapsto \bar{\alpha} \end{cases}$$

is a surjective homomorphism. If $\alpha \in \ker \tau$, *then* $B(\alpha) \subseteq \text{rad } \bar{V}$. *If* (V, Q) *is regular, then* $\ker \tau = \{\alpha \in G \mid B(\alpha) \subseteq \text{rad } \bar{V}\}$.

The proof of this given in [**19**] holds in the case of general characteristic. Indeed, the last part of the lemma can easily be shown to be true under the weaker condition that $\text{rad } V \cap \bar{V} = \{0\}$.

We next state

(7.2) *Let* $\sigma_x \in T$. *Then* $\bar{\sigma}_x = \bar{1}$ *if and only if* $x \in \text{rad } \bar{V}$.

The proof of this is left to the reader.

From 7.2, we see that the set $T^* := \{\bar{\sigma}_x \mid \sigma_x \in T \text{ and } x \notin \text{rad } \bar{V}\}$ is a system of generators for the group $\bar{G}$. We remark that $\bar{G}$ is the group $0^*(\bar{V}, \bar{Q})$, where $\bar{Q} := Q|_{\bar{V}}$. For the pair $(\bar{G}, T^*)$, we can invoke Theorem 6.1 to state

(7.3) *The relations between elements of* T^* *of types* (a), (b), (c) *and* (d) (*as described in* § 2) *form a system of defining relations for the group* $\bar{G}$.

8. The relation theorem for (G, T). We first sharpen the concept of equivalence discussed in § 3. If

$$\sigma_{a_1} \ldots \sigma_{a_m}, \sigma_{b_1} \ldots \sigma_{b_k} \in T,$$

we write

$$\sigma_{a_1} \ldots \sigma_{a_m} \approx \sigma_{b_1} \ldots \sigma_{b_k}$$

if the two words are equivalent, and if the fundamental relations whose insertion or deletion transforms the first into the second all have associated subspaces that lie in $\bar{V}$. Clearly

$$\sigma_{a_1} \ldots \sigma_{a_m} \gtrsim \sigma_{b_1} \ldots \sigma_{b_k}$$

implies

$$\sigma_{a_1} \ldots \sigma_{a_m} \approx \sigma_{b_1} \ldots \sigma_{b_k}.$$

(8.1) *Let* $\bar{V}$ *be an isotropic subspace of* V, *and let* (G, T) *be the group associated with* V, $\bar{V}$ (*as described in* § 7). *Then the fundamental relations of length* 2, 3 *and* 4 *between elements of* T *form a defining system of relations for* G.

Proof. Since $\bar{V}$ is isotropic, we have $\sigma_x\sigma_y \approx \sigma_y\sigma_x$ for all $x, y \in T$. We first show that 8.1 is valid in case $Q \text{ rad } \bar{V} = 0$.

(a) Suppose $Q \text{ rad } \bar{V} = 0$. Then $\bar{V}$ contains no non-trivial singular vectors. We now proceed by induction on $\dim \bar{V}$. If $\dim \bar{V} \leqq 2$, then the result is an immediate consequence of 4.2. We now assume that $\dim \bar{V} \geqq 3$.

Now let $\sigma_{a_1} \ldots \sigma_{a_k} = 1$ be some given relation, and let H be some hyperplane of $\bar{V}$. Since any $\sigma_{a_i}, \sigma_{a_j}$ commute, we may assume that $a_1, \ldots, a_l \in H$, and that $a_{l+1}, \ldots, a_k \notin H$ for some suitable l with $0 \leqq l \leqq k$. If $l = k$, our claim follows by the induction hypothesis. Clearly, $k - l = 1$ cannot happen as otherwise $B(\sigma_{a_1} \dots \sigma_{a_k}) \neq 0$. Hence we must have $k - l \geqq 2$. We may assume $\sigma_{a_l} \neq \sigma_{a_{l+1}}$. But then

$$\dim (\langle a_{l+1}, a_{l+2} \rangle \cap H) = 1, \quad \text{say}$$
$$\langle a_{l+1}, a_{l+2} \rangle \cap H = \langle a_{l+1}' \rangle.$$

If $a_{l+1}' \in \operatorname{rad} V$, then $\sigma_{a_{l+1}}\sigma_{a_{l+2}} \approx \sigma_{a_{l+2}'}$; if $a_{l+1}' \notin \operatorname{rad} V$, then

$$\sigma_{a_{l+1}}\sigma_{a_{l+2}} \approx \sigma_{a_{l+1}'}\sigma_{a_{l+2}'}.$$

We can repeat this procedure to obtain $\sigma_{a_1} \dots \sigma_{a_k} \approx \sigma_{b_1} \dots \sigma_{b_m}$, where $b_1, \dots, b_{m-1} \in H$. But as noted above, this implies $b_m \in H$, and hence the result follows from the induction hypothesis.

(b) Now suppose $\dim Q \operatorname{rad} \bar{V} = n$. Suppose that the result 8.1 has been proved if $\dim Q \operatorname{rad} \bar{V} = n - 1$. Clearly, $n < \dim \bar{V}$ as otherwise there is nothing to prove, and hence there exists a hyperplane H of $\bar{V}$ such that

$$\dim (H \cap Q \operatorname{rad} \bar{V}) = n - 1.$$

If in addition $Q \operatorname{rad} \bar{V} \cap \operatorname{rad} V \neq 0$, we may choose this hyperplane H such that there exists a vector $r \in (Q \operatorname{rad} \bar{V} \cap \operatorname{rad} V)\backslash H$ (This can be done as we may exclude any given vector r from H, by just taking a complement of r in $\bar{V}$).

Now, suppose $\sigma_{a_1} \dots \sigma_{a_k} = 1$. As in (a), we may assume that the a_i's have been ordered so that $a_1, \dots, a_l \in H$, and $a_{l+1}, \dots, a_k \notin H$, for some l, $0 \leqq 1 \leqq k$. We proceed by induction on $v = k - 1$. For $v = 0 (k = 1)$, the result holds. If $v \neq 0$, then again $v \geqq 2$. If a_{l+1}, a_{l+2} are linearly dependent, then $\sigma_{a_{l+1}} = \sigma_{a_{l+2}}$, and the result holds by the induction hypothesis on v. If a_{l+1}, a_{l+2} are linearly independent, we let

$$\langle a_{l+1}' \rangle = H \cap \langle a_{l+1}, a_{l+2} \rangle.$$

If a_{l+1}' is non-singular, we proceed as in (a) to deduce the result. Thus we may assume that a_{l+1}' is singular, and that $\langle a_{l+1}, a_{l+j} \rangle$ is singular for all j. Thus, we may assume that the subspace $H \cap L$ is singular, where $L = \langle a_{l+1}, \dots, a_k \rangle$. Now $\sigma_{a_1} \dots \sigma_{a_k} = 1$, and so

$$\sigma_{a_1} \dots \sigma_{a_l} = \sigma_{a_{l+1}} \dots \sigma_{a_k} = \alpha$$

where $B(\alpha) \subseteq H \cap L$ is singular. We may assume that $\alpha \neq 1$, as otherwise we are done by induction.

We now consider two cases.

Case 1. $B(\alpha) \subsetneqq Q \operatorname{rad} V$. In this case, there exists $s \in B(\alpha)$ such that $s \notin \operatorname{rad} V$ (by Lemma 2.2). Choose $a \in V$ such that $\alpha(a) = a + s$. If $B(\alpha)$ were contained in $a^\perp$, we would have $f(\alpha(x) + x, a) = 0$ for all $x \in V$, implying

$$f(\alpha(x), a) = f(x, a) = f(\alpha(x), \alpha(a)) = f(\alpha(x), a + s)$$
$$= f(\alpha(x), a) + f(\alpha(x), s) \quad \text{for all } x \in V,$$

and hence $f(\alpha(x), s) = 0$ for all $x \in V$. But this would imply that $s \in \text{rad } V$, contrary to the choice of s. Hence $B(\alpha) \subsetneq a^{\perp}$, and so there exists $t \in B(\alpha)$ with $f(a, t) = 1$. From $Q(\alpha(a)) = Q(a)$, we see that $f(a, s) = 0$. Hence $\langle s, t \rangle \cap \text{rad } V = 0$. Now let $\tau(x) = x + f(x, s)t + f(x, t)s$, and consider $\alpha^* = \alpha\tau$. Since $F(\tau) = \langle s, t \rangle^{\perp}$, and since $\langle s, t \rangle \subseteq B(\alpha)$ implies $F(\alpha) \subseteq B(\alpha)^{\perp} \subseteq \langle s, t \rangle^{\perp} = F(\tau)$, we see that $F(\alpha) \subseteq F(\alpha\tau)$. Also $\alpha\tau(a) = \alpha(a + s) = a + s + s = a$. Hence $F(\alpha\tau) \neq F(\alpha)$. Now, using 2.9 and 4.10 we see that for any σ_{h_1} with $h_1 \in \langle a_1, \ldots, a_l \rangle$, and σ_{b_1} with $b_1 \in L$, we can find σ_{h_i}, σ_{b_i} $(i = 2, 3, 4)$, $h_i \in \langle a_1, \ldots, a_l \rangle \subseteq H$, $b_i \in L$ such that

$$\tau = \sigma_{h_1}\sigma_{h_2}\sigma_{h_3}\sigma_{h_4} = \sigma_{b_4}\sigma_{b_3}\sigma_{b_2}\sigma_{b_1} \quad \text{and} \quad \sigma_{h_1}\sigma_{h_2}\sigma_{h_3}\sigma_{h_4}\sigma_{b_1} \gtrsim \sigma_{b_4}\sigma_{b_3}\sigma_{b_2}.$$

Thus

$$\sigma_{a_1} \ldots \sigma_{a_k} \approx \sigma_{a_1} \ldots \sigma_{a_l}\sigma_{h_1}\sigma_{h_2}\sigma_{h_3}\sigma_{h_4}\sigma_{b_4}\sigma_{b_3}\sigma_{b_2}\sigma_{b_1}\sigma_{a_{l+1}} \ldots \sigma_{a_k},$$

where

$$\alpha^* = \sigma_{a_1} \ldots \sigma_{a_l}\sigma_{h_1}\sigma_{h_2}\sigma_{h_3}\sigma_{h_4}, \quad \text{and} \quad F(\alpha^*) \supset F(\alpha),\ F(\alpha^*) \neq F(\alpha).$$

If $B(\alpha^*) \subsetneq \text{rad } V$, we may continue this process of increasing $F(\alpha^*)$ until we either reach the case that $\alpha^*|_H = 1$ and we have reduced the case to the induction hypothesis, or the case given by

Case 2. $B(\alpha) \subseteq \text{rad } V$. In this case, let $B(\alpha) = \langle s_1, \ldots, s_p \rangle$, so that

$$\alpha(x) = x + f(x, c_1)s_1 + \ldots + f(x, c_p)s_p,$$

where $c_1, \ldots, c_p \in L \backslash \text{rad } V$. We treat two cases.

(α) c_1 is non-singular. Let $\lambda = Q(c_1)$. Let $\alpha^* = \alpha\sigma_{c_1}\sigma_{c_1+\lambda s_1}$. Then

$$\sigma_{c_1}\sigma_{c_1+\lambda s_1}(x) = x + f(x, c_1)s_1,$$

and so

$$\alpha\sigma_{c_1}\sigma_{c_1+\lambda s_1}(x) = x + f(x, c_2)s_2 + \ldots + f(x, c_p)s_p.$$

Here again $F(\alpha^*) \supseteq F(\alpha)$, $F(\alpha^*) \neq F(\alpha)$. Now, by our choice of H, we know there exists $r \in Q \text{ rad } V \cap \bar{V}$, with $r \subsetneq H$. Now, $\langle c_1, r \rangle \cap H = \langle b^* \rangle$, and so $b^* = \xi c_1 + \zeta r$. By a suitable norming, we can replace b^* by $b = r + c_1$, $b \in H$, $Q(b) = Q(c_1)$. We let $b' = b + \lambda s_1$. Then $Q(b') = Q(b)$. Now $\sigma_b\sigma_{b'}\alpha = \alpha^*$, and hence we have

$$\sigma_{a_1} \ldots \sigma_{a_k} \approx \sigma_{a_1} \ldots \sigma_{a_l}\sigma_{c_1}\sigma_{c_1+\lambda s_1}\sigma_b\sigma_{b'}\sigma_{a_{l+1}} \ldots \sigma_{a_k},$$

where again we have increased the dimension of the fix, i.e.

$$F(\alpha^*) \supseteq F(\alpha),\ F(\alpha^*) \neq F(\alpha).$$

(β) c_1 is singular. We choose $\sigma_{c'}$, $c' \in L$. By 2.8, there exist c'', d', $d'' \in \langle c', c_1, s_1 \rangle \subseteq L$ such that $\sigma_{c'}\sigma_{c''}\sigma_{d'}\sigma_{d''}(x) = x + f(x, c_1)s_1$. But then let $\alpha^* = \alpha\sigma_{c'}\sigma_{c''}\sigma_{d'}\sigma_{d''}$.

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Now, let $\langle c_1, r\rangle \cap H = \langle e_1\rangle$ so that $e_1 = c_1 + r$ (after norming). Since $f(x, c_1) = f(x, e_1)$, we see that $\alpha(x) = x + f(x, e_1)s_1 + \ldots + f(x, c_p)s_p$.

We now repeat the procedure outlined above, only we work in H to obtain $\alpha^* = \sigma_{e'}\sigma_{e''}\sigma_{f'}\sigma_{f''}\alpha$, with $e', e'', f', f'' \in H$. By using 4.10 and 4.8, we get

$$\sigma_{c'}\sigma_{c''}\sigma_{d'}\sigma_{d''}\sigma_{e'}\sigma_{e''}\sigma_{f'}\sigma_{f''} \gtrsim \emptyset,$$

and hence

$$\sigma_{a_1} \ldots \sigma_{a_k} \gtrsim \sigma_{a_1} \ldots \sigma_{a_l}\sigma_{c'}\sigma_{c''}\sigma_{d'}\sigma_{d''} \,.\, \sigma_{e'}\sigma_{e''}\sigma_{f'}\sigma_{f''}\sigma_{a_{l+1}} \ldots \sigma_{a_k},$$

where the fix has been increased, i.e. $F(\alpha^*) \supseteq F(\alpha)$, $F(\alpha^*) \neq F(\alpha)$.

Thus we see that by repeating this process, we finally obtain

$$\sigma_{a_1} \ldots \sigma_{a_k} \gtrsim \sigma_{y_1} \ldots \sigma_{y_s}\sigma_{z_1} \ldots \sigma_{z_t} \text{ where } y_i \in H,\ z_i \in L, \text{ and}$$

$$\alpha^{**} = \sigma_{y_1} \ldots \sigma_{y_s} = \sigma_{z_1} \ldots \sigma_{z_t} = 1.$$

Now, by induction hypothesis, we have

$$\sigma_{y_1} \ldots \sigma_{y_s} \approx \emptyset,\ \sigma_{z_1} \ldots \sigma_{z_t} \approx \emptyset,$$

and hence $\sigma_{a_1} \ldots \sigma_{a_k} \approx \emptyset$, as claimed. This completes the proof of 8.1.

We now let $\bar{V}$ be any subspace of V. We define G, T and T^* as in § 7, and state

(8.2) *Let* $\bar{\sigma}_a, \bar{\sigma}_b, \bar{\sigma}_c, \bar{\sigma}_d \in T^*$. *Then we have*

(i) $\bar{\sigma}_a\bar{\sigma}_b = \bar{1}$ *implies* $\sigma_a\sigma_b = 1$.

(ii) *If* a, b, c, d *are linearly dependent, then* $\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_c\bar{\sigma}_d = \bar{1}$ *implies* $\sigma_a\sigma_b\sigma_c\sigma_d = 1$.

(iii) $\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_c = \bar{1}$ *implies that either* $\sigma_a\sigma_b\sigma_c = 1$ *or, that* $\sigma_a\sigma_b\sigma_c = \sigma_x$ *with* $x \in$ rad $\bar{V}\backslash Q$ rad $\bar{V}$.

(iv) *If* a, b, c *are linearly independent, and if* $\langle a, b, c\rangle \cap$ rad $\bar{V}$ *is singular, then* $\bar{\sigma}_a\bar{\sigma}_b\bar{\sigma}_c\bar{\sigma}_d = 1$ *implies that*

$$\dim B(\sigma_a\sigma_b\sigma_c\sigma_d) \leqq 2, \quad \textit{and} \quad B(\sigma_a\sigma_b\sigma_c\sigma_d) \subseteq Q \text{ rad } \bar{V}.$$

The proof is an easy consequence of the results in § 2 and § 7.

Now suppose $\sigma_{a_1} \ldots \sigma_{a_k} = 1$ is some given relation between elements of T. If for $\sigma_x \in T$ we have $x \in$ rad $\bar{V}$, then we have $\sigma_x\sigma_y \approx \sigma_y\sigma_x$ for all $\sigma_y \in T$. Hence we can assume that in the relation given above, we have ordered the σ_i's so that $a_1, \ldots, a_l \notin$ Rad $\bar{V}$, and $a_{l+1}, \ldots, a_k \in$ rad $\bar{V}$ for some suitable l with $0 \leqq l \leqq k$. Then of course $\bar{\sigma}_{a_1} \ldots \bar{\sigma}_{a_l} = \bar{1}$. By (7.3), this relation between elements of T^* is a consequence of the basic relations of length 2, 3 or 4 between elements of T^*. Thus, we may now use 8.2 to write

$$\sigma_{a_1} \ldots \sigma_{a_k} \approx r_1 \ldots r_t\sigma_{y_1} \ldots \sigma_{y_m},$$

where each $y_i \in$ rad $\bar{V}$, and where each r_i is a relation between elements of T^* of length 2, 3 or 4.

Now choose some fixed $\sigma_{x_0} \in T$. If $r_i = \sigma_x\sigma_x$ for some $\sigma_x \in T$, then $r_i \approx$

$\sigma_{x_0}\sigma_{x_0}$ with $r_i\sigma_{x_0}\sigma_{x_0} \approx \emptyset$. If $r_i = \sigma_x\sigma_y\sigma_z$, then by 4.7, $\sigma_x\sigma_y\sigma_z \approx \sigma_{x_0}\sigma_{y'}\sigma_{z'}$ with

$$\sigma_x\sigma_y\sigma_z\sigma_{z'}\sigma_{y'}\sigma_{x_0} \approx \emptyset.$$

Finally, if $r_i = \sigma_x\sigma_y\sigma_z\sigma_w$, then by 4.3, $\sigma_{x_0}r_i \approx \sigma_{y'}\sigma_{z'}\sigma_{w'}$. By 8.2, all the x', y', $w' \in \langle x_0 \rangle + Q$ rad $\bar{V}$. Thus we see that $\sigma_{a_1} \ldots \sigma_{a_k} \approx \sigma_{b_1} \ldots \sigma_{b_m}$, where $b_i \in \langle x_0 \rangle +$ rad $\bar{V}$, where of course, $\langle x_0 \rangle +$ rad $\bar{V}$ is an isotropic subspace of $\bar{V}$. Now we invoke 8.1 to conclude that $\sigma_{a_1} \ldots \sigma_{a_k} \approx \emptyset$. Thus we have proved the following theorem in the case dim $V < \infty$.

THEOREM 8.3. *Let $\bar{V}$ be any subspace of a metric vector space (V, Q), V being a vector space over a field K of characteristic 2, with $K \neq GF(2)$. Let T be the set of simple isometries $\sigma_x \in S$ for which $x \in \bar{V}$, and let G be the group generated by T. Then every relation in G between elements of T is a consequence of fundamental relations of length 2, 3 or 4 between elements of T.*

To prove that 8.3 also holds when dim $V = \infty$ follows along the same lines as the proof of 6.1.

Let $\sigma_{a_1} \ldots \sigma_{a_k}$ be a relation between elements of T. Let $U = \langle a_1, \ldots, a_k \rangle$. By 1.1, there exists some finite-dimensional subspace A of V with $U \subseteq A$ and $A + A^\perp = V$. Now Theorem 8.3 holds for A and U instead of V and $\bar{V}$, and then Lemma 1.1 (b) gives the desired result.

As in Corollary 6.2, we can now sharpen the Theorem in special cases. However, we omit the detailed discussion of these cases.

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Memorial University,
Corner Brook, Newfoundland;
Technische Hochschule,
Darmstadt, W. Germany