Can. J. Math. Vol. **46** (2), 1994 pp. 324–342

CHINBURG'S THIRD INVARIANT IN THE FACTORISABILITY DEFECT CLASS GROUP

D. HOLLAND

ABSTRACT. Chinburg's third invariant $\Omega(N/K,3) \in \mathrm{Cl}(Z[\Gamma])$ of a Galois extension N/K of number fields with group Γ is closely related to the Galois structure of unit groups and ideal class groups, and deep unsolved problems such as Stark's conjecture.

We give a formula for $\Omega(N/K,3)$ modulo $D(Z\Gamma)$ in the *factorisability defect class group*, reminiscent of analytic class number formulas. Specialising to the case of an absolutely abelian, real field N, we give a natural conjecture in terms of Hecke factorisations which implies the vanishing of the invariant in the defect class group.

We prove this conjecture when N has prime-power conductor using Euler systems of cyclotomic units, Ramachandra units and Hecke factorisation. This supports a general conjecture of Chinburg, which in our situation specialises to the statement that $\Omega(N/K,3) = 0$ for such extensions.

We also develop a slightly extended version of Euler systems of units for general abelian extensions, which will be applied to abelian extensions of imaginary quadratic fields elsewhere.

1. **Introduction.** In this paper we investigate (the image of) Chinburg's third invariant $\Omega(N/K,3)$ (defined in [C1]) of a Galois extension N/K of number fields, in a certain factorisability defect class group. The results might be regarded as the 'multiplicative' analogue of those obtained in [H-W3] (see also [H1]) for the 'additive' invariant $\Omega(N/K,2)$. Thus, this paper and [H1, H-W3] together provide a model for applying the defect class group to obtain refined information about invariants in class groups of integral group rings.

The results here concerning $\Omega(N/K,3)$ only apply in a restricted number of cases (as far as explicit computation of the invariant goes), because of their intimate relation to Stark's conjecture, and Kolyvagin's Euler systems of units, about which there is little general information at the present time. Nonetheless, the method is open to much greater (though conjectural) generalisation.

In the first three §§, a module-theoretic formula for the invariant (for arbitrary N/K) is computed. This formula involves a formal (permutation) lattice, the units and the ideal class group of N as modules (plus a *Hecke cohomology* class (see §2), which is often zero, for example if N is totally real)—each term is independent of the choices made in defining $\Omega(N/K,3)$.

Received by the editors June 16, 1992.

AMS subject classification: Primary 11R33; secondary 11R65, 16E30, 11R18.

Key words and phrases: factorisability, defect class group, Euler systems, multiplicative Galois structure, Chinburg conjecture

In §4 we put the formula into a form suitable for relating to 'canonical factorisation' and 'Euler systems of units'. Our combination of these methods is the main new technique of this paper.

Canonical factorisation (or canonical factor equivalence) now features in different forms in many papers: we refer the interested reader to [B2], where it appeared for the first time.

In §5 we give a summary of the (well-known) basic properties of Kolyvagin's Euler systems of units: we also extend the Euler systems to intermediate fields, to overcome technical problems, and include the prime 2.

The basic aim is to use Euler systems to put the formula for $\Omega(N/K, 3)$ from §3 into a computable form. This approach requires a class-number formula involving special units. There are other limitations on this method, however: *cf.* [Ru2], where the connections with Stark's conjecture ([St], see also [T], Chapter I, Conjecture 5.1) are explored.

In §6, the case N totally real and $N/\mathbf{Q}$ abelian is considered. We make a natural conjecture which implies (in these cases) the vanishing of $\Omega(N/K, 3)$ in the defect class group. We are able to prove the conjecture in the *primary* case (see §6 for details). Further examples verifying the conjecture for non-primary real abelian $N/\mathbf{Q}$ will appear elsewhere.

We shall treat totally imaginary abelian extensions of $\mathbf{Q}$ and abelian extensions of quadratic imaginary fields elsewhere.

For a different approach, the reader can compare this work with [B1, F1], where parametrizations of Galois structure are obtained from a character function with values which are the quotient of Fröhlich's integral regulator by an L-function value at zero, for abelian extensions of $\mathbf{Q}$ or a quadratic imaginary field. (Finer results, on local freeness of units modulo torsion, for real abelian extensions, were obtained in [F2].)

The results of [B1, F1] are parallel to ours because the character functions they use are factorisations of Fröhlich's module defect functions, rather than the functions naturally occurring from factorisability defect relative groups in our approach. This leads to different module-theory invariants, even though the classifying groups are isomorphic. Another difference is that the integral regulator and L-value at zero appear as the main objects of study, rather than Chinburg's invariant: in our approach the L-values appear more indirectly via the analytic class number formula of Ramachandra, but are still an essential ingredient.

A. Fröhlich tells me that he is preparing an extension of his earlier approach, using the module defect for arbitrary finite groups, and monomial representations, in a somewhat similar way to the construction of factorisability defect groups of [H-W1, H-W2], which may clarify the connections with our approach.

ACKNOWLEDGEMENTS. I have benefitted greatly from conversations (sometimes by electronic mail) with David Burns, Ted Chinburg, Manfred Kolster, Victor Snaith, David Solomon, Larry Washington and Steve Wilson, and it is a pleasure to acknowledge their assistance here. Many thanks are due to the referee whose constructive suggestions allowed me to make an opaque first version of this paper (I hope) more transparent.

2. **Chinburg's third invariant in a defect class group.** Suppose that Γ is a finite group and $\mathbf{Z}[\Gamma]$ its integral group ring.

We shall use the factorisability defect class group $\mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$ of the category $\mathcal{D}$ of finitely generated $\mathbf{Z}[\Gamma]$-modules to house our invariants. We briefly recall the details from [H-W1] and [H-W2]. Let V be the Γ-set $\cup_\Delta \Gamma/\Delta$, disjoint union over all subgroups of Γ, $\Lambda_V = \mathrm{End}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[V])$ and $\mathrm{FD} = \mathcal{G}_0^t(\Lambda_V)$ be the Grothendieck group of finite Λ_V-modules, taken with respect to exact sequences. We define the compensated category $\mathcal{D}_{\mathrm{fd}} = \mathcal{D} \times \mathrm{FD}$, an *extensional category* in the sense of Heller ([He]): the extensional structure consists of exact sequences in $\mathcal{D}$ paired with triples of morphisms in FD: we only allow zero and identity morphisms in FD.

Let $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$ be the Grothendieck group of the extensional category $\mathcal{D}_{\mathrm{fd}}$. We write $[M] + [a]$ for the class in $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$ of the object $(M, a) \in \mathcal{D}_{\mathrm{fd}}$. We may use the relation in $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$ implied by a short exact sequence $E = (0 \longrightarrow A \longrightarrow B \xrightarrow{\alpha} C \longrightarrow 0)$ in $\mathcal{D}$ provided we compensate by adding its *factorisability defect* $\mathrm{fd}(E) \in \mathrm{FD}$. That is, we have the equality

$$[A] + [C] - [B] = [\mathrm{fd}(E)] \in \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}}), \tag{2.1}$$

where $\mathrm{fd}(E)$ will be defined below. For this reason we use the following "diagram convention": we refer to a diagram

$$0 \longrightarrow A \longrightarrow B \xrightarrow{\alpha} C \longrightarrow 0 \quad [\mathrm{fd}(E)]$$

consisting of a short exact sequence on the left and its factoring defect class (in $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$, not FD) on the right, as a *relation*. When we refer to the diagram, the reader is to understand this as a mnemonic for the relation 2.1 in $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$.

Denote by $H^n_\Gamma(-)$ the homology of the derived functors of

$$- \bigotimes_{\mathbf{Z}[\Gamma]} \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}[V], \mathbf{Z}) \text{ and } \mathrm{Hom}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[V], -).$$

Note that, just as with Tate cohomology groups, we allow n to run over all of $\mathbf{Z}$. These *Hecke cohomology* groups are the groups denoted $\mathrm{Ext}^n_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[V], -)$ defined in [H2], 2.1. We have

$$\mathrm{fd}(E) = \mathrm{coker}\big(\alpha^0_\Gamma \colon H^0_\Gamma(B) \longrightarrow H^0_\Gamma(C)\big). \tag{2.2}$$

An endomorphism of $\mathbf{Z}[V]$ induces an endomorphism of a projective resolution, where maps are determined up to homotopy, and so induces endomorphisms of the Hecke cohomology groups in the standard way. By homological algebra, this gives a natural Λ_V-module action on the Hecke cohomology groups.

If we project onto $\Lambda_{\Gamma/\Delta}$-modules, we recover the ordinary Tate cohomology groups (over a subgroup Δ of Γ) as $\Lambda_{\Gamma/\Delta}$-modules.

Since it is easier to work in

$$\mathrm{FDS}(\Gamma) = \oplus_\Delta \mathcal{G}_0^t(\Lambda_{\Gamma/\Delta}) \tag{2.3}$$

in practise, where the sum is over all subgroups Δ of Γ, I give a few remarks on the relations between these groups.

Let fds be the corresponding defect map. Then fd is equivalent to fds (in the sense of [H-W1] §1).

FD has the formal advantage of induction (and restriction and inflation) being well-defined $\mathrm{FD}(\Delta) \longrightarrow \mathrm{FD}(\Gamma)$ in such a way as to extend ordinary induction on $\mathbf{Z}[\Delta]$-modules: formally, these change of group maps on the defect groups FD turn the corresponding functors into defect preserving functors ([H-W2], 1.10). For example, if E is a short exact sequence of $\mathbf{Z}[\Delta]$-modules, we have the equation

$$\mathrm{fd}_\Gamma\big(\mathrm{ind}_\Delta^\Gamma(E)\big) = \mathrm{ind}_\Delta^\Gamma\big(\mathrm{fd}_\Delta(E)\big). \tag{2.4}$$

Unfortunately these results do not immediately extend to the group FDS. However, using the idempotent of Λ_V corresponding to projection onto $\Lambda_{\Gamma/\Delta}$, we obtain an injective homomorphism

$$\mathrm{FD}(\Gamma) \longrightarrow \mathrm{FDS}(\Gamma)$$

([H-W2] 1.8(i, ii), see also 1.3). So we might for instance replace $\mathrm{FD}(\Gamma)$ by its image in $\mathrm{FDS}(\Gamma)$, to ensure that induction is well-defined, and yet we only have to consider one subgroup at a time, that is $\Lambda_{\Gamma/\Delta}$-modules. We take this approach occasionally for the purposes of calculation.

Finally, $\mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$ is the kernel of the localisation (completion) map from $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$ to $\prod_p \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}_p})$. Here, and in the rest of the paper, subscript p denotes p-adic completion. Let $\mathrm{Cl}(\mathbf{Z}[\Gamma])$ be the locally free class group of $\mathbf{Z}[\Gamma]$. From [H-W2], 2.3, there is a natural surjection $\mathrm{Cl}(\mathbf{Z}[\Gamma]) \longrightarrow \mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$ (by $[X]-[Y] \longmapsto [X]-[Y]$, with X, Y locally free of equal rank) with kernel $D(\mathbf{Z}[\Gamma])$, the *kernel group*. Let us apply this to obtain a preliminary formula for Chinburg's third invariant in $\mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$. First we introduce some notation to be used in the rest of this paper.

Let N/K be a normal extension of number fields with Galois group Γ. Let $S = S(N)$ be a finite, Γ-stable set of places of N including the infinite places $S_\infty(N)$ and the ramified places in N/K, and such that the $S(F)$-class number (where $S(F)$ consists of the places of F below those in $S(N)$) of intermediate fields F between N and K is 1. Let $\mathbf{Z}[S]$ be the free abelian group on the set S and let X_S be the kernel of the $\mathbf{Z}[\Gamma]$-map from $\mathbf{Z}[S]$ to $\mathbf{Z}$ which sends each place in S to 1. Let U_S be the $\mathbf{Z}[\Gamma]$-module of S-units in N.

Also let σ_Γ be the trace element of $\mathbf{Z}[\Gamma]$ and $J_\Gamma = (\sigma_\Gamma - |\Gamma|)\mathbf{Z}[\Gamma]$, a two-sided ideal contained in the augmentation ideal I_Γ of $\mathbf{Z}[\Gamma]$. The defining exact sequence for Chinburg's third invariant $\Omega(N/K, 3)$ ([C1], Lemma 3.1 or [T], I.5.1) is

$$0 \longrightarrow U_S \longrightarrow A_3 \longrightarrow B_3 \longrightarrow X_S \longrightarrow 0. \tag{2.5}$$

This sequence has extension class induced from that of the *fundamental class* for N/K (in the sense of class field theory), as described in [T], I.5.1. In formula 2.5 A_3 and B_3 are finitely generated cohomologically trivial $\mathbf{Z}[\Gamma]$-modules. As such they determine classes in $\mathcal{K}_0(\mathbf{Z}[\Gamma])$ by resolution by projectives. By the Dirichlet S-unit theorem, $A_3 \otimes \mathbf{Q}$ and

$B_3 \otimes \mathbf{Q}$ are isomorphic, so that $[A_3] - [B_3]$ lies in $\mathrm{Cl}(\mathbf{Z}[\Gamma])$. So we define, following Chinburg ([C3], Definition 3.1)

$$\Omega(N/K,3) = [A_3] - [B_3] \in \mathrm{Cl}(\mathbf{Z}[\Gamma]). \tag{2.6}$$

Tensor with X_S the exact sequences of $\mathbf{Z}[\Gamma]$-modules (of the opposite hand) $I_\Gamma \longrightarrow \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}$ to obtain the relation

$$0 \longrightarrow X_S \otimes I_\Gamma \longrightarrow X_S \otimes \mathbf{Z}[\Gamma] \longrightarrow X_S \longrightarrow 0 \quad [H^0_\Gamma(X_S)] \tag{2.7}$$

because $H^0_\Gamma(\mathbf{Z}[\Gamma]) = 0$. One can find an exact sequence 2.5 with the required extension class in which $B_3 = X_S \otimes \mathbf{Z}[\Gamma]$. This choice gives the relation

$$0 \longrightarrow U_S \longrightarrow A_3 \longrightarrow X_S \otimes I_\Gamma \longrightarrow 0 \quad \text{zero} \tag{2.8}$$

because $[H^1_\Gamma(U_S)] = 0$ as in [T], II.6.8. Here we have used formula 2.2 as a working definition of fd, combined with the Hecke cohomology long exact sequence.

Thus we obtain (using the projection $\mathrm{Cl}(\mathbf{Z}[G]) \to \mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$)

$$\begin{aligned} \Omega(N/K,3) &\overset{2.6}{=\!=} [A_3] - \big[X_S \otimes \mathbf{Z}[\Gamma]\big] \\ &\overset{2.8}{=\!=} [U_S] + [X_S \otimes I_\Gamma] - \big[X_S \otimes \mathbf{Z}[\Gamma]\big] \\ &\overset{2.7}{=\!=} [U_S] - [X_S] + [H^0_\Gamma(X_S)] \in \mathrm{Cl}(\mathcal{D}_{\mathrm{fd}}). \end{aligned} \tag{2.9}$$

Recall that $\mathrm{Cl}(\mathcal{D}_{\mathrm{fd}}) \cong \mathrm{Cl}(\mathbf{Z}[\Gamma])/D(\mathbf{Z}[\Gamma])$, and that $D(\mathbf{Z}[\Gamma])$ lies in the kernel of the Cartan map $h\colon \mathcal{K}_0(\mathbf{Z}[\Gamma]) \longrightarrow \mathcal{G}_0(\mathbf{Z}[\Gamma])$. So formula 2.9 sharpens 8.2 of [C1]:

$$\Omega(N/K,3) = [U_S] - [X_S] \in \mathcal{G}_0(\mathbf{Z}[\Gamma]).$$

In a similar way, an "amusing formula" of Chinburg, which he interprets as an analogue for Galois structure of analytic class number formulae, provides the motivation for the work of the next section—see in particular the discussion after formula 3.15.

Note that we use the same symbol for the image of $\Omega(N/K,3)$ in different classifying groups, for simplicity of notation: as we specify which group, this won't cause confusion.

3. **Switching from S-units to units.** We continue the analysis by switching from S-units to ordinary units.

We write U and X for U_S and X_S where S contains only the infinite places, S_f for the finite places in S, I_{S_f} for the fractional ideals of N supported on the places in S_f and $\mathcal{P}_{S_f}$ for the principal ideals in I_{S_f}.

Thus we have relations

$$0 \longrightarrow U \longrightarrow U_S \longrightarrow \mathcal{P}_{S_f} \longrightarrow 0, \quad [H^1_\Gamma(U)] \tag{3.1}$$

because $H^1_\Gamma(U_S) = 0$, and

$$0 \longrightarrow X \longrightarrow X_S \longrightarrow \mathbf{Z}[S_f] \longrightarrow 0 \quad [H^1_\Gamma(X)] - [H^1_\Gamma(X_S)] \tag{3.2}$$

because the Hecke cohomology group $H^1_\Gamma(\mathbf{Z}[S_f])$ vanishes, and (by the choice of S), if Cl_N is the ideal class group of N,

$$0 \longrightarrow \mathcal{P}_{S_f} \longrightarrow I_{S_f} \longrightarrow \mathrm{Cl}_N \longrightarrow 0 \quad [H^1_\Gamma(\mathcal{P}_{S_f})] \tag{3.3}$$

because $I_{S_f} \cong \mathbf{Z}[S_f]$).

Then

(3.4)

$$\begin{aligned}[U_S] - [X_S] &= [U] - [X] - [\mathrm{Cl}_N] - [\mathrm{fd}(3.1)] + [\mathrm{fd}(3.2)] + [\mathrm{fd}(3.3)] \\ &= [U] - [X] - [\mathrm{Cl}_N] - [H^1_\Gamma(U)] + [H^1_\Gamma(X)] - [H^1_\Gamma(X_S)] + [H^1_\Gamma(\mathcal{P}_{S_f})].\end{aligned}$$

Let us denote by T^V the group $\mathrm{Map}_\Gamma(V, T)$ for a Γ-set T. This is naturally isomorphic to $\mathrm{Hom}_{\mathbf{Z}[\Gamma]}(\mathbf{Z}[V], T)$, hence a Λ_V-module. Then N^V is (as a ring) isomorphic to the direct sum of fields N^Δ for each subgroup Δ of Γ. Let O_{N^V} be the maximal order in N^V, and define $\mathrm{Cl}(N^V)$ to be the locally free class group of O_{N^V}. By the idèlic description of class groups, if $\mathcal{J}(N^V)$ denotes the idèles and $\mathcal{U}(O_{N^V})$ the unit idèles of N^V,

$$\begin{aligned}\mathrm{Cl}(N^V) &\cong \frac{\mathcal{J}(N^V)}{(N^V)^\times \mathcal{U}(O_{N^V})} \\ &\cong \frac{\mathcal{J}(N)^V}{(N^\times)^V \mathcal{U}(O_N)^V}.\end{aligned}$$

Because these isomorphisms are natural, $\mathrm{Cl}(N^V)$ becomes a Λ_V-module in a natural way. We can also define I_{N^V} as the invertible ideal group of O_{N^V}, namely

$$I_{N^V} = \frac{\mathcal{J}(N^V)}{\mathcal{U}(O_{N^V})} \cong \frac{\mathcal{J}(N)^V}{\mathcal{U}(O_N)^V}.$$

Hence I_{N^V} is a Λ_V-submodule of

$$I_N^V = \left(\frac{\mathcal{J}(N)}{\mathcal{U}(O_N)}\right)^V,$$

and

$$\mathrm{Cl}(N^V) \cong I_{N^V} / \mathcal{P}_{N^V}$$

where $\mathcal{P}_{N^V}$ is the group of principal invertible ideals of O_{N^V}, a Λ_V-submodule of $\mathcal{P}_N^V$. The same holds if we restrict to ideals supported in $S_f(N)$. Each of these objects gives, under the forgetful functor, modules over $\oplus_\Delta \Lambda_{\Gamma/\Delta}$, where each summand is the $\Lambda_{\Gamma/\Delta}$-module given by replacing V by Δ: thus $\mathrm{Cl}(N^V)$ maps to $\oplus_\Delta \mathrm{Cl}(N^\Delta)$.

It follows that we have the following equalities in $\mathrm{FD}(\Gamma)$:

$$\begin{aligned}[H^1_\Gamma(\mathcal{P}_{S_f(N)})] &= [(\mathrm{Cl}_N)^V] - [(I_{S_f(N)})^V / (\mathcal{P}_{S_f(N)})^V] \\ &= [(\mathrm{Cl}_N)^V] - [(I_{S_f(N)})^V / I_{S_f(N^V)}] \\ &\quad - [I_{S_f(N^V)} / \mathcal{P}_{S_f(N^V)}] + [(\mathcal{P}_{S_f(N)})^V / \mathcal{P}_{S_f(N^V)}] \\ &= [(\mathrm{Cl}_N)^V] - [(I_{S_f(N)})^V / I_{S_f(N^V)}] - [\mathrm{Cl}_{N^V}] + [H^1_\Gamma(U)]\end{aligned} \tag{3.5}$$

https://doi.org/10.4153/CJM-1994-016-0 Published online by Cambridge University Press

where the last equality comes from the analogues of formulas 3.1 and 3.3 with N replaced by N^Δ.

Let $w \in S_f(N^\Delta)$ and let $x|w$ be a place of N. Let Δ_x be its decomposition group in N/N^Δ of order d_w. As the decomposition groups are conjugate for different choice of x, d_w only depends on w. Also let $\mathrm{tr}_\Delta(w) = \sum_{x|w} x \in \mathbf{Z}[S_f]$.

From the element $H^0_\Gamma(\mathbf{Z}[S_f])$ of $\mathrm{FD}(\Gamma)$ we obtain $\Lambda_{\Gamma/\Delta}$-modules by the forgetful functor:

$$(3.6)\qquad [\hat{H}^0(\Delta, \mathbf{Z}[S_f])] = \Big[\bigoplus_{w\in S_f(N^\Delta)} (\mathbf{Z}/d_w\mathbf{Z})\,\mathrm{tr}_\Delta(w)\Big].$$

Let $e(N/N^\Delta, w)$ be the ramification index of w in the extension N/N^Δ). If we identify w with $e(N/N^\Delta, w)tr_\Delta(w)$ (to obtain the natural inclusion $I_{S_f(N^D)} \subset I_{S_f(N)}$), then we can define a $\Lambda_{\Gamma/\Delta}$-module

$$(3.7)\qquad e(N)(\Delta) \stackrel{\mathrm{def}}{=\!=} (I_{S_f(N)})^\Delta / I_{S_f(N^\Delta)} \cong \bigoplus_{w\in S_f(N^\Delta)} \big(\mathbf{Z}/e(N/N^\Delta, w)\mathbf{Z}\big)\,\mathrm{tr}_\Delta(w)$$

which does not depend on the choice of S_f, since S_f contains the ramified primes. We denote by e_N the corresponding element of $\mathrm{FD}(\Gamma)$, which does exist from the above remarks.

Let $f(N/N^\Delta, w)$ be the residue class degree of w in N/N^Δ. We get a $\Lambda_{\Gamma/\Delta}$-module $f_{S_f}(\Delta)$ by replacing $e(N/N^\Delta, w)$ by $f(N/N^\Delta, w)$ in the RHS of formula 3.6. From the equation $d_w = e(N/N^\Delta, w)f(N/N^\Delta, w)$ it follows that $\big[\hat{H}^0(\Delta, \mathbf{Z}[S_f])\big] = [e(N)(\Delta)] + [f_{S_f}(\Delta)] \in \mathcal{G}_0^t(\Lambda_{\Gamma/\Delta})$.

We can therefore define $f_{S_f} = \big[H^0_\Gamma(\mathbf{Z}[S_f])\big] - e(N) \in \mathrm{FD}(\Gamma)$.

Next we shall show that $[f_{S_f}] = 0$ in $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$. For this we need to do some preparatory work. There is a standard relation

$$(3.8)\qquad 0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}[\Gamma]/\sigma_\Gamma \longrightarrow 0 \quad \text{zero}$$

where the first map is multiplication by σ_Γ. Next consider the relation

$$(3.9)\qquad 0 \longrightarrow I_\Gamma \longrightarrow \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z} \longrightarrow 0 \quad [H^0_\Gamma(\mathbf{Z})].$$

If Γ is cyclic, then $I_\Gamma \cong \mathbf{Z}[\Gamma]/\sigma_\Gamma$, and so by formula 3.8

$$(3.10)\qquad [H^0_\Gamma(\mathbf{Z})] = 0 \in \mathrm{Cl}(\mathcal{D}_{\mathrm{fd}}) \quad \text{if } \Gamma \text{ is cyclic.}$$

Choose $v \in S_f(K)$, $x|v$ in $S_f(N)$. Let $\overline{\Gamma_x}$ be the quotient of Γ_x (the decomposition group of x in N/K or, what is the same, the Γ-stabiliser of x) by the inertia subgroup I_x of x. Put $\overline{\Gamma_x}$ in place of Γ in formula 3.8. By standard theory, $\overline{\Gamma_x}$ is cyclic. So we replace Γ by $\overline{\Gamma_x}$ in formula 3.9 to obtain a relation in the appropriate defect Grothendieck group. Next we inflate to Γ_x and induce up to Γ, and take the direct sum over $v \in S_f(K)$. These operations being well-defined on defect groups (with defect map fd!) by [H-W2], 1.10, we obtain a relation (with zero defect class since we started off with a sequence whose defect had zero class)

$$(3.11)\qquad 0 \longrightarrow \oplus_v \operatorname{ind}_{\Gamma_x}^{\Gamma} I_{\overline{\Gamma_x}} \longrightarrow \oplus_v \operatorname{ind}_{\Gamma_x}^{\Gamma} \mathbf{Z}[\overline{\Gamma_x}] \longrightarrow \oplus_v \operatorname{ind}_{\Gamma_x}^{\Gamma} \mathbf{Z} \longrightarrow 0 \quad \text{zero.}$$

It will be enough to show (because of the injection $\mathrm{FD}(\Gamma) \longrightarrow \mathrm{FDS}(\Gamma)$) that the sequence in formula 3.11 has Δ-component of defect $f_{S_f}(\Delta)$. It suffices to check the direct summand indexed by v.

Well, $\mathbf{Z}[\Gamma/\Gamma_x]^\Delta = \oplus_c(\sum_b bc\Gamma_x)\mathbf{Z}$, where c runs over coset representatives for $\Delta \setminus \Gamma/\Gamma_x$ and b runs over a set B of coset representatives for Δ/Δ_{cx}, and similarly $\operatorname{ind}_{\Gamma_x}^{\Gamma} \mathbf{Z}[\overline{\Gamma_x}]^\Delta = \oplus_c\big(\sum_b b(c \otimes \mathbf{Z}[\overline{\Gamma_x}])\big)^\Delta$. Let $\alpha = \sum_b b(c \otimes r_{b\Delta_{cx}}) \in \big(\sum_b b(c \otimes \mathbf{Z}[\overline{\Gamma_x}])\big)^\Delta$. Let ${}^c x = cxc^{-1}$. Thus $\Delta_{cx} = {}^c\Delta_x$. For each $\delta \in \Delta$ we have $\delta b = b_\delta\, {}^c g_\delta$ for some $g_\delta \in \Delta_x$ and $b_\delta \in B$. A short calculation shows that

$$(3.12)\qquad r_{\delta b \Delta_{cx}} = g_\delta r_{b\Delta_{cx}}.$$

If ϵ is the augmentation $\mathbf{Z}[\overline{\Gamma_x}] \longrightarrow \mathbf{Z}$, then the image of α under $\operatorname{ind}_{\Gamma_x}^{\Gamma} \epsilon$ is $\epsilon(r_{\Delta_{cx}}) \sum_b bc\Gamma_x$, because $\epsilon(g_h r_{b\Delta_{cx}}) = \epsilon(r_{b\Delta_{cx}})$. But, if ${}^c g \in \Delta_{cx}$, $g \in \Gamma_x$, then ${}^c gc \otimes r_{\Delta_{cx}} = c \otimes g r_{\Delta_{cx}}$, thus from formula 3.12 (with $b = 1$) $r_{b\Delta_{cx}} = r_{\Delta_{cx}} \in \mathbf{Z}[\overline{\Gamma_x}]^{\Delta_{cx}}$. This is the same thing as saying that $|\overline{\Delta_{cx}}|$ divides $\epsilon(r_{b\Delta_{cx}})$. But $f(N/N^\Delta, cx) = |\overline{\Delta_{cx}}|$, and there is a bijection between the double cosets $\Delta \setminus \Gamma/\Gamma_x$ and the places $w|v$ in N^Δ, if we identify w as before with $e(N/N^\Delta, w)\,\mathrm{tr}_\Delta(cx)$. This shows (*cf.* formula 3.7 and following remarks) that the required defect is indeed $f_{S_f}(\Delta)$. (We have gone through Mackey induction and Schapiro's lemma explicitly so as to retain the $\Lambda_{\Gamma/\Delta}$-module structure: if we just quoted these theorems we would only have an isomorphism of groups). Thus $[f_{S_f}] = 0 \in \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$.

So from formulas 3.6 and 3.7 it follows that

$$(3.13)\qquad \big[H^0_\Gamma(\mathbf{Z}[S_f])\big] = [e_N] + [f_{S_f}] = [e_N] \in \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}}).$$

Next, we note from formula 3.2, and the analogue with S replaced by S_∞,

$$(3.14)\qquad [H^0_\Gamma(X_S)] - [H^1_\Gamma(X_S)] = [H^0_\Gamma(X)] - [H^1_\Gamma(X)] + \big[H^0_\Gamma(\mathbf{Z}[S_f]\big].$$

Thus we have

$$(3.15)\qquad \begin{aligned}
\Omega(N/K,3) &\overset{2.9}{=\!=} [U_S] - [X_S] + [H^0_\Gamma(X_S)] \\
&\overset{3.4}{=\!=} [U] - [X] - [\mathrm{Cl}_N] - [H^1_\Gamma(U)] + [H^1_\Gamma(X)] + [H^1_\Gamma(\mathcal{P}_{S_f})] \\
&\qquad - [H^1_\Gamma(X_S)] + [H^0_\Gamma(X_S)] \\
&\overset{3.14}{=\!=} [U] - [X] - [\mathrm{Cl}_N] - [H^1_\Gamma(U)] + [H^1_\Gamma(\mathcal{P}_{S_f})] \\
&\qquad + [H^0_\Gamma(X)] + \big[H^0_\Gamma(\mathbf{Z}[S_f]\big] \\
&\overset{3.5}{=\!=} [U] - [X] - [\mathrm{Cl}_N] + [H^0_\Gamma(X)] + \big[H^0_\Gamma(\mathbf{Z}[S_f]\big] \\
&\qquad + [(\mathrm{Cl}_N)^V] - [\mathrm{Cl}_{N^V}] - [(I_{S_f(N)})^V / I_{S_f(N^V)}] \\
&\overset{3.13}{=\!=} [U] - [X] - [\mathrm{Cl}_N] + [(\mathrm{Cl}_N)^V] - [\mathrm{Cl}_{N^V}] + [H^0_\Gamma(X)]
\end{aligned}$$

which finally gives a formula explicitly independent of S.

In the same way as discussed in the paragraph following formula 2.9, 3.15 sharpens Proposition 3.4 of [C1]:

$$h\big(\Omega(N/K,3)\big) = [U] - [X] - [\mathrm{Cl}_N] \in \mathcal{G}_0(\mathbf{Z}[\Gamma]).$$

This formula motivated the results of this section.

What's more, the form of formula 3.15 suggests a relation with Hecke factorisation and the analytic class number formula, which will be made precise in §§5, 6. Essentially the idea is to introduce a module (perhaps a lattice) of special units $\mathcal{E}$, and produce a cancellation—using Hecke factorisation, Euler systems and an analytic class number formula involving special units—of all the terms in this formula, to leave only $[X]$, $[\mathcal{E}]$ and $[H^0_\Gamma(X)]$ as remaining terms. In a classical case, all these terms cancel to give zero, as predicted by Chinburg's third conjecture.

4. **Hecke factorisations and the defect class group.** In this section we recall some more details of defect groups in order to relate *Hecke factorisation* (defined below) to a representative for $\Omega(N/K,3)$ in $\mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$.

We write $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}},\otimes\mathbf{Q})$ for the Heller ([He]) relative group, which is $\mathcal{K}_\#$ of the fibre category of the functor $\otimes\mathbf{Q}$ from $\mathcal{D}_{\mathrm{fd}}$ to the category of finitely generated $\mathbf{Q}[\Gamma]$-modules. It is generated by classes $[M,f,N]+[a]$ where $a \in \mathrm{FD}$, $M,N \in \mathcal{D}$ and $f\colon M\otimes\mathbf{Q} \xrightarrow{\sim} N\otimes\mathbf{Q}$.

Let $\pi_{\mathrm{fd}}\colon \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}},\otimes\mathbf{Q}) \longrightarrow \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}})$ be the connecting homomorphism in the appropriate Heller exact sequence for $\otimes\mathbf{Q}$, defined by $[M,f,N]+[a] \longmapsto [N]-[M]+[a]$ ([H-W1] 1.3(ii)). If $f\colon X\otimes\mathbf{Q} \longrightarrow U\otimes\mathbf{Q}$ is any $\mathbf{Q}[\Gamma]$-isomorphism, define

(4.1*a*) $\quad \mathrm{cl}(f) = [X,f,U] - [0,0,\mathrm{Cl}_N] + [(\mathrm{Cl}_N)^V] - [\mathrm{Cl}_{N^V}] + [H^0_\Gamma(X)] \in \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}},\otimes\mathbf{Q}).$

Then by formula 3.15

(4.1*b*) $$\Omega(N/K,3) = \pi_{\mathrm{fd}}\big(\mathrm{cl}(f)\big).$$

As in [H-W2], before 1.7, denote by $\mathcal{H}_p$ the set of triples (p,Δ,ω), where p is a prime number, Δ is a subgroup of Γ and ω an idempotent of $\Lambda_{\Gamma/\Delta,p}$. We identify $\mathcal{G}_0^t(\mathbf{Z}_p)$ with the corresponding relative group $\mathcal{G}_0(\mathbf{Z}_p,\otimes\mathbf{Q})$ as in [H-W2] 1.1b. Then define $\mathrm{FD}_{\mathcal{H},p} = \mathrm{Map}\big(\mathcal{H}_p, \mathcal{G}_0^t(\mathbf{Z}_p)\big)$. Let $\mathcal{H} = \cup_p\mathcal{H}_p$, disjoint union over prime numbers p, and similarly let $\mathrm{FD}_{\mathcal{H}} = \oplus_p \mathrm{FD}_{\mathcal{H},p}$. By [H-W2] 1.7 there is a projection homomorphism onto a direct summand:

(4.3) $$\mathcal{K}_0(\mathcal{D}_{\mathrm{fd},p},\otimes\mathbf{Q}) \xrightarrow{\mathcal{F}_p} \mathrm{FD}_{\mathcal{H},p} \text{ by}$$
$$[N_1,h,N_2]+[a] \longmapsto \big[: (p,\Delta,\omega) \longmapsto [\omega N_1^\Delta, h, \omega N_2^\Delta]\big] + [a]$$

for each p, which induces a projection $\mathcal{F}\colon \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}},\otimes\mathbf{Q}) \longrightarrow \mathrm{FD}_{\mathcal{H}}$, since by [H-W1] 3.1 there is an isomorphism $\mathcal{K}_0(\mathcal{D}_{\mathrm{fd}},\otimes\mathbf{Q}) \cong \oplus_p \mathcal{K}_0(\mathcal{D}_{\mathrm{fd},p},\otimes\mathbf{Q})$.

By [H-W2] 1.1c, there is an isomorphism $x \longmapsto |x|$ of $\mathcal{G}_0(\mathbf{Z}_p, \otimes\mathbf{Q})$ with $I_{\mathbf{Q}_p}$ (a generalised module index). Thus, if $[M_1, f, M_2] \in \mathcal{G}_0(\mathbf{Z}_p, \otimes\mathbf{Q})$, and $t(M_i)$ and $\overline{M_i}$ are the torsion subgroup of M_i and its image in $M_i \otimes \mathbf{Q}$, respectively, then

$$|M_1, f, M_2| = \frac{|t(M_2)|[\overline{M_2} : f(\overline{M_1}) \cap \overline{M_2}]}{|t(M_1)|[f(\overline{M_1}) : f(\overline{M_1}) \cap \overline{M_2}]}.$$

Let subscript Δ denote projection $\mathrm{FD} = \mathcal{G}_0^t(\Lambda_V) \longrightarrow \mathcal{G}_0^t(\Lambda_{\Gamma/\Delta})$.

Paraphrasing [H-W2] formula 2.4b, we say that $[M_1, g, M_2] + [a] \in \mathcal{K}_0(\mathcal{D}_{\mathrm{fd}}, \otimes\mathbf{Q})$ has a $\mathcal{H}$-(or *Hecke*) *factorisation* if there is some $\alpha \in \mathcal{J}(\mathbf{Q}[\Gamma])$ such that

$$(4.4) \qquad |\omega(M_1)_p^\Delta, g_p, \omega(M_2)_p^\Delta|\,|\omega(a_p)_\Delta| = \big(\det{}_{\mathbf{Q}_p}(\alpha_p | \mathbf{Q}_p[\Gamma]e_\Delta\omega)\big) \quad \forall (p, \Delta, \omega) \in \mathcal{H}.$$

Let U be a field realising the characters of Γ and Galois over $\mathbf{Q}$. Let $R_\Gamma = \mathcal{K}_0(U[\Gamma])$ be the character ring of Γ. Let AI be the associated ideal (or content) map $\mathcal{J}(U) \longrightarrow I_U$ and let Det be the well-known generalised determinant map, which factorises through reduced norm and so switches from the idèlic- to the Hom-description notation. In the above situation we say that $AI\big(\mathrm{Det}(\alpha)\big) \in \mathrm{Hom}_{\mathrm{Gal}(U/\mathbf{Q})}(R_\Gamma, I_U)$ is the $\mathcal{H}$-factorisation of $[M_1, g, M_2] + [a]$.

Let C be the centre of $\mathbf{Q}[\Gamma]$ and $\nu\colon \mathcal{J}(\mathbf{Q}[\Gamma]) \longrightarrow \mathcal{J}(C)$ be the continuous map induced by reduced norm. From [H-W2] 2.6(i) and 2.8(i) and (iii) (we take the set S there to be all the prime divisors of $|\Gamma|$) and by 4.2, for any choice of f, $\mathrm{cl}(f)$ has a $\mathcal{H}$-factorisation $AI\big(\mathrm{Det}(\alpha)\big)$, and

$$(4.5) \qquad \Omega(N/K, 3) = \mathrm{cls}_{\mathcal{D}_{\mathrm{fd}}}\big(\nu(\alpha)\big)$$

where $\mathrm{cls}_{\mathcal{D}_{\mathrm{fd}}}$ is the epimorphism $\mathcal{J}(C) \longrightarrow \mathrm{Cl}(\mathbf{Z}[\Gamma]) \longrightarrow \mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$ of [H-W2] 2.3, with kernel $C^\times \nu(\mathcal{U}(\mathbf{Z}[\Gamma]) \prod_{p \nmid |\Gamma|} O_{C,p}^\times$, which gives the idèlic description of $\mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$ (O_C is the unique maximal order in C).

Explicitly, we have, for each $(p, \Delta, \omega) \in \mathcal{H}$,

$$(4.6) \qquad |\omega(X_p)^\Delta, f, \omega(U_p)^\Delta|\,|\omega(\mathrm{Cl}_{N^\Delta})_p|^{-1}|\omega\hat{H}^0(\Delta, X_p)| = \big(\det{}_{\mathbf{Q}_p}(\alpha_p | \mathbf{Q}_p[\Gamma]e_\Delta\omega)\big)$$

after taking note of a cancellation of two terms.

5. **Euler systems of units.** In this section, let F/K_0 be a finite abelian extension of number fields and let L be an intermediate field (so $F \supset L \supset K_0$) with $G = \mathrm{Gal}(F/L)$ and let $M > 1$ be a large power of the prime number p (to be specified later). We do not assume that p is odd. Let μ_n denote the group of n-th roots of unity in $\mathbf{C}$ for a positive integer n.

As in previous sections, write O_F for the ring of integers in F, U_F for the group of units of O_F and Cl_F for the ideal class group of F.

Throughout this section, l will denote a finite prime of K_0 and λ a finite prime of L. We write $\mathbf{N}l$ for $|O_{K_0}/l|$, the absolute norm of l. Write $K_0(l)$ for the ray class field modulo l over K_0. Write H_0 for the Hilbert class field of K_0.

https://doi.org/10.4153/CJM-1994-016-0 Published online by Cambridge University Press

Let $\mathcal{S}_M(K_0)$ be the set of finite primes l of K_0 which split completely in F/K_0 and for which $M|(\mathbf{N}l-1)/w(l)$, where $w(l)$ is the order of the image of U_{K_0} in $(O_{K_0}/l)^\times$. Also denote by $\mathcal{S}_M^s(K_0)$ the set of squarefree ideals supported in $\mathcal{S}_M(K_0)$. We also allow O_{K_0} to lie in $\mathcal{S}_M^s(K_0)$. If $a \neq 0$ is an ideal of O_{K_0}, denote by $\mathcal{S}_M^s(a, K_0)$ the subset of $\mathcal{S}_M^s(K_0)$ consisting of ideals prime to a. Let $(K_0)_M = K_0\big(\mu_M, (U_{K_0})^{1/M}\big)$.

Write $\mathcal{S}_M(L)$ for the primes of L lying over primes in $\mathcal{S}_M(K_0)$. Similarly define $\mathcal{S}_M^s(L)$ and $\mathcal{S}_M^s(a, L)$.

LEMMA 5.1 ([RU2] 1.1). *If l splits completely in F/K_0 then $l \in \mathcal{S}_M(K_0)$ if and only if $M \mid [K_0(l) : H_0]$, and this holds if and only if l splits completely in $(K_0)_M/K_0$.* ■

Suppose we are given a tower of extensions fields $F[\lambda]/F$ for $\lambda \in \mathcal{S}_M(L)$ satisfying the following axioms:

EF(i) $F[\lambda]/L$ is abelian,
EF(ii) $F[\lambda]/F$ is unramified at finite places not dividing λ,
EF(iii) $F[\lambda]/F$ is totally ramified at primes dividing λ,
EF(iv) $\big[F[\lambda]/F\big] = M$.

Then for $\rho \in \mathcal{S}_M^s(L)$ we denote by $F[\rho]$ the composite of the fields $F[\lambda]$ for $\lambda|\rho$.

By an *Euler system for* $(F/L, M)$ we shall mean, for some ideal $a \neq 0$ of O_L, a set α of maps from $\mathcal{S}_M^s(a, L)$ to $(\bar{F})^\times$ satisfying the following axioms:

ES(i) if $\rho \in \mathcal{S}_M^s(a, L)$, then $\alpha(\rho) \in U_{F[\rho]}$,
ES(ii) if $\lambda\rho \in \mathcal{S}_M^s(a, L)$, then

$$\mathcal{N}_{F[\lambda\rho]/F[\rho]}\alpha(\lambda\rho) = \alpha(\rho)^{\mathrm{Fr}_\lambda - 1},$$

where Fr_λ is the Frobenius of λ in $\mathrm{Gal}(F[\rho]/L)$,
ES(iii) if $\lambda\rho \in \mathcal{S}_M^s(a, L)$, then $\alpha(\lambda\rho) \equiv \alpha(\rho)^{(\mathbf{N}\lambda-1)/M}$ modulo all primes dividing λ.

We let $\mathrm{ES}_{F/L,M}$ denote the set of all Euler systems for $(F/L, M)$. It is closed under multiplication, inverses and the action of Ω_F. Given $\alpha \in \mathrm{ES}_{F/L,M}$, if a is the corresponding ideal we denote also by $\mathcal{S}_M^s(\alpha, L)$ the set $\mathcal{S}_M^s(a, L)$ above, which is the domain of α.

Note that other definitions of Euler systems for units have been given; this one is closely related to that in [Ru3]. See also [Ru1] and [Ru2]. We have made the definition relative to subfields so that one only needs the condition $F \supset H_0$ rather than $F \supset H_L$ for an Euler system for $(F/L, M)$.

To exhibit a tower of fields $F[l]$ we choose $F[l]$ to be the unique (cyclic) extension of order M in $FK_0(l)/F$, provided $F \supset H_0$. If $\lambda \in \mathcal{S}_M(L)$, and $\lambda|l$ where $l \in \mathcal{S}_M(K_0)$, then we simply define $F[\lambda] = F[l]$. Note that we can identify Fr_l and Fr_λ since l splits completely in F/K_0.

Note that $\mathrm{Gal}\big(K_0(l)/H_0\big) \cong (O_{K_0}/l)^\times/\operatorname{im} U_{K_0}$ by class field theory (*cf.* the well-known real cyclotomic case with $K_0 = \mathbf{Q}$), and, because $F \supset H_0$, $\mathrm{Gal}\big(FK_0(l)/F\big)$ identifies with $\mathrm{Gal}\big(K_0(l)/H_0\big)$ by restriction. Ramification considerations show that the axioms EF are satisfied (indeed, EF(ii) extends to the infinite primes). Also, the fields $F[l]$ are linearly disjoint over F.

For every $\rho \in \mathcal{S}_M^s(L)$, we let $G_\rho = \mathrm{Gal}(F[\rho]/F) \cong \prod_{\lambda|\rho} G_\lambda$ and let σ_λ be a generator of G_λ (which is cyclic of order M). Let N_λ be the trace element of $\mathbf{Z}[G_\lambda]$ and define $D_\lambda = \sum_{i=0}^{M-1} i\sigma_\lambda^i \in \mathbf{Z}[G_\lambda]$. This operator D_λ is chosen to satisfy

$$(5.2) \qquad (\sigma_\lambda - 1)D_\lambda = M - N_\lambda \in \mathbf{Z}[G_\lambda].$$

Also define $D_\rho = \prod_{\lambda|\rho} D_\lambda$. We may identify G_λ with $\mathrm{Gal}(F[\rho]/F[\rho/\lambda])$, the inertia group of λ in G_ρ.

PROPOSITION 5.3 ([RU3], PROPOSITION 2.2). *For each $\alpha \in \mathrm{ES}_{F/L,M}$ there is a canonical map*

$$\kappa = \kappa_\alpha: \mathcal{S}_M^s(\alpha) \longrightarrow F^\times/(F^\times)^M$$

such that, for every $\rho \in \mathcal{S}_M^s(\alpha, L)$,

$$\kappa(\rho) \equiv \alpha(\rho)^{D_\rho}\big(\mathrm{mod}(F[\rho]^\times)^M\big). \qquad \blacksquare$$

Let I_λ denote the subgroup of I_F consisting of ideals supported over λ, written additively. If $y \in F^\times$, let $[y]_\lambda$ denote the projection into I_λ/MI_λ of the principal ideal generated by y. Note that $[y]_\lambda$ is also well-defined for $y \in F^\times/(F^\times)^M$.

PROPOSITION 5.4 ([RU3], PROPOSITION 2.3). *Let $\lambda \in \mathcal{S}_M(L)$. There is a unique $\mathbf{Z}[G]$-isomorphism*

$$\phi_\lambda: \frac{(O_F/\lambda O_F)^\times}{\big((O_F/\lambda O_F)^\times\big)^M} \longrightarrow \frac{I_\lambda}{MI_\lambda}$$

which makes the following diagram commute:

$$\begin{array}{ccccc} & & F[\lambda]^\times & & \\ x \longmapsto \{\overline{x^{1-\sigma}}\lambda\}^{1/d} & \swarrow & & \searrow & x \longmapsto [x^N\lambda]\lambda \\ \frac{(O_F/\lambda O_F)^\times}{((O_F/\lambda O_F)^\times)^M} & & \xrightarrow{\phi_\lambda} & & \frac{I_\lambda}{MI_\lambda} \end{array}$$

where $\overline{x^{1-\sigma_\lambda}}$ is the reduction of $x^{1-\sigma_\lambda}$ (which is a unit at each prime $\lambda'|\lambda$ of $F[\lambda]$) in

$$\bigoplus_{\lambda'|\lambda} O_{F[\lambda]}/\lambda' \cong O_F/\lambda O_F$$

and $d = (\mathbf{N}\lambda - 1)/M$. $\blacksquare$

Note that the theory of ramification in tame, local Galois extensions is here being used (in the well-definedness of the left hand diagonal map) and that ϕ_λ may be considered as a logarithm modulo λ.

We shall also write ϕ_λ for the induced homomorphism

$$\phi_\lambda: \{y \in F^\times/(F^\times)^M : [y]_\lambda = 0\} \longrightarrow I_\lambda/MI_\lambda.$$

To see that this extension makes sense, note that $[y]_\lambda = 0$ means that we can choose a representative for y which is a unit at primes dividing λ.

PROPOSITION 5.5 ([K], THEOREM 5, *cf.* [RU3], PROPOSITION 2.4). *Suppose* $\alpha \in \mathrm{ES}_{F/L,M}$, κ *is the map defined in formula 5.3,* $\rho \in \mathcal{S}_M^s(\alpha, L)$, *and* λ *is a finite prime of L.*

(i) If $\lambda \not| \rho$ *then* $[\kappa(\rho)]_\lambda = 0$,

(ii) If $\lambda | \rho$ *then* $[\kappa(\rho)]_\lambda = \phi_\lambda\big(\kappa(\rho/\lambda)\big)$. ■

Note that we can dispense with the condition "$\rho \neq O_L$" in [Ru3], because $\kappa(O_L) = \alpha(O_L)$ is a unit. Even with the slightly different axiom system of [Ru3], where $\alpha(O_L)$ is not assumed to be a unit, nevertheless it is a unit at primes dividing λ (by ES(iii)), if $\lambda \in \mathcal{S}_M(L)$, which is all that is needed. Indeed, $\rho = O_L$ is used in the applications there (Theorem 3.2 of [Ru3]).

Let $m = m(F)$ be the order of $\mu_M(F)$. For the rest of this section we shall suppose that $p \not| \; |G|$. Let χ be an irreducible $\mathbf{Q}_p$-character of $G = \mathrm{Gal}(F/L)$. Let e_χ be the idempotent

$$e_\chi = |G|^{-1} \sum_{g \in G} \chi(g) g^{-1}$$

of $\mathbf{Z}_p[G]$ corresponding to χ (this makes sense even when χ is not 1-dimensional since G is abelian, so the simple $\mathbf{Q}_p[G]$ modules occur with multiplicity 1 in $\mathbf{Q}_p[G]$). If V is any $\mathbf{Z}[G]$-module, write V^χ for the χ-isotypic component of the p-adic completion V_p of V, so that $V^\chi = e_\chi(V_p)$.

LEMMA 5.6. *(i) The natural map*

$$F^\times/(F^\times)^M \longrightarrow F(\mu_M)^\times/\big(F(\mu_M)^\times\big)^M$$

is injective, unless $p = 2$ *and* $m = 2$, *in which case the kernel is of order* 2.

(ii) In any case, if $\chi \neq 1$, *and* $p \not| \; |G|$, *then the map*

$$\big(F^\times/(F^\times)^M\big)^\chi \longrightarrow F(\mu_M)^\times/\big(F(\mu_M)^\times\big)^M$$

is injective.

PROOF. (ii) is an obvious consequence of (i) and orthogonality of idempotents, since $\{\pm 1\}$ is G-trivial and $\chi \neq 1$.

Let $F' = F(\mu_M)$. The sketch proof of (i) (when p is odd) in [Ru3], Lemma 2.5, shows that the kernel is isomorphic to $H^1\big(\mathrm{Gal}(F'/F), \mu_M\big)$, which has the same order as

$$\hat{H}^0\big(\mathrm{Gal}(F'/F), \mu_M\big) = \mu_m/\mathcal{N}_{F'/F}(\mu_M)$$

since μ_M is finite and F'/F is cyclic. This also applies in the case $p = 2$. Thus, (i) is plain if $m = 1$, and we may assume that $m > 1$. Then $\mathrm{Gal}(F'/F) \cong \mathrm{Gal}\big(\mathbf{Q}(\mu_M)/\mathbf{Q}(\mu_m)\big)$ by restriction, because $F \cap \mathbf{Q}(\mu_{pm})$ is a proper subfield of $\mathbf{Q}(\mu_{pm})$ containing $\mathbf{Q}(\mu_m)$. Since $[\mathbf{Q}(\mu_{pm}) : \mathbf{Q}(\mu_m)] = p$, (as $m > 1$) $F \cap \mathbf{Q}(\mu_{pm}) = \mathbf{Q}(\mu_m)$. Therefore $F \cap \mathbf{Q}(\mu_M) = \mathbf{Q}(\mu_m)$.

Thus $\mathcal{N}_{F'/F}(\mu_M) = \mathcal{N}_{\mathbf{Q}(\mu_M)/\mathbf{Q}(\mu_m)}(\mu_M)$. However, if $r > 1$, (recall $\zeta_n = \exp(2\pi i/n)$)

$$\begin{aligned} \mathcal{N}_{\mathbf{Q}(\mu_{p^r})/\mathbf{Q}(\mu_{p^{r-1}})}\zeta_{p^r} &= \prod_{a=0}^{p-1} \zeta_{p^r}^{1+ap^{r-1}} \\ &= \zeta_{p^{r-1}} \zeta_p^{p(p-1)/2} \\ &= \begin{cases} \zeta_{p^{r-1}}, & \text{if } p \text{ odd,} \\ -\zeta_{2^{r-1}}, & \text{if } p = 2. \end{cases} \end{aligned}$$

Thus, as $-\zeta_{2^{r-1}}$ has the same order as $\zeta_{2^{r-1}}$ if $r > 2$, and it is 1 if $r = 2$, (i) follows. ∎

Let A be the p-part of Cl_F.

THEOREM 5.7. *Suppose that* $p \nmid |G|$, χ *is non-trivial on* $\mathrm{Gal}\big(F/F \cap L(\mu_M)\big)$, $M > 1$ *is a power of* p, $\beta \in \big(F^\times/(F^\times)^M\big)^\chi$, $\tilde{A}$ *is a* $\mathbf{Z}[G]$*-quotient of* A^χ. *Let* m *be the order of* β *in* $F^\times/(F^\times)^M$, W *the* $\mathbf{Z}[G]$*-submodule of* $\big(F^\times/(F^\times)^M\big)^\chi$ *generated by* β, H *the unramified extension of* F *corresponding to* $\tilde{A}$, *and* $H' = H \cap F(\mu_M, W^{1/M})$. *Then there is a* $\mathbf{Z}[G]$*-generator* c' *of* $\mathrm{Gal}(H'/F)$ *such that for every* $c \in \tilde{A}$ *whose restriction to* H' *is* c', *there are infinitely many primes* λ' *of* F *of degree* 1 *such that*

(i) the projection of the class of λ' *into* $\tilde{A}$ *is* c,

(ii) the prime λ *of* L *below* λ' *lies in* $\mathcal{S}_M(L)$,

(iii) $[\beta]_\lambda = 0$ *and there exists* $u \in \big((\mathbf{Z}/M\mathbf{Z})[G]^\chi\big)^\times$ *such that* $\phi_\lambda(\beta) = u(M/m)\lambda'$.

PROOF. This is very similar to the proof of Theorem 3.1 in [Ru3]. The same idea occurs (without details) in [Ru2], proof of Theorem 3.2: we give the details for completeness.

Let $F' = F(\mu_M)$ and let $L' = L(\mu_F)$. Since G acts via $\chi \neq 1$ on $\mathrm{Gal}(H/F)$ but trivially on $\mathrm{Gal}(F'/F)$ (as $\mathrm{Gal}(F'/L)$ is abelian), we have $H \cap F' = F$. By Lemma 5.6(ii), Kummer theory gives an isomorphism

$$\mathrm{Gal}\big(F'(W^{1/M})/F'\big) \cong \mathrm{Hom}(W, \mu_M).$$

By the definition of m we have $W \cong (\mathbf{Z}/m\mathbf{Z})[G]^\chi$, so $\mathrm{Gal}\big(F'(W^{1/M})/F'\big)$ is cyclic over $\mathbf{Z}[\mathrm{Gal}(F'/L)]$. Fix a $\mathbf{Z}[\mathrm{Gal}(F'/L)]$-generator τ of $\mathrm{Gal}\big(F'(W^{1/M})/F'\big)$ and let c' be the restriction of τ to H'; then c' generates $\mathrm{Gal}(H'/F) \cong \mathrm{Gal}(H'F'/F')$ over $\mathbf{Z}[G]$. Let $c \in \tilde{A} \cong \mathrm{Gal}(H/F)$ be any class whose restriction to H' is c'. Fix an automorphism σ of $\mathrm{Gal}(HF'(W^{1/M}, U_{K_0}^{1/M})/F$ (this extension being Galois as a composite of H and a splitting field over F) satisfying

$$\sigma|_H = c, \quad \sigma|_{F'(W^{1/M})} = \tau \text{ and } \sigma|_{F'(U_{K_0}^{1/M})} = 1.$$

Since $\mathrm{Gal}(F/F \cap L') \cong \mathrm{Gal}(F'/L')$ acts (via χ) non-trivially on $\mathrm{Gal}\big(F'(W^{1/M})/F'\big)$ and trivially on $\mathrm{Gal}\big(F'(U_{K_0}^{1/M})/F'\big)$, $F'(W^{1/M}) \cap F'(U_{K_0}^{1/M}) = F'$, so the second and third conditions are independent. Similarly, $\mathrm{Gal}(F'/L')$ acts trivially on $\mathrm{Gal}\big(F'(U_{K_0}^{1/M})/F'\big)$ (as $\mathrm{Gal}\big(F'(U_{K_0}^{1/M})/L'\big)$ is abelian) and non-trivially (via χ) on HF'/F', so that $F'(U_{K_0}^{1/M}) \cap H = F$ (because $H \cap F' = F$). As c and τ agree on $F'(W^{1/M}) \cap H = H'$, the first condition can therefore be achieved simultaneously with the other two.

Let λ' be a prime of F of degree one whose Frobenius in $\mathrm{Gal}(HF'(W^{1/M}, U_{K_0}^{1/M})/F$ is the conjugacy class of σ, and such that the prime λ of L below λ' is unramified in $F'(W^{1/M})/L$. (The Čebotarev theorem guarantees the existence of infinitely many such λ'). The verification that λ' satisfies (i), (ii) and (iii) is now the same as in the proof of Theorem 3.1 in [Ru3]. ∎

https://doi.org/10.4153/CJM-1994-016-0 Published online by Cambridge University Press

The condition on χ can be weakened in the only two cases so far where explicit Euler systems of units are known to exist—though Stark's conjecture predicts more examples (see [Ru2]) and was "empirically verified" in [St] for some extensions with K_0 real quadratic or cubic.

COROLLARY 5.8. *If U_{K_0} is finite, then Theorem 5.7 holds with the hypothesis "χ non-trivial on* $\mathrm{Gal}\big(F/F\cap L(\mu_M)\big)$*" weakened to "$\chi \neq 1$", provided either $p > 3$, or $p = 3$ and $|U_{K_0}| = 2$. If $p = 2$ and $|U_{K_0}| = 2$, with the same condition that $\chi \neq 1$ we obtain Theorem 5.7 with M replaced by $M/2$ in Theorem 5.7(ii) (provided M is chosen larger than 2).*

PROOF. In this case U_{K_0} consists of roots of unity, hence $K_0 = \mathbf{Q}$ or a quadratic imaginary field, and $|U_{K_0}| = 2, 4,$ or 6. Also, $F(\mu_M, U_{K_0}^{1/M})/L$ is abelian, and the hypothesis that $\chi \neq 1$ is sufficient to show that $H \cap F(\mu_M, U_{K_0}^{1/M}) = F$. If $p > 3$, or $p = 3$ and $|U_{K_0}| = 2$ then $F(\mu_M, U_{K_0}^{1/M}) = F(\mu_M)$, so the same proof as above now works with the weaker restriction on χ.

On the other hand, if $p = 2$ and $|U_{K_0}| = 2$, if we simply replace the field $F(\mu_M, U_{K_0}^{1/M})$ with $F(\mu_M)$ in the proof of Theorem 5.7, we find that λ splits completely in $F(\mu_M)/L$, so $M|(\mathbf{N}\lambda - 1)$, and therefore (as $w(\lambda) = 1$ or 2) $\frac{1}{2}M|(\mathbf{N}\lambda - 1)/w(\lambda)$, so $\lambda \in \mathcal{S}_{M/2}(L)$. ∎

For the applications to index theorems for p-adic eigenspaces of Cl_F, we define

$$\mathcal{ES}_{F/L,M} = U_F \cap \{\alpha(O_L) : \alpha \in \mathrm{ES}_{F/L,M}\}.$$

Note that $\mathcal{ES}_{F/L,M}$ only depends on F/K_0, and in the case $K_0 = \mathbf{Q}$, it turns out that $(\mathcal{ES}_{F/L,M})_p$ is independent of M provided M is sufficiently large, and $p \nmid [F : \mathbf{Q}]$.

THEOREM 5.9. *Suppose that $p \nmid [F : L]$ and that $\chi \neq 1$. Suppose that either the hypotheses of Theorem 5.8 are satisfied (so that $p = 2$ or 3 may or may not be excluded) or that χ is non-trivial on $F \cap L(\mu_M)$.*

Further suppose that M is divisible by $p[U_F^\chi : \mathcal{ES}_{F/L,M}^\chi]|\,\mathrm{Cl}_F^\chi\,|$ (or twice this if $p = 2$). In particular we suppose that $[U_F^\chi : \mathcal{ES}_{F/L,M}^\chi]$ is finite. Then $|\,\mathrm{Cl}_F^\chi\,|$ divides $[U_F^\chi : \mathcal{ES}_{F/L,M}^\chi]$.

PROOF. The proof is the same as that of Theorem 3.2 in [Ru3], with Theorem 3.1 of [Ru3] replaced by our Theorem 5.8. ∎

Note that we are assuming that $F \supset H_0$: without this assumption one would need instead to assume that $p \nmid |H_0|$ in addition (*cf.* [B1]).

The careful reader of Theorems 5.8 and 5.9 will find that only the hypotheses $p \nmid [F : L]$, $\chi \neq 1$ (and M sufficiently large) are needed in Theorem 5.9, when $K_0 = \mathbf{Q}$—in this case the existence of Galois-stable groups of cyclotomic units of finite index in U_F shows that $[U_F^\chi : \mathcal{ES}_{F/L,M}^\chi]$ is finite.

6. **The real abelian case.** In this section we apply the results of §§1–5 to the case where $K = \mathbf{Q}$ and $N \subset \mathbf{Q}(\zeta_n)^+$, where ζ_n is a primitive n-th root of unity in $\mathbf{C}$, and we suppose that $n \not\equiv 2 \pmod 4$. We let σ_a (for $(a,n) = 1$) be the element of $\mathrm{Gal}\big(\mathbf{Q}(\zeta_n)/\mathbf{Q}\big)$ which raises n-th roots of unity to the power a.

We shall need the lattice of cyclotomic units $\mathrm{Ra}_{\mathbf{Q}(\zeta_n)^+}$ discovered by Ramachandra ([Ra]; see also [Wa], 8.3)) generated by the multiplicatively independent units ν_a defined as follows.

Let $n = \prod_{i=1}^{s} q_i$ as a product of distinct prime-powers q_i. Let I run over all subsets of $\{1,\ldots,s\}$ except $\{1,\ldots,s\}$ and let $n_I = \prod_{i\in I} q_i$. Let $\xi_{n,a} = 1 - \zeta_n^a$. For $1 < a < n/2$, $(a,n) = 1$ define

$$\nu_a = \zeta_n^{d_a} \prod_I \frac{\xi_{n,an_I}}{\xi_{n,n_I}}, \quad d_a = \frac{1-a}{2}\sum_I n_I. \tag{6.1}$$

There is an isomorphism

$$I_{\mathrm{Gal}(\mathbf{Q}(\zeta_n)^+/\mathbf{Q})} \cong \mathrm{Ra}_{\mathbf{Q}(\zeta_n)^+} \text{ by } \quad 1 - \{\sigma_a, \sigma_{-a}\} \mapsto \nu_a. \tag{6.2}$$

A brief calculation shows that this is a $\mathrm{Gal}(\mathbf{Q}(\zeta_n)^+/\mathbf{Q})$-isomorphism (assuming the independence of the ν_a). We can therefore uniquely define Ramachandra units Ra_N for each real abelian field N by

$$\mathrm{Ra}_N = (\mathrm{Ra}_{\mathbf{Q}(\zeta_n)^+})^{\mathrm{Gal}(\mathbf{Q}(\zeta_n)^+/N)} = \mathcal{N}_{\mathrm{Gal}(\mathbf{Q}(\zeta_n)^+/N)}(\mathrm{Ra}_{\mathbf{Q}(\zeta_n)^+}), \tag{6.3}$$

where n is the conductor of N, that is, the smallest integer f with $N \subset \mathbf{Q}(\zeta_f)$. We obtain an explicit free generating set of Ra_N as an abelian group in the obvious way, and we have:

THEOREM 6.5. *Let N be an arbitrary subfield of $\mathbf{Q}(\zeta_n)^+$. Let χ run over the non-trivial characters of $\Gamma = \mathrm{Gal}(N/\mathbf{Q})$ (considered as characters of $\mathrm{Gal}\big(\mathbf{Q}(\zeta_n)/\mathbf{Q}\big)$ by inflation). Then*

$$[U_N : \mathrm{Ra}_N.\{\pm 1\}] = |\,\mathrm{Cl}_N\,| \prod_{\chi\neq 1} \prod_{r \nmid f_\chi} \big(\phi(r^e) + 1 - \chi(\sigma_r)\big)$$

where r runs over the prime divisors of the conductor n of N not dividing the conductor f_χ of χ and r^e is the highest power of r dividing n.

PROOF. The case where $N = \mathbf{Q}(\zeta_n)^+$ is Theorem 8.3 of [Wa]. The proof in general is a straightforward extension (see p. 152 of [Wa]). ■

Let the function h be defined on subgroups Δ of Γ by

$$h(\Delta) = [U_F : (\mathrm{Ra}_N)^\Delta.\{\pm 1\}]\,|\,\mathrm{Cl}_F\,|^{-1}, \quad F = N^\Delta. \tag{6.6a}$$

From Theorem 6.5, h is $\mathbf{Q}$-factorisable (in the sense of [F3]), with a factorisation of h being given by the function $g = g_N \in \mathrm{Hom}_{\mathrm{Gal}(U/\mathbf{Q})}(R_\Gamma, I_U)$ defined on absolutely irreducible characters χ by

$$g(\chi) = \prod_{r \nmid f_\chi} \big(\phi(r^e) + 1 - \chi(\sigma_r)\big). \tag{6.6b}$$

Note that Ra_N does not have Galois descent. For example, suppose that $N/\mathbf{Q}$ is real cyclic of degree 6 and conductor 35. If $\mathrm{ord}(\phi) = 3$ then $g_N(\phi) = \big(5 - \phi(\sigma_5)\big) \neq 1$. However, if H has order 2 then ϕ is inflated from $\mathrm{Gal}(N^H/\mathbf{Q})$ and $g_{N^H}(\phi) = 1$ since N^H has conductor 7. From the index formula Theorem 6.5

$$[U_{N^H} : (\mathrm{Ra}_N)^H\{\pm 1\}]/h_{N^H} = g_N(\phi)g_N(\phi^2)$$

and

$$[U_{N^H} : (\mathrm{Ra}_{N^H})\{\pm 1\}]/h_{N^H} = 1$$

therefore

$$(6.7) \qquad [\mathrm{Ra}_{N^H} : (\mathrm{Ra}_N)^H] = g_N(\phi)g_N(\phi^2) \neq 1.$$

Let $f: X_N \otimes \mathbf{Q} \xrightarrow{\sim} U_N \otimes \mathbf{Q}$ be induced by the composite isomorphism $X_N \cong I_\Gamma \cong \mathrm{Ra}_N$, where the first isomorphism holds since S_∞ splits completely and the second is formula 6.2. Then $\Omega(N/\mathbf{Q}, 3) = \pi_{\mathrm{fd}}\big(\mathrm{cl}(f)\big)$ by 4.1.

From the relation

$$0 \longrightarrow \mathbf{Z} \xrightarrow{.2} \mathbf{Z} \longrightarrow \{\pm 1\} \longrightarrow 0 \quad \text{zero}$$

it follows that $\Omega(N/\mathbf{Q}, 3) = \pi_{\mathrm{fd}}\big(\mathrm{cl}(f) - [\mathbf{Z}, .2, \mathbf{Z}]\big)$. However (if $(p, \Delta, \omega) \in \mathcal{H}$), by formula 4.6,

$$(6.8) \qquad \begin{aligned} \big|\mathcal{F}\big(\mathrm{cl}(f) - [\mathbf{Z}, .2, \mathbf{Z}]\big)(p, \Delta, \omega)\big| &= [\omega(U_F)_p : \omega(\mathrm{Ra}_N^\Delta . \{\pm 1\})_p)] |\omega(\mathrm{Cl}_F)_p|^{-1} \\ &= \big(\det{}_{\mathbf{Q}_p}(\alpha_p | \mathbf{Q}_p[\Gamma] e_\Delta \omega)\big) \end{aligned}$$

for some $\alpha \in \mathcal{J}(\mathbf{Q}[\Gamma])$. If we compare formula 6.8 with 6.6 we are led to

CONJECTURE 6.9. *The function g of formula 6.6b is the Hecke factorisation of* $\mathrm{cl}(f) - [\mathbf{Z}, .2, \mathbf{Z}]$, *where* $\mathrm{cl}(f)$ *is defined in formula 6.6a.*

Since g has principal values and Γ is abelian, g lies in the denominator of the ideal-theoretic Hom-description of $\mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$. So Conjecture 6.9 implies the truth of

CONJECTURE 6.10. *If N is real abelian then* $\Omega(N/K, 3) = 0 \in \mathrm{Cl}(\mathcal{D}_{\mathrm{fd}})$.

NOTE. We have apparently only established this in the case $K = \mathbf{Q}$, but by functorial properties of $\Omega(3)$ ([C2]) this suffices for $K \neq \mathbf{Q}$.

Next we shall apply the results of §5 in the case where F is a real abelian field (so $K_0 = \mathbf{Q}$).

If the conductor of F is a prime power, we say F is *primary*. Otherwise we say F is *split*. Recall that $F \supset L$, $G = \mathrm{Gal}(F/L)$ and that $\mathcal{ES}_{F/L,M}$ was defined at the end of §5. Also observe that Ra_F does have Galois descent when F is primary.

THEOREM 6.11. *If F is primary, M is as in Theorem 5.9 and $p \nmid [F:L]$ then*

$$|\,\mathrm{Cl}_F^{\chi}\,| = [U_F^{\chi} : \mathcal{ES}_{F/L,M}^{\chi}] = [U_F^{\chi} : (\mathrm{Ra}_F.\{\pm 1\})^{\chi}]$$

for each $\mathbf{Q}_p$-irreducible character χ of G (including $\chi = 1$).

PROOF. In this case Theorem 6.5 shows that $|(\mathrm{Cl}_F)_p| = [(U_F)_p : (\mathrm{Ra}_F.\{\pm 1\})_p]$. If $\chi = 1$ then $\mathrm{Cl}_F^{\chi} = (\mathrm{Cl}_F)_p^G$, and by the genus formula relating $(\mathrm{Cl}_F)^G$ and Cl_L (*cf.* formula 3.5), since $p \nmid [F:L]$,

$$|(\mathrm{Cl}_F)_p^G| = |(\mathrm{Cl}_L)_p| \overset{6.5}{=} [(U_L)_p : (\mathrm{Ra}_L.\{\pm 1\})_p] = [(U_F)_p^G : (\mathrm{Ra}_F.\{\pm 1\})_p^G],$$

since $U_F^G = U_L$ and $\mathrm{Ra}_L = \mathrm{Ra}_F^G$.

As $(\mathrm{Cl}_F)_p = \oplus_{\chi} \mathrm{Cl}_F^{\chi}$, and similarly for the other index, and by Theorem 5.9 (for $\chi \neq 1$) and the above for $\chi = 1$, the orders of the factors divide (because $\mathrm{Ra}_F.\{\pm 1\} \subset \mathcal{ES}_{F/L,M}$; see [Ru3], Proposition 1.2 and [Ru1], §1), they must be equal by the index formula. ∎

Note that this shows $(\mathrm{Ra}_F)_p = (\mathcal{ES}_{F/L,M})_p$ for F primary, if $p \nmid [F:L]$, and M is sufficiently large: a kind of local determination of cyclotomic units (first noted by Rubin, and true for all real cyclic extensions with Ra_N replaced by a lattice of units introduced by Gillard in [Gi]).

COROLLARY 6.12. *Conjecture 6.9 (and so also Conjecture 6.10) is true if N is primary.*

PROOF. By the uniqueness property of canonical factorisation (see [H1], 2.11) the $\mathcal{H}$-factorisation g' of $\mathrm{cl}(f) - [\mathbf{Z}, .2, \mathbf{Z}]$ is uniquely determined by the values $g'_p\big(\mathrm{ind}_C^G \inf_{C_\times}^C(\chi)\big)$ where g'_p is the local component of g' ([H1], 2.15), $C = C_p \times C_\times$ (C_p the p-part of C) is a cyclic subgroup of G and χ an irreducible $\mathbf{Q}_p$-value character of $C_\times$. So it suffices to apply Theorem 6.11 in the case $F/L = N^{C_p}/N^C$, which shows that $g' = g = 1$. ∎

REMARKS 6.13. (i) Ted Chinburg and David Burns pointed out to me that an equivalent form of Conjecture 6.10 for $[N:\mathbf{Q}]$ a prime power can also be deduced from [F1].

(ii) In [Bu3], Burns obtains the full Chinburg third conjecture for certain real abelian extensions of prime-power degree. Similar results will hold in the analogous elliptic cases, that is, for certain abelian extensions of quadratic imaginary fields, and will appear elsewhere. These examples show very clearly to what degree the module-theory will generate approximations to the Chinburg conjecture and to what degree the extension class data is required to obtain the full conjecture.

(iii) It is illuminating to consider the work of [R-W] (see also [G-W]) in this light, where invariants independent of extension class of certain finite Galois N/K are introduced.

NOTE ADDED IN PROOF. A proof of Conjecture 6.9 (and hence also Conjecture 6.10) has been obtained in [G-H])

References

[B1] D. J. Burns, *On integral Stark conjectures (and multiplicative Galois module structures)*, J. reine angew. Math. **422**(1991), 141–163.

[B2] ______, *Canonical factorisability and a variant of Martinet's conjecture*, J. London Math. Soc. (2) **44**(1991), 24–46.

[Bu3] ______, *On Multiplicative Galois structure invariants*, (1992), preprint.

[C1] T. Chinburg, *On the Galois structure of algebraic integers and S-units*, Invent. Math. **74**(1983), 321–349.

[C2] ______, *Multiplicative Galois module structure*, J. London Math. Soc (2), **29**(1984), 23–33.

[C3] ______, *Exact sequences and Galois module structure*, Ann. Math. **121**(1985), 351–376.

[F1] A. Fröhlich, *L-values at zero and multiplicative Galois module structure (also Galois Gauss sums and additive Galois module structure)*, J. reine angew. Math. **397**(1989), 42–99.

[F2] ______, *Units in real, abelian fields*, J. reine angew. Math. **429**(1992), 191–217.

[F3] ______, *Module Defect and Factorisability*, Ill. J. Math., **32**(1988), 407–421.

[Gi] R. Gillard, *Remarques sur les unités cyclotomiques et les unités elliptiques*, J. Number Theory **11**(1979), 21–48.

[G-H] C. Greither and D. Holland, *Chinburg's third invariant in the defect class group for abelian extensions of Q*, preprint 1993

[G-W] K. W. Gruenberg and A. Weiss, *Galois invariants for units*, (1992), preprint.

[He] A. Heller, *Some exact sequences in algebraic K-theory*, Topology **3**(1965), 389–408.

[H1] D. Holland, *Additive Galois structure and Chinburg's invariant*, J. reine angew. Math. **425**(1992), 193–218.

[H2] ______, *Homological equivalences of modules and their projective invariants*, J. London Math. Soc. (2), **43**(1991), 396–411.

[H-W1] D. Holland and S. M. J. Wilson, *Localization and class groups of module categories with exactness defect*, Proc. Camb. Phil. Soc., **113** (1993) 233-251.

[H-W2] ______, *Factor equivalence of rings of integers and Chinburg's second invariant in the defect class group*, J. London Math. Soc., to appear.

[H-W3] ______, *Fröhlich's and Chinburg's conjectures in the factorizability defect class group*, J. reine angew. Math. **442**(1993), 1–17.

[K] V. A. Kolyvagin, *Euler systems (1988)*. In: The Grothendieck Festschrift (Vol. 2), Prog. in Math. **86**, Boston, Birkauser, (1990).

[Ra] K. Ramachandra, *On the units of cyclotomic fields*, Acta Arith. **12**(1966), 165–173.

[R-W] J. Ritter and A. Weiss, *On the local Galois structure of S-units and applications*, (1992), preprint.

[Ru1] K. Rubin, *Appendix: The Main Conjecture*. In: Cyclotomic Fields I and II (Combined Second Edition), Grad. Texts Math. **121**, Springer-Verlag, Berlin-Heidelberg-New York, (1990).

[Ru2] ______, *Stark units and Kolyvagin's Euler systems*, J. reine angew. Math. **425**(1992), 141–154.

[Ru3] ______, *The "main conjecture" of Iwasawa theory for imaginary quadratic fields*, Invent. Math. **103** (1991), 25–68.

[St] H. M. Stark, *L-functions at $s = 1$.IV. First Derivatives at $s = 0$*, Advances in Math. **35**(1980), 197–235.

[T] J. Tate, *Les Conjectures de Stark sur les Fonctions L d'Artin en $s = 0$*, Prog. in Math. **47**, Basel-Stuttgart-Boston, 1984.

[Wa] L. C. Washington, *Introduction to cyclotomic fields*, Grad. Texts Math. **83**, Springer-Verlag, Berlin-Heidelberg-New York, (1982).

Department of Mathematics and Statistics
McMaster University
1280 Main Street West
Hamilton, Ontario
L8S 4K1