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BOUNDARY VALUE PROBLEMS IN PLASTICITY THEORY*

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The techniques for solving boundary value problems in plasticity theory are reviewed. In illustration, three application areas are discussed

(a) to the mechanics of glaciers,

(b) to the analysis of strip rolling processes, and

(c) to computing the collapse of isotropic plates.

1. Introduction

The mathematical theory of plasticity is concerned with the permanent deformations of solids. It has found applications in many branches of engineering including structural mechanics, metal forming processes, soil - and geo-mechanics, fracture mechanics and tribology. The type of boundary values problems encountered in plasticity theory have a number of unusual features. For example the governing equations in the plastic regions are usually hyperbolic, but unlike most areas of application of such equations, solutions have to be found in finite regions of space rather than in semi-infinite space-time domains. In addition, as will be seen, there is seldom enough boundary data to formulate a Cauchy problem. Another problem arises

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from the fact that it is only rarely that the entire body will deform plastically so that the boundary of the plastically deforming region has to be found as part of the solution.

In this paper I will attempt to amplify these difficulties and to review the methods which have been successfully employed to overcome them. In illustration three problems will be discussed in some detail:

(a) the modelling of a glacier;

(b) the analysis of the deformation in the roll gap of the hot strip-rolling process;

(c) the collapse mechanisms of isotropically reinforced concrete slabs.

As will be seen the solution to these problems all contain a strong element of trial and error, and to solve them it is first necessary to build up a good physical feeling for the nature of plastic deformations.

2. Elements of plasticity theory

Two idealised models commonly used to describe the stress-strain behaviour of a ductile solid in a simple tension test are illustrated in Figure 1. In the first, elastic-plastic, model the material behaves as a

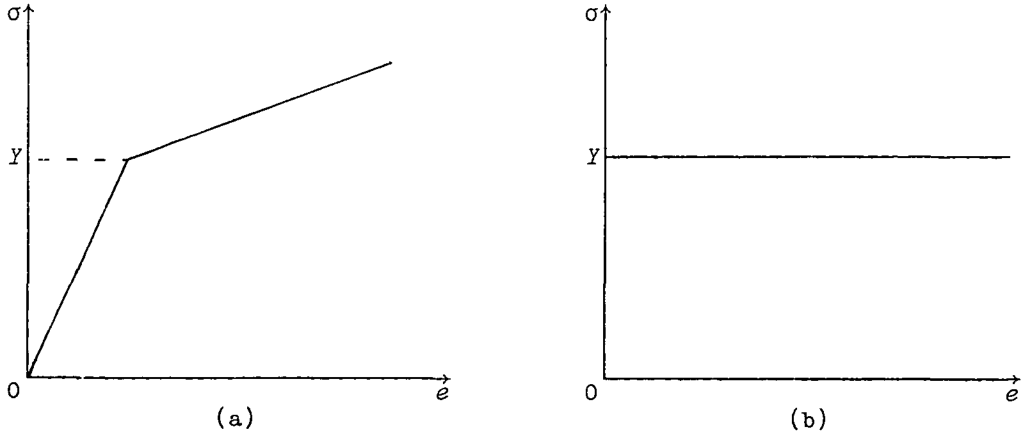

FIGURE 1. Stress-strain relation in tensile test

(a) Elastic/plastic model
(b) Rigid perfectly plastic model.

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linear elastic solid until the stress reaches a critical yield value Y after which it hardens plastically. If the applied stress is removed during the elastic phase of the loading, the strain returns to zero. However if the stress is removed after it has exceeded the critical value Y, only the elastic part of the strain is recovered. In the second, simplified, rigid/perfectly plastic model the elastic strains are ignored and the material remains rigid until the stress reaches the yield value after which it deforms plastically but at the same stress level. Because of the relative simplicity of this latter model it is possible to develop a theory in which semi-analytical solutions can be obtained without resort to too expensive and time-consuming computations. Such simplified solutions frequently lead to a good understanding of the basic mechanics of the phenomena or process involved, even if they give no information on secondary effects such as "elastic springback or recovery". The extra information yielded by using the more accurate but computationally expensive models is frequently only of marginal practical value - see for example the recent review by Johnson [11].

In a general three-dimensional situation the concept of yield is described by a yield function $f(\sigma_{ij})$. If $f(\sigma_{ij}) < 0$ the behaviour is elastic (or rigid) and only when $f(\sigma_{ij}) = 0$ can the material deform plastically. The equation $f(\sigma_{ij}) = 0$ defines a (convex) yield-surface in 6 dimensional stress space. The plastic deformation is such that the (plastic) strain-rate tensor e_{ij} is proportional to $\partial f/\partial\sigma_{ij}$ and can hence be represented by a 6-dimensional "vector" directed along the outward normal to the yeild surface. If the material is isotropic this geometric interpretation can be formulated in 3-dimensional principal stress space. If the stress state is further restricted as in plane stress or plate theory the yield surface can be given a 2-dimensional representation (see Figure 6, p.). As is explained in the next section, under plane strain conditions the theory is particularly simple, since the governing equations for the stress and velocity fields turn out to be always hyperbolic. The analysis of problems with axial symmetry or under plane stress conditions, such as small deformation plate theory, is more complicated because the governing equations are variously elliptic, hyperbolic or parabolic depending on the position of the current stress point on the yield locus. Problems of this type are discussed in Section 4.

3. Plane strain theory

Under plane strain conditions both the commonly employed yield conditions of von Mises and Tresca reduce to the statement that the material yields when the maximum shear stress reaches a maximum value k say. Elementary stress analysis shows that this maximum shear stress is attained on line segments whose directions bisect the two principal axes. The trajectories of these maximum shear stress directions form two mutually orthogonal families of curves - called sliplines, and labelled α- and β- quadrant. When referred to the curvilinear co-ordinate slipline network it is found that the (quasi-static) equilibrium equations reduce to a pair of ordinary differential equations, showing that the sliplines are the characteristics of the equilibrium equations, which integrate to give the invariant Hencky relations:

(1) $$p \pm 2k\psi = \text{constant on } \alpha\text{- }/\beta\text{- lines}$$

where p is the in-plane mean pressure (first stress invariant) and ψ is the anticlockwise rotation from an arbitrary chosen reference axis to the α-slipline. Once the slipline network has been found and provided the pressure is known at one point of the network, the Hencky relations can be used to find the pressure and hence the complete stress state at all points in the plastically deforming region. A simple geometrical consequence of (1), known as Hencky's first theorem, is illustrated in Figure 2. This states that the angle turned through in going along a slipline from A to B is equal to that in going from C to D, that is, $\psi_B - \psi_A = \psi_D - \psi_C$. The same is true for the other family of sliplines since $\psi_D - \psi_B = \psi_C - \psi_A$. Two simple examples of orthogonal networks possessing this so called "Henchy's-Prandtl property" which occur frequently in solutions are

(a) a constant state region in which both families of sliplines consist of parallel straight lines and in which the stress state is uniform, and

(b) a centred fan region consisting of radial straight lines and

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concentric circular arcs.

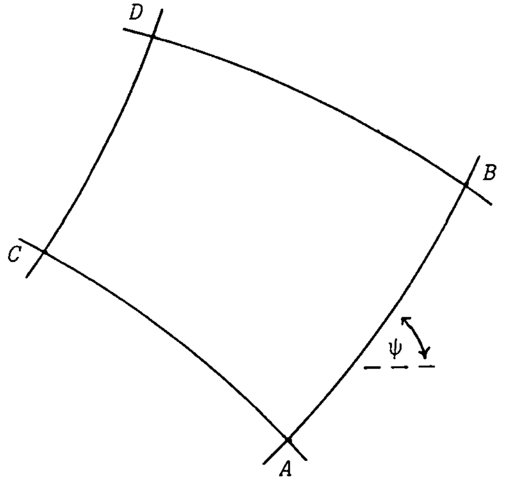

FIGURE 2. Hencky's first theorem.

The complete slipline field solution to a given problem will consist of a number of subregions in each of which the solution is analytic. The solution in any such subregion can be constructed using one of the three basic constructions for second order quasilinear hyperbolic systems [9]:

(a) the Cauchy problem in which two pieces of information (for example two traction components) are given on a non-characteristic curve,

(b) the Riemann or Characteristic Initial Value Problem where two sliplines, one from each family are given, and

(c) the Goursat or Mixed Problem where one piece of information is given on a non-characteristic curve together with the shape of one slipline.

These constructions can occasionally be formulated analytically but usually one has to resort to numerical procedures. The traditional method has been to use finite-difference approximations [10] but recently more powerful methods have been developed using a double-power series form of solution of the governing equations [6]. There are several possible geometrical quantities which can be used to define a slipline. One of the most convenient is the radius of curvature, for if ρ_α and ρ_β are the radii

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of curvature of the α- and β-lines they are found to satisfy the telegraph equation system

(2) $$\rho_{\alpha,\beta} - \rho_\beta = 0 , \quad \rho_{\beta,\alpha} + \rho_\alpha = 0 ,$$

where α and β are the angles turned through along the α- and β-lines respectively. As shown by Ewing [6] this sytem has the double-power series solution

(3) $$\rho_\alpha(\alpha, \beta) = \sum_{m,n=0}^{\infty} a_n \alpha_{m+n}(-\beta)_m + b_n(-\alpha)_m \beta_{m+n+1} ,$$

$$\rho_\beta(\alpha, \beta) = \sum_{m,n=0}^{\infty} -a_n \alpha_{m+n+1}(-\beta)_m + b_n(-\alpha)_m \beta_{m+n} ,$$

(where $\alpha_m = \alpha^m/m!$, and so on) which reduce to

(4) $$\rho_\alpha(\alpha, 0) = \sum_{n=0}^{\infty} a_n \alpha_n \quad \text{and} \quad \rho_\beta(0, \beta) = \sum_{n=0}^{\infty} b_n \beta_n$$

on the two base characteristics $\beta = 0$ and $\alpha = 0$. Thus if the two sets of coefficients $\{a_n\}$ and $\{b_n\}$ are known, (3) gives the solution at any point. This is hence the solution to the Characteristics Initial Value Problem. That for the Cauchy and Mixed Problems can be formulated similarly. Collins [1] has shown that if a slipline is thought of as being represented by the column vector of coefficients in the series expansions in (4), then most of the basic constructions met with in computing complete solutions can be represented by a handful of basic matrix operators. These ideas have been incorporated in a systematic computational package of sub-routines by Dewhurst and Collins [5].

The straight forward "marching in" procedure for solving Cauchy problems can really only be used when the plastically deforming region spreads to a stress free surface where the normal and tangential traction components are specified zero. This is a common situation in geomechanics problems but occurs only rarely in metal forming analyses, where the typical boundary condition at a tool/workpiece interface involves both velocity and traction components.

An unusual example of the application of slipline theory in geomechanics is Nye's [*14*] rigid/plastic model of a valley glacier. Although this model is highly simplified it introduced many basic ideas and was the first serious attempt at understanding the mechanics of glaciers. Nye took as his model a parallel sided slab of rigid/plastic material deforming under plane strain conditions and resting on a slope inclined at an angle α to the horizontal (Figure 3, p. 128). On a long time scale the constitutive law appropriate for ice deformation is a power creep law (Norton's law or Glen's law as it is known in the glaciological literature). The strain-rate and stress deviator tensors are proportional:

(5) $$\dot{e}_{ij} = \lambda \sigma'_{ij}$$

and the proportionality constant λ is determined by the power law relation $\dot{e} = (\sigma/A)^n$ between the second invariants of the strain-rate $\dot{e} = (\frac{1}{2}\dot{e}_{ij}\dot{e}_{ij})^{\frac{1}{2}}$ and of the stress deviator $\sigma = (\frac{1}{2}\sigma'_{ij}\sigma'_{ij})^{\frac{1}{2}}$. The power n is one for a linear viscous fluid but is in the range 3-4 for ice. The rigid/perfectly plastic material is obtained in the limit as $n \to \infty$. The real behaviour of ice is hence intermediate between that of these two idealised model materials.

The motion of a glacier is due to gravity which was ignored in the derivation of the basic slipline equations (1). However since the material behaviour is unaffected by the hydrostatic part of the stress, the influence of gravity is easily accounted for by superposing a hydrostatic stress equal to ρg times the vertical height, where ρ is the ice density. This has the effect only of adding a linearly varying normal traction on the top free surface of the ice, which can now be treated as weightless. The solution to the Cauchy problem of finding the slipline field adjacent to a rectilinear surface with zero shear traction and linearly varying normal traction can be found analytically. It occurs in the solution for compression between long rough plates [*10*].

The slipline field consists of two families of cycloids, the curves of one family running together to form a straight line envelope on the glacier bed where the shear traction is maximal. The ice thickness is predicted to be

(6) $$h = h_0 \operatorname{cosec} \alpha \cong h_0/\alpha$$

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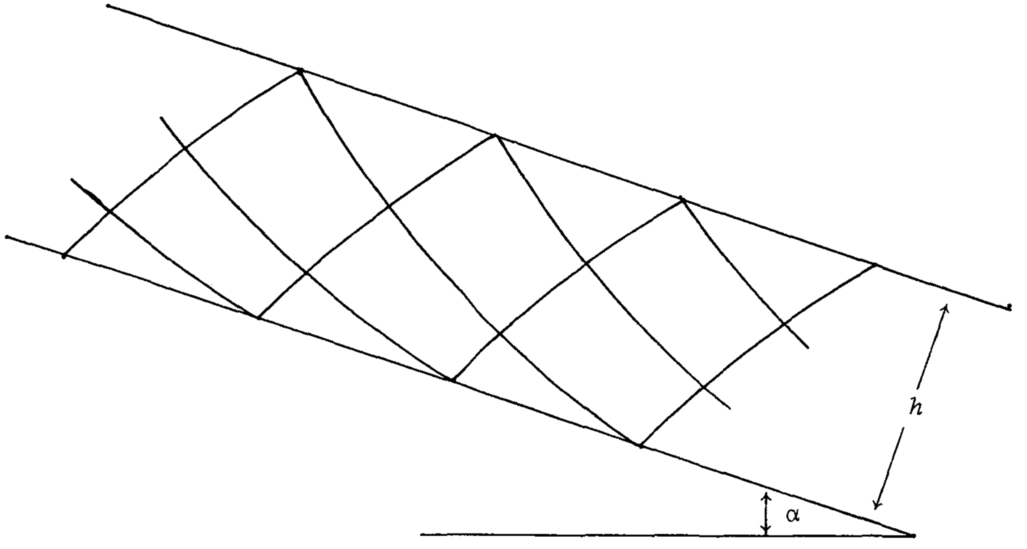
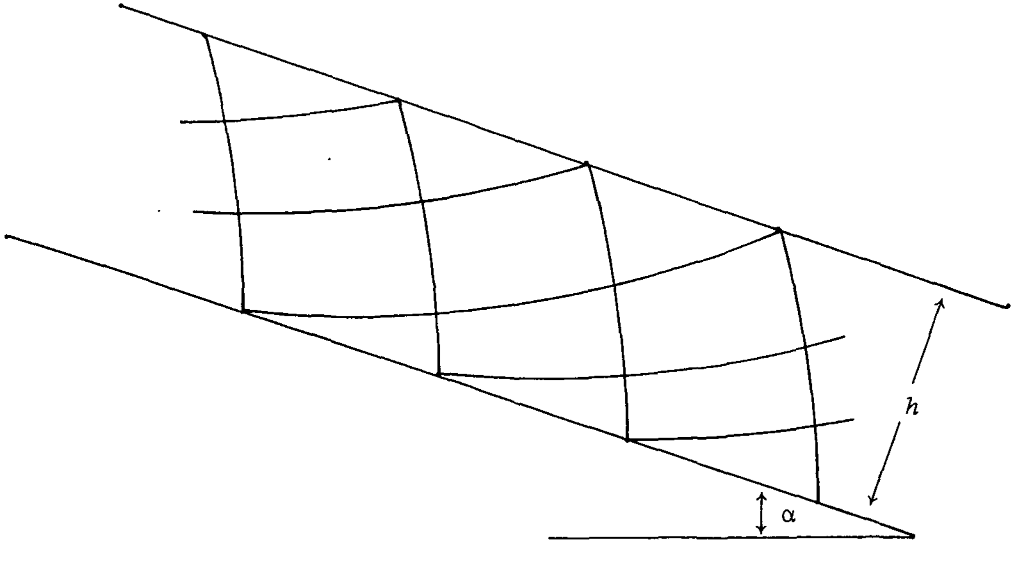


(a)



(b)

FIGURE 3. Slip solution for glacier
(a) Extending flow
(b) Compressive flow.

where $h_0 = k/\rho g$ is a characteristic length ($\cong$ 11 metres) . This model hence predicts that the ice thickness is independent of the rate of snow accumulation and is inversely proportional to the bed slope. Both these predictions are largely borne out in practice [15].

There are in fact two slipline solutions as shown in Figure 3. The correct solution is the one which leads to a positive dissipation-rate prediction when associated with a velocity field. Here it is found that the "active" solution in Figure 3 (a) is applicable when the normal velocity on the free surface is directed *inwards* corresponding to conditions on the upper parts of a glacier in the *accumulation zone* where there is a net gain in material over the year. The "passive" solution of Figure 3 (b) corresponds to the *ablation zone* on the lower slopes of the glacier. It is of interest to note that the longitudinal stress component σ_{xx} is everywhere compressive in the "passive" solution, but is tensile to a depth of approximately $2h_0$ ($\cong$ 23 metres) below the surface in the "active" solution. This explains the observation that crevasses occur most frequently in the accumulation zone and provides a reasonably accurate estimate of their depth. Since in hyperbolic systems small disturbances are propagated along the characteristics, it follows that disturbances caused by small protuberances on the bed will be transmitted along the sliplines and appear on the surface somewhat displaced from the position of the protuberance.

Since this early paper by Nye many other theoretical studies have been made with improved and more complicated models. None of them however have made such significant advances to our basic understanding of glacier mechanics. For a recent review see Paterson [15].

As remarked above, in the analysis of metal forming problems, it is necessary to consider the velocity field as well as the stress field, because of the presence of boundary conditions on velocity components. A velocity solution has to satisfy two conditions:

(a) as a consequence of isotropy (or flow rule) it follows that the principal axes of stress and strain-rate coincide, so that the slipline directions are the directions of maximum shear strain-rate; and

(b) since the material is incompressible the velocity field is isochoric.

An analysis of these two conditions [*10*] shows that the slipline directions are

(a) the characteristics of the velocity equations (as well as of the stress equilibrium equations), and

(b) the directions of zero extension rate.

This latter condition, when formulated mathematically, gives the ordinary differential equations along the slipline characteristics (Geiringer's equations):

(7) $du - v d\psi = 0$ on β-lines and $dv + u d\psi = 0$ on β-lines

or equivalently

(8) $u,_{\alpha} - v = 0$ and $v,_{\beta} + u = 0$

where (u, v) are the velocity components in the (α, β) directions.

Two simple consequences of these equations are of particular importance.

(A) In a constant state region, where ψ is uniform, it follows that both velocity components can be constant so that the material is undergoing a rigid translation.

(B) In a centred fan, with straight α-lines say, it follows that u is constant along each α-line and the change in v between any two α-lines is constant along their lengths. Thus if the material is undergoing a rigid translation on one side of a fan, the resulting velocity field is compatible with a rigid translation at the other extreme of the fan.

In order to illustrate the type of argument used to construct the slipline solution to metal forming processes consider the bar drawing process illustrated in Figure 4. The boundary conditions on the smooth die face are that the tangential traction and the normal velocity components are both zero. There is hence insufficient boundary data to formulate a Cauchy problem for the stress or for the velocity fields. Since the shear traction on AB face is zero, all sliplines must intersect this line at $45°$. *If* the normal traction were known to be uniform, on AB , then we

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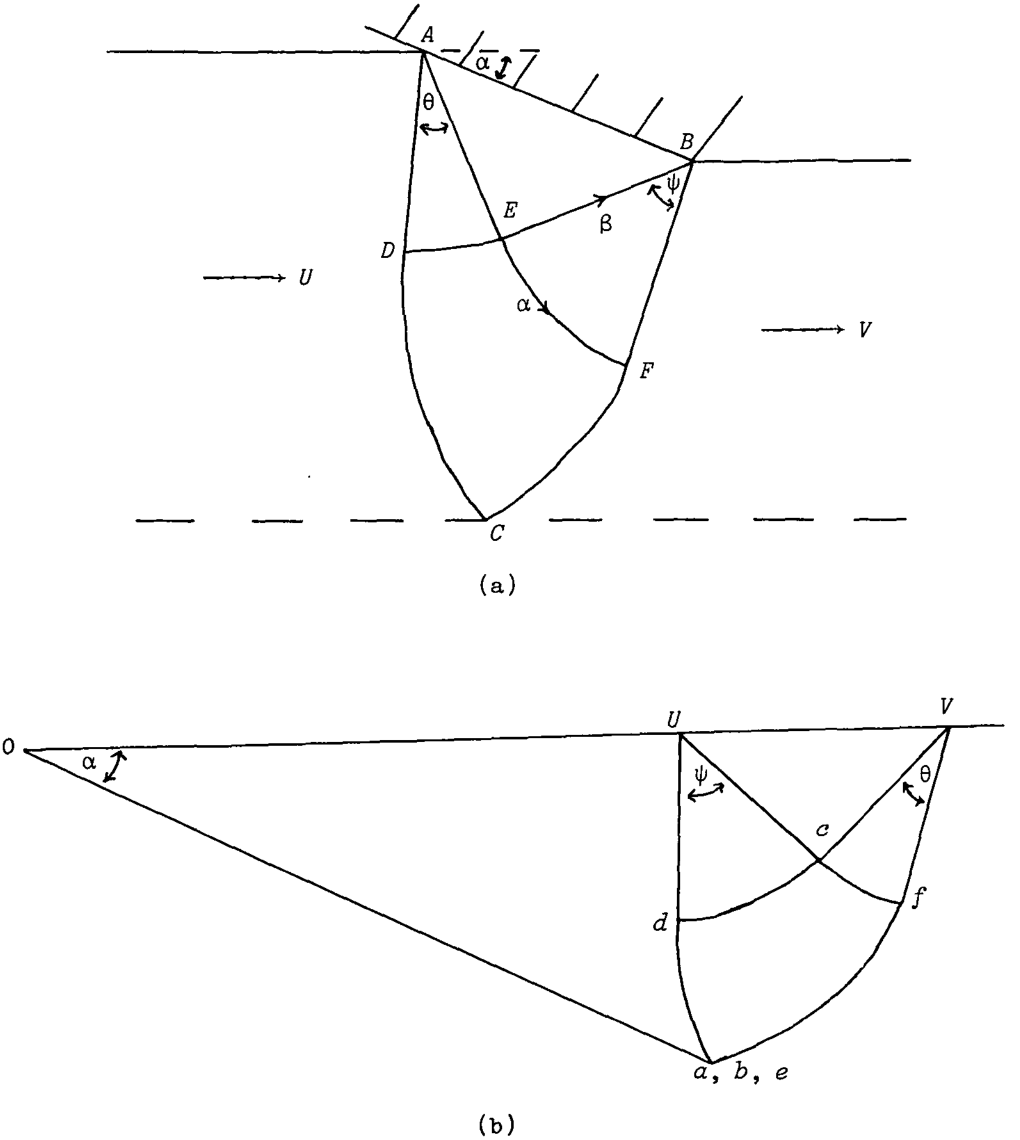

FIGURE 4. Slipline solution for bar drawing

(a) Slipline solution
(b) Hodograph.

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would have a Cauchy problem for stress and in fact *ABE* would have to be a constant state region. The regions *ADE* and *BFE* would then have to be centred fans and *EFCD* would be the field between the two circular arcs *ED* and *EF* . Now from lemma (B) above it follows that the rigid motion at entry is "communicated" through the fan *ADE* so that the material in the constant state triangle *ABE* slides rigidly over the die face, and furthermore that this rigid motion is further "communicated" through the exit fan *BEF* and is compatible with the rigid motion of the drawn strip. It is thus for these *kinematic* reasons that the pressure distribution on the die face must be uniform. Any other distribution would lead to curved sliplines and the resulting velocity distribution on the exit slipline would not be consistent with the rigid motion of the drawn strip. This semi-inverse solution procedure is typical of plasticity theory. There is a uniqueness theorem, though a rather weak one, which guarantees the correctness of solutions obtained in this way. One further feature of this velocity solution is worthy of comment. There are tangential velocity discontinuities across the sliplines *ADC* and *BFC* . It is easily shown from Geiringer's equations that the magnitude of such a discontinuity must be constant along the length of the slipline. In real materials such discontinuities correspond to narrow bands of finite thickness through which the velocity is continuous but changes rapidly.

Many similar solutions based on the use of centred fans and constant change regions can be found in the literature [*10*], [*12*]. The rigid regions in these solutions are always either at rest or translating rigidly. New problems appear if the rigid region is rotating as in strip rolling processes (Figure 5). The shape of the starting slipline cannot now be straight and has to be determined by solving an integral equation (usually of Fredholm type) and requires the simultaneous consideration of the stress and velocity distributions. In order to discuss this latter class of solutions it is necessary to introduce the concept of a hodograph diagram.

There is a strong duality between the stress and velocity fields as is made evident by the fact that the sliplines are the characteristics of both. This duality is most strikingly demonstrated by the introduction of the hodograph or velocity diagram. As a point P traces out a slipline field in the physical plane, its image point p , with position vector

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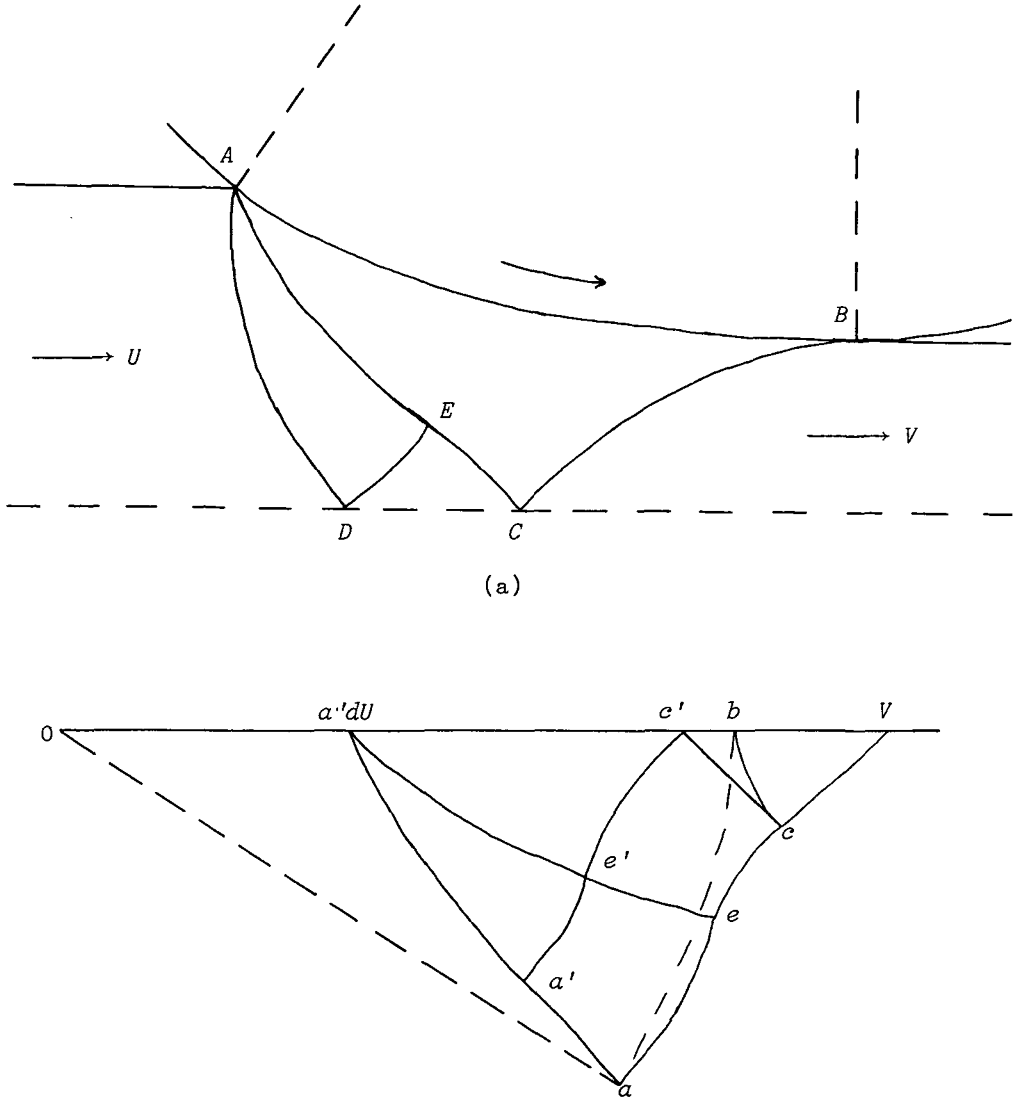

FIGURE 5. Slipline solution for hot rolling of strip

(a) slipline solution
(b) Hodograph

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v - the velocity at *P* - traces out an image network in the velocity plane. Since the extension rate is zero in a slipline direction it follows that as *P* traces out an α- (or β-) line, *p* traces out an image α'- (or β'-) line which is everywhere *orthogonal* to the original slipline at corresponding points. It follows therefore that the hodograph network also forms a Hencky-Prandtl net. The hodograph of the deformation associated with the bar drawing solution is shown in Figure 4 (b). As can be seen it closely resembles the original slipline field - this is a common feature of solutions.

The mapping between the physical and hodograph planes is not (1-1) . For example a rigid translating region in the physical plane is mapped into a single point in the hodograph diagram (for example, the image points *U*, *V* and *abe* of the rigid regions at entry, exit and in *ABE* respectively). Similarly since the magnitude of a velocity discontinuity is constant along its length, such a discontinuity will be represented by two parallel curves in the hodograph diagram. If the material on one side of such a discontinuity is translating rigidly, its image will be a single point. The image of the velocity field on the other side of the discontinuity will hence have to be arcs of circles, for example *cd* and *cf* in Figure 4 (b).

Consider now the strip rolling solution shown in Figure 5. Only the upper half of the solution is illustrated, *DC* lies on the central line of symmetry on which the shear stress must be zero, so that all sliplines meet this line of symmetry at $45°$. In *AECB* the material rotates rigidly attached to the roll face and then crosses the circular arc velocity discontinuity *CB* at exit. This velocity discontinuity is reflected back up *CEA* from the line of symmetry. The material deforms continuously through the singular field *AED* and through *DEC* . Note that the entry slipline *AD* cannot be straight since otherwise the material when it emerges from *AE* would have to translate rather than rotate as explained above.

Assuming for the moment that the shape of *AD* is known, then the singular field *ADE* is uniquely determined as it is a degenerate case of the characteristic initial value problem where one of the base sliplines has degenerated into a point, namely *A* . The remainder of the network

DEC can then be determined as a mixed problem. Thus a slipline field of this general form can be found for an arbitrary choice of starting slipline *AD* . However only one such field will lead to a velocity solution which is compatible with a rigid rotation in *AECB* . Consider now the associated hodograph shown in Figure 5 (b). The images of the rigid strip at entry and exit are the points *U* and *V* . The image of the rotating rigid region *AECBA* is a geometrically similar region *aecba* , scaled up by a factor of ω , the angular speed of the rolls, and turned through 90° in the direction of rotation. This result follows immediately from the expression $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for the velocity of a rigidly rotating body. The tramline region *aecc'e'a'* represents the velocity discontinuity across *AEC* . Since the normal velocity on *CD* is zero by symmetry, the velocity solution in *CED* can be constructed as the solution to a mixed problem. In other words the region *c'e'd* can be constructed in the hodograph diagram. The field between *e'a* at *e'd* can now be determined as a characteristics initial value problem. In general the resulting field will not have a singular point at *d* . However this must be so since the velocity distribution on *AD* is a horizontal rigid translation. It is this final "consistency condition" in the hodograph diagram which determines the shape of the initial slipline *AD* in the physical plane.

When formulated analytically this class of problems lead to Fredholm integral equations. The matrix operator formulation of slipline constructions mentioned earlier [*1*, 5] was devised to compute such solutions. A large number of problems, with an essentially similar mathematical structure, have been successfully solved using this method including applications to machining, symmetrical and unsymmetrical extrusion, indenting, cutting and forging, rolling friction as well as to symmetrical and unsymmetrical strip rolling. A recent general review of the procedure has been given by Collins [*4*]. This matrix procedure is limited to linear problems however and there is at present no corresponding universally applicable procedure for problems involving non-linear integral equations which are encountered in problems involving Coulomb friction boundary conditions, such as cold rolling, and in the axisymmetric extension of the above theory.

The solutions obtained from rigid/plastic analyses have yielded much information of practical value. They would be even more useful if work hardening could be incorporated. This is not easy to do however due to the difficulty of understanding how the velocity discontinuities "open-up" in the presence of hardening [4].

4. Rigid/plastic plate theory

The problems encountered in plane strain are compounded when it comes to other two-dimensional situations such as axial symmetry, plane stress or plate theory, since the governing equations are now variously hyperbolic, elliptic or parabolic depending on the position of the stress point on the yield locus. As a representative class of problems consider the deformation of isotropically reinforced concrete plates which are most frequently modelled by a rigid/plastic material with a square yield criterion in principal moment space - see Figure 6 (a). The material remains rigid as long as the principal bending moments M_α, M_β satisfy $-M < M_\alpha \, , \; M_\beta < M$. The corresponding yield locus for an isotropic metal plate is a hexagon (Figure 6 (b)) assuming the Tresca yield condition or an ellipse if the Mises criterion is employed. When the plate yields, the ratio of the local principal curvature rates (K_α, K_β) is given by the direction of the outward normal to the yield surface.

Referred to the α- and β- principal moment trajectories or "yield lines" the (quasi-static) equilibrium equations are

$$(9) \qquad \begin{cases} \dfrac{\partial M_\alpha}{\partial s_\alpha} + \dfrac{M_\alpha - M_\beta}{\rho_\beta} - Q_\alpha = 0 \; , \\[2ex] \dfrac{\partial M_\beta}{\partial s_\beta} + \dfrac{M_\beta - M_\alpha}{\rho_\alpha} - Q_\beta = 0 \; , \\[2ex] \dfrac{\partial Q_\alpha}{\partial s_\alpha} + \dfrac{\partial Q_\beta}{\partial s_\beta} + \dfrac{Q_\alpha}{\rho_\alpha} + \dfrac{Q_\beta}{\rho_\beta} = -q \; , \end{cases}$$

where (Q_α, Q_β) is the shear force vector, q is the applied load/unit area and $\rho_\alpha = -\partial\psi/\partial s_\alpha$ and $\rho_\beta = \partial\psi/\partial s_\beta$ are the in plane radii of curvature of the yield lines, which just as for sliplines are inclined at the angles ψ and $\psi + \pi/2$ to an arbitrarily chosen x-axis.

For the square yield condition there are three basic plastic regimes.

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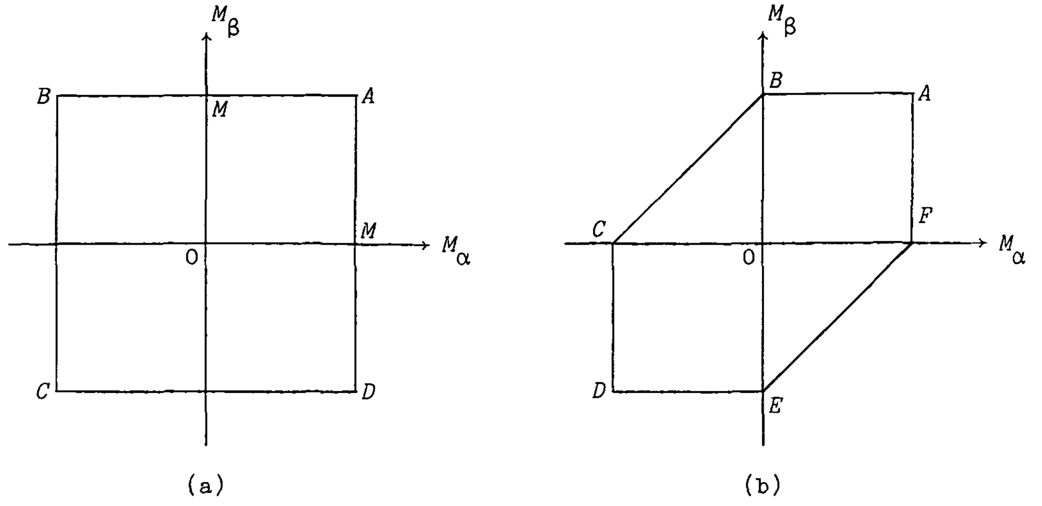

FIGURE 6. Yield loci for plates

(a) Square locus
(b) Tresca locus.

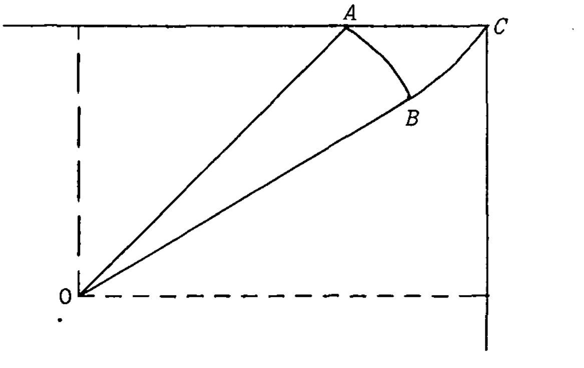

FIGURE 7. Yield line solution for collapse of simply supported rectangular plate with central point load (upper right quadrant illustrated).

Regime A. Since the two principal moments are now equal, the equilibrium equations require that $Q_\alpha = Q_\beta = q = 0$. This "isotropic" regime occurs only rarely in solutions and then only at isolated points.

Regime AB. From the normality rule it follows that K_α is zero. The plate is hence deforming instantaneously into a developable surface. The α-trajectories are straight lines and ρ_α is infinite. With these simplifications, the equilibrium equations, which are parabolic, can be integrated explicitly. Massonet [13] and others have derived a large number of solutions for simply supported plates under a variety of loading conditions in which the plastic region is entirely in this regime; the collapse mechanisms consisting of fans with one family of yield lines straight.

Regime B. At this point $M_\alpha = -M$, $M_\beta = M$ so that the first two equilibrium equations imply that $Q_\alpha = -2M/\rho_\beta$ and $Q_\alpha = 2M/\rho_\beta$, which on substitution in the third equation (9) gives a purely geometrical restriction on the shape of the yield line network. In the special case where $q = 0$, so that the only applied loads are point or edge loads, it can be shown [2], [7], [16] that this relation implies that the yield-line network forms a Hencky-Prandtl network. There is hence a precise analogy between plate solutions in this regime and the classical theory of slip-line fields. An example of a solution which exhibits this analogy is shown in Figure 7 (page). The deforming region consists of a centred fan OAB and its extension into ABC , where the simply supported boundary AC plays the rôle of a shear stress free surface. The governing equations are still hyperbolic when $q \neq 0$ and much the same solution techniques are applicable [3], [8].

In addition to the above static analogy there is a kinematic analogy between plate and plane strain problems [2]. The analogue of the plane strain velocity vector in plate theory is the angular velocity vector $\omega = (\omega_\alpha, \omega_\beta)$. It is readily shown [2] that since the yield line directions are principal curvature-rate trajectories they are also the directions of zero twist rate, so that

(10) $d\omega_\alpha - \omega_\beta d\psi = 0$ on α-lines, $d\omega_\beta + \omega_\alpha d\psi = 0$ on β-lines,

which are precise analogues of Geiringer's equations (7). The concept of a hodograph diagram carries over to plate problems as well and several examples are given by Collins [2].

For a Tresca material there is an additional plastic regime corresponding to the side *BC* of the yield locus (Figure 6 (b)) on which $K_\alpha = -K_\beta$ and the yield line network forms an isometric net [16]. The governing equations in this regime are elliptic.

In order to show that a postulated rigid/plastic solution is the unique exact solution and not just an upper bound collapse mechanism, it is necessary to find at least one statically admissible stress field in the postulated rigid regions. If these regions are unbounded, as is frequently the case in forming operations, such stress fields can usually be found without much difficulty. The problem is very much harder if the rigid regions are finite as in plate problems. The search for exact solutions in plastic plate theory is hence difficult and full of frustration, which has led Wood [17] to postulate the non-existence of solutions to a certain class of plate problems, typified by the uniformly loaded, clamped, square plate. The exact solution to this problem was eventually demonstrated by Fox [8], after much computational effort, who showed that the deforming zones contained both parabolic (*AB*) and hyperbolic (*B*) regions.

5. Conclusions

This paper has attempted to outline some of the difficulties inherent in constructing solutions in plasticity theory. Attention has been limited to the classical approach to boundary value problems by attempting to solve the governing partial differential equations rather than by setting up a variational formulation and using finite element techniques. Whilst these latter techniques clearly have much to offer they do not give the same physical feeling for the nature of the solutions which is inherent in the intimate connection between the mathematical structure of the governing equations (that is the characteristics) and the associated stress and deformation fields. To the author it is this intertwining of the mathematics and the mechanics which provides the chief fascination of the subject.

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