ON REGIONS OMITTED BY UNIVALENT FUNCTIONS II

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1. Introduction. Let S denote the family of functions $f(z)$ regular and univalent in $|z| < 1$, with the expansion $f(z) = z + a_2 z^2 + \ldots$ about $z = 0$, and let A_f denote the area of the intersection of the open circle $|w| < 1$ with D_f, the image of $|z| < 1$ under $f(z)$. A few years ago one of the authors **(1)** proved that if

(1) $$A = \inf_{f \in S} \{A_f\}$$

then

(2) $$.5000\pi \leqslant A < .7728\pi.$$

Recently Jenkins **(2)**, by a rather ingenious application of circular symmetrization, improved the lower bound for A to

(3) $$.5387\pi < A.$$

By introducing only a slight change in Jenkins' work we obtain a still better lower bound, namely

(4) $$.62\pi < A,$$

and at the same time we generalize Jenkins' result on the maximal length of arc on a circle $|w| = r$ left uncovered by D_f.

2. Values omitted by functions in $S(s)$. This section is devoted to proving the generalization just mentioned. Let s be the distance from the origin to the boundary of D_f. It is well known that $\frac{1}{4} \leqslant s \leqslant 1$ and that the extreme values correspond to the unique functions $f(z) = z/(1 + e^{i\theta}z)^2$, and $f(z) = z$, respectively. If we form the set $S(s)$ of all functions of S corresponding to a given fixed s, then as s runs from $\frac{1}{4}$ to 1 the sets $S(s)$ will exhaust S.

THEOREM 1. *Let $f(z) \in S(s)$, $\frac{1}{4} \leqslant s \leqslant 1$ and let $L(r, s)$ be the length of arc of the circle $|w| = r$ not covered by D_f. If*

(6) $$s \leqslant r \leqslant s/(2s^{\frac{1}{2}} - 1)$$

then

(7) $$L(r, s) \leqslant 2r \cos^{-1} (4s^{\frac{1}{2}} - 1 - r(8 - 4s^{-\frac{1}{2}})) \equiv \phi(r, s)$$

and this inequality is sharp in both variables.

Received February 23, 1954. The results presented in this paper were obtained independently and almost simultaneously by the two authors. Their results were identical up to and including equation (25), where it becomes necessary to estimate the maximum of the transcendental function defined by equations (24) and (25). Goodman obtained $A > .6028\pi$, while Reich by a suitable transformation, equation (27), and considerably more computation achieved $A > .62\pi$. In the case of Reich the work forms part of a Ph.D. thesis supervised by Professor E. F. Beckenbach at the University of California, Los Angeles.

Remarks. It is clear that if $r < s$, then $L(r, s) = 0$, and it will develop in the proof that if $r \geqslant s/(2s^{\frac{1}{2}} - 1)$ the entire circle $|w| = r$ may be omitted. In the range (6) the function

(8) $$a(r, s) \equiv 4s^{\frac{1}{2}} - 1 - r(8 - 4s^{-\frac{1}{2}})$$

satisfies $|a(r, s)| \leqslant 1$, and the principal branch of $\cos^{-1}$ is to be used in (7). In the special case $r = s$, (7) becomes

(9) $$L(r, r) \leqslant 2r \cos^{-1}(8r^{\frac{1}{2}} - 8r - 1),$$

a result obtained by Jenkins. The bound (9) is valid for all functions of S, but equality can occur only if $r = s$, since $\phi(r, s)$ is an increasing function of s for each r in $\frac{1}{4} \leqslant s \leqslant r \leqslant 1$.

Proof. We begin by constructing an explicit conformal mapping. Consider, in the ζ-plane, the domain D bounded by $|\zeta| = 1$ together with the two portions of the real axis $\zeta_1 = -\rho_1 \leqslant \zeta \leqslant -1$ and $1 \leqslant \zeta \leqslant \infty$, and distinguish the point $\zeta_0 = -\rho_0$ $(\rho_0 > \rho_1 > 1)$ within this domain. The function $\eta = \zeta + \zeta^{-1} + 2$ maps this domain on the η-plane slit along the real axis from $\eta_1 = -(\rho_1 + \rho_1^{-1} - 2) < 0$ to $+\infty$. The point ζ_0 goes into $\eta_0 = -(\rho_0 + \rho_0^{-1} - 2) < \eta_1$. The function $W = (\eta - \eta_1)^{\frac{1}{2}}$ (taking the positive determination on the upper side of the positive real axis) maps the preceding domain on the upper half W-plane, η_0 going into $W_0 = i(\eta_1 - \eta_0)^{\frac{1}{2}}$. Finally $z = -(W - W_0)/(W - \bar{W}_0)$ maps the latter domain on $|z| < 1$, W_0 going into $z = 0$. An elementary calculation shows that

(10) $$\left.\frac{d\zeta}{dz}\right|_{z=0} = \frac{4\rho_0(\rho_0 - 1)}{\rho_0 + 1} - \frac{4\rho_0^2(\rho_1 - 1)^2}{\rho_1(\rho_0^2 - 1)} > 0.$$

On the other hand the function $Z = \rho_0(\zeta^2 + \rho_0\zeta)/(\rho_0\zeta + 1)$ maps D on a domain D^* bounded by an arc on $|Z| = \rho_0$ placed symmetrically with respect to the positive real axis, together with the portions

$$Z_1 = \frac{\rho_0\rho_1(\rho_0 - \rho_1)}{\rho_0\rho_1 - 1} \leqslant Z \leqslant \infty$$

of the latter. The point ζ_0 goes into $Z = 0$. By an elementary calculation

$$\left.\frac{dZ}{d\zeta}\right|_{\zeta=\zeta_0} = \rho_0^2(\rho_0^2 - 1).$$

As a function of z, Z maps $|z| < 1$ on the domain D^* and

(11) $$\left.\frac{dZ}{dz}\right|_{z=0} = \frac{4\rho_0^3}{(\rho_0 + 1)^2} - \frac{4\rho_0^4(\rho_1 - 1)^2}{\rho_1(\rho_0^2 - 1)^2} \equiv K.$$

Thus the function $w = Z/K$ belongs to $S(s)$, where

(12) $$s = \frac{Z_1}{K} = \frac{\rho_1^2(\rho_0^2 - 1)^2}{4\rho_0^2(\rho_0\rho_1 - 1)^2}.$$

It maps $|z| < 1$ onto a domain $\tilde{D}(r, s)$ bounded by the portion $s \leqslant z \leqslant \infty$ of the real axis and an arc of the circle $|w| = r$, placed symmetrically with respect to the real axis. An easy computation yields

(13) $$r = \frac{\rho_0}{K} = \frac{\rho_1(\rho_0^2 - 1)^2}{4\rho_0^2(\rho_0\rho_1 - 1)(\rho_0 - \rho_1)}.$$

To determine the length of arc on $|w| = r$ on the boundary of $\tilde{D}(r, s)$ we note that the end points of the arc in D^* are the images of the points on $|\zeta| = 1$ where $dZ/d\zeta = 0$, i.e., the solutions of $\rho_0\zeta^2 + 2\zeta + \rho_0 = 0$. These are the points $\zeta = (-1 \pm i(\rho_0^2 - 1)^{\frac{1}{2}})/\rho_0$ and their images are

$$Z = (\rho_0^2 - 2 \pm 2i(\rho_0^2 - 1)^{\frac{1}{2}})/\rho_0.$$

The angle subtended by this arc is $\theta = 2\cos^{-1}(1 - 2/\rho_0^2)$ (the principal branch of $\cos^{-1}$ being used). It remains to determine ρ_0 as a function of s. Solving equations (12) and (13) simultaneously, yields

(14) $$\rho_0 = \frac{s^{1/4}}{((2r - s^{\frac{1}{2}})(2s^{\frac{1}{2}} - 1))^{\frac{1}{2}}},$$

and

(15) $$\rho_1 = \frac{s^{3/4}}{r}\left(\frac{2r - s^{\frac{1}{2}}}{2s^{\frac{1}{2}} - 1}\right)^{\frac{1}{2}}.$$

The conditions $\frac{1}{4} \leqslant s \leqslant 1$, $s \leqslant r$ required by the domain $\tilde{D}(r, s)$, imply that $2r - s^{\frac{1}{2}} \geqslant 0$ and $2s^{\frac{1}{2}} - 1 \geqslant 0$, so that (14) and (15) determine positive values of ρ_0 and ρ_1. The construction of a function mapping $|z| < 1$ onto $\tilde{D}(r, s)$ can be effected in the manner described if ρ_0 and ρ_1 determined by (14) and (15) satisfy

(16) $$\rho_0 > \rho_1 > 1.$$

The condition $\rho_1 > 1$ leads from (15) to the equivalent condition

(17) $$(r - s)(r + s - 2rs^{\frac{1}{2}}) > 0,$$

and the requirement that $\rho_0 > \rho_1$ yields

(18) $$\rho_0 - \rho_1 = \frac{s^{1/4}(r + s - 2rs^{\frac{1}{2}})}{r((2r - s^{\frac{1}{2}})(2s^{\frac{1}{2}} - 1))^{\frac{1}{2}}} > 0,$$

and since the other factors are positive, both (17) and (18) are satisfied if and only if

(19) $$r + s - 2rs^{\frac{1}{2}} > 0.$$

This yields the right side of (6) and also gives

(20) $$s > 2r^2 - r - 2r(r^2 - r)^{\frac{1}{2}}.$$

This last expression coincides with the lower bound given in a theorem due to Pick and Nevanlinna (proved again in **(1)**) whereby a bounded function

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$|g(z)| < r$, $g(z) \in S$, cannot omit a point with modulus less than the left side of (20). This shows that if $r \geqslant s/(2s^{\frac{1}{2}} - 1)$ then the entire circle $|w| = r$ may be omitted by $f(z)$.

We have thus shown that if r and s are given, subject to the restriction (6), and $\frac{1}{4} \leqslant s \leqslant 1$, then (14) and (15) determine ρ_0 and ρ_1 satisfying (16), and the function $f(z)$ mapping $|z| < 1$ onto $\tilde{D}(r, s)$ can be constructed as described. For this function the arc omitted on $|w| = r$ has length

$$2r \cos^{-1}(1 - 2/\rho_0^2)$$

and using (14) this yields $\phi(r, s)$, the left side of (7).

The proof that $\phi(r, s)$ is a maximum for $L(r, s)$ in the class of functions $S(s)$ is almost identical with that given by Jenkins **(2)**, in the special case $r = s$, and hence will not be reproduced here.

3. A lower bound for A. We now apply Theorem 1 to obtain a new lower bound for A. We consider the various subsets $S(s)$ of S and (with the notation of §1) set

(21) $$A(s) = \inf_{f \in S(s)} \{A_f\},$$

and hence

(22) $$A = \inf_{\frac{1}{4} \leqslant s \leqslant 1} \{A(s)\}.$$

Let $B(s) = \pi - A(s)$, then by Theorem 1

(23) $$B(s) < B^*(s) \equiv \int_s^1 \phi(r, s)\, dr.$$

This integral can be evaluated in terms of the elementary functions:

(24) $$B^*(s) = F(\alpha, \beta, 1) - F(\alpha, \beta, s)$$

where

(25) $$F(\alpha, \beta, r) = \alpha^{-2}\{(\alpha^2 r^2 - \beta^2 - \tfrac{1}{2}) \cos^{-1}(\alpha r + \beta) + \tfrac{1}{2}(3\beta - \alpha r)(1 - (\alpha r + \beta)^2)^{\frac{1}{2}}\},$$

where $\alpha = 4s^{-\frac{1}{2}} - 8$ and $\beta = 4s^{\frac{1}{2}} - 1$. All that remains is to determine the maximum value of $B^*(s)$ for s in the interval $(\frac{1}{4}, 1)$. This, unfortunately, leads to a transcendental equation and we are forced to make a dull and detailed set of computations to secure a valid numerical bound for $B^*(s)$.

A table of $B^*(s)$ partially reproduced at the end of the paper, seems to indicate that $B^*(s) < 1.18 < .376\pi$, which would imply (4). To prove (4) it is however sufficient to show that

(26) $$B^*(s) < 1.19 < .38\pi.$$

The proof of (26) involved a great deal of tedious computation, but, briefly, was accomplished as follows.

In (23) change the variable of integration from r to t where

$$t^2 = 1 - 2s^{\frac{1}{2}} + r(4 - 2s^{-\frac{1}{2}}) = \tfrac{1}{2}\{1 - a(r, s)\} \geqslant 0.$$

Also put $\xi = 2s^{\frac{1}{2}} - 1$. As a result of these transformations we can write

$$B^*(s) = G(\xi) = \frac{(1+\xi)^2}{2\xi^2}\int_\xi^{\gamma(\xi)} (\xi + t^2)\, t \sin^{-1} t\, dt \tag{27}$$

where $0 < \xi \leqslant 1$ for $\frac{1}{4} < s \leqslant 1$, and

$$\gamma(\xi) = \left(\frac{3\xi - \xi^2}{1+\xi}\right)^{\frac{1}{2}}.$$

The formula (27) was found convenient for estimation because of the convexity properties of $\gamma(\xi)$ and the inverse sine. The next step was to find bounds for $\Delta G(\xi) = G(\xi + \Delta\xi) - G(\xi)$ as a function of ξ and $\Delta\xi$, useful for small $|\Delta\xi|$. This is a straightforward task, starting from (27), but we do not reproduce the result here as it is quite complicated and uninteresting. The last step of the procedure was to evaluate $G(\xi)$ at a series of about forty unequally spaced mesh points $\{\xi_i\}$ in [0, 1], and to apply the upper bound that had been obtained for $|\Delta G(\xi)|$ to show that $G(\xi)$ did not exceed 1.19 in any interval (ξ_i, ξ_{i+1}).

4. A supplementary remark. The upper bound on A stated in (2) shows that there exist $f \in S$ which omit an area

$$\pi - A_f > .7137 \tag{28}$$

in $|w| < 1$. Let $\tilde{S}$ be the subset of S for which (28) holds. We now prove

THEOREM 2. *$f \in \tilde{S}$ implies that D_f contains the circle $|w| \leqslant .295$.*

Proof. We show that $G(\xi) < 0.7135$ for $0 < \xi < 0.0868$.

Since

$$\sin^{-1} t \leqslant \frac{\sin^{-1}\gamma(\xi)}{\gamma(\xi)}\, t, \qquad 0 \leqslant t \leqslant \gamma(\xi),$$

we have, using (27),

$$G(\xi) \leqslant \frac{(1+\xi)^2 \sin^{-1}\gamma(\xi)}{2\xi^2\gamma(\xi)}\int_0^{\gamma(\xi)} (\xi t^2 + t^4)\, dt = \frac{(7+\xi)(3-\xi)\sin^{-1}\gamma(\xi)}{15}$$

$$\leqslant \frac{7+\xi}{5}\sin^{-1}\gamma(\xi) < 0.7135,$$

if $\xi \leqslant 0.0868$.

COROLLARY. *If there exists a function $g \in S$ for which $A_g = A$, then D_g contains the circle $|w| \leqslant 0.295$.*

s	$B^*(s)/\pi$	$A^*(s)/\pi \equiv 1-B^*(s)/\pi$
.25	.00000	1.00000
.30	.22639	.77361
.35	.29716	.70284
.40	.33799	.66201
.45	.36164	.63863
.50	.37305	.62695
.55	.37457	.62543
.60	.36741	.63259
.65	.35218	.64782
.70	.32908	.67092
.75	.29806	.70194
.80	.25881	.74119
.85	.21081	.78919
.90	.15315	.84685
.95	.08425	.91574
1.00	.00000	1.00000

References

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2. J. A. Jenkins, *On values omitted by univalent functions*, Amer. J. Math., *75* (1953), 406–408.

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