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Interior h^1 Estimates for Parabolic Equations with LMO Coefficients

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Abstract. In this paper we establish *a priori* h^1-estimates in a bounded domain for parabolic equations with vanishing LMO coefficients.

1 Introduction

We consider the uniformly parabolic operator $\mathcal{P}$ with discontinuous coefficients

$$\mathcal{P}u = u_t - \sum_{i,j=1}^{n} a_{ij}(x) D_{ij}u, \quad \text{a.e } x = (x', t) \in Q_T,$$

where $Q_T = \Omega \times [0, T]$, $T > 0$, and $\Omega \in \mathbb{R}^n$ is a bounded domain $D_i u = \partial u/\partial x_i$, $u_t = D_t u = \frac{\partial u}{\partial t}$, $D_{ij}u = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $Du = (D_1 u, \dots, D_n u, D_t u)$. The principal part of the operator is symmetric and uniformly elliptic, *i.e.*,

$$a_{ij} = a_{ji}, \quad \lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \lambda^{-1}|\xi|^2,\ \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in Q_T,$$

with the constant $\lambda \in (0, 1]$. It is well known that if $a_{ij} \in C(Q_T)$, the following interior estimates hold:

$$\|u_t\|_{L^p(\Omega' \times [0,T])} + \|D^2 u\|_{L^p(\Omega' \times [0,T])} \le C(\|\mathcal{P}u\|_{L^p(Q_T)} + \|u\|_{L^p(Q_T)}),\ 1 < p < \infty, \tag{1.1}$$

where $\Omega' \Subset \Omega$ and the constant C depends only on $n, p, \lambda, \Omega', T, \Omega$, and the moduli of continuity of the coefficients a_{ij}.

Recently the estimate was improved in [1], where it was shown that (1.1) is still true if $a_{ij} \in C(Q_T)$ is replaced by the weak condition $a_{ij} \in VMO$, the function space of vanishing mean oscillation, first introduced by D. Sarason [12].

On the other hand, it is well known that the corresponding estimate (1.1) does not hold at the endpoint $p = 1$. A natural question is whether it is suitable for $L^1(Q_T)$ space to be replaced by $h^1(Q_T)$ (local Hardy space). It is true for elliptic operators, as was shown in [14]. For parabolic operators, by contrast, there are no corresponding results for the operator $\mathcal{P}$ in the Hardy space.

The main purpose of the present work is show that

$$\|u_t\|_{h^1(\Omega' \times [0,T])} + \|D^2 u\|_{h^1(\Omega' \times [0,T])} \le C\left(\|\mathcal{P}u\|_{h^1(Q_T)} + \|u\|_{h^1(Q_T)}\right),$$

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where $\Omega' \Subset \Omega$ and the constant C depends only on $n, p, \lambda, T, \Omega', \Omega$ and the coefficients $a_{ij} \in$ LMO. The crucial point of our investigations is the establishment of suitable integral estimates of singular integral operators and their commutators with variable parabolic Calderón–Zygmund (PCZ) kernel. The expansion of the kernel into spherical harmonics allows us to reduce our considerations over integral operators with constant PCZ kernel possessing good enough regularity.

The paper is organized as follows. In Section 2 we introduce some notations and definitions, and we recall and prove some preliminary results. The key estimate is proved in Section 3. In Section 4 we give the main results.

2 Definitions and Preliminary Results

Fabes and Rivière [6] introduced the parabolic distance

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + t^2}}{2}}, \quad d(x, y) = \rho(x - y),$$

where $x = (x', t) = (x_1, \dots, x_n, t)$. A ball with respect to the metric d centered at $x_0 = (x_0', t_0)$ and of radius r is simply an ellipsoid

$$B_r(x_0) = \Big\{ x \in \mathbb{R}^{n+1} : \frac{|x' - x_0'|^2}{r^2} + \frac{(t - t_0)^2}{r^4} < 1 \Big\}.$$

Obviously, the unit sphere with respect to this metric coincides with the sphere in $\mathbb{R}^{n+1}$, a.e $\partial B_1(0) = S^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| = \big(\sum_{i=1}^n x_i^2 + t^2 \big)^{1/2} = 1\}$.

It is well known that C. Fefferman and E. M. Stein [7] established the real theory of Hardy space in $\mathbb{R}^n$. Furthermore, A. P. Calderón and A. Torchinsky [2–4] established the real theory of parabolic Hardy space and obtained the atomic decomposition theory for the parabolic Hardy space. Later, D. Goldberg [8] established the theory of local Hardy space. Now we briefly give the definition and the atomic decomposition theory for $H^1(\mathbb{R}^{n+1})$ and the local Hardy space $h^1(\mathbb{R}^{n+1})$.

Let $\phi \in C_0^\infty$ be a mollifier with supp $\phi \subset B(0, 1)$, $\phi \geq 0$, $\int \phi = 1$. In this article we keep such a ϕ fixed.

Definition 2.1 For a locally integral function f, define

$$M_\phi f(x) = \sup_{0<t} |f * \phi_t(x)|, \quad H^1 = \{f \in L^1 | M_\phi f \in L^1\},$$

$$m_\phi f(x) = \sup_{0<t<1} |f * \phi_t(x)|, \quad h^1 = \{f \in L^1 | m_\phi f \in L^1\},$$

equipped with norms $\|f\|_{H^1} = \|M_\phi f\|_{L^1}$ and $\|f\|_{h^1} = \|m_\phi f\|_{L^1}$, where $\phi_t(x) = t^{-(n+2)}\, \phi(t^{-1}x', t^{-2}x_{n+1})$.

A locally integrable function a with compact support contained in a ball $B = B_r$ is called an H^1-atom if $\|a\|_\infty \leq |B|^{-1} \leq C_n r^{-(n+2)}$ and $\int a(x)\, dx = 0$.

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Define a to be of type (a) if (i) $\|a\|_\infty \le C_n r^{-(n+2)}$, $r \le 1$ and (ii) $\int a(x)dx = 0$. Define a to be of type (b) if (ii) holds, and (i) holds only for $r > 1$. We also call both type (a) and (b) atoms h^1 atoms.

Theorem 2.2 *Suppose $f \in H^1$. There exist $\{\lambda_j\}_{j=1}^\infty$, $\{\mu_j\}_{j=1}^\infty \in l^1$, and H^1 atoms $\{a_j\}_{j=1}^\infty$, such that*

$$f = \lim_{N\to+\infty} \sum_{j=1}^{N} \lambda_j a_j = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where the convergence is in L^1. Moreover, $\|f\|_{H^1} \sim \inf \sum |\lambda_j|$, where the infimum is taken over all possible decompositions of f.

For an h^1 function there is a similar atomic decomposition. See [8].

Theorem 2.3 *Suppose $f \in h^1$. There exist $\{\lambda_j\}_{j=1}^\infty$, $\{\mu_j\}_{j=1}^\infty \in l^1$, type* (a) *atoms $\{a_j\}_{j=1}^\infty$, and type* (b) *atoms $\{b_j\}_{j=1}^\infty$ such that*

$$f = \lim_{N\to+\infty} \sum_{j=1}^{N} \lambda_j a_j + \lim_{N\to+\infty} \sum_{j=1}^{N} \mu_j b_j = \sum_{j=1}^{\infty} \lambda_j a_j + \sum_{j=1}^{\infty} \mu_j b_j,$$

where the convergence is in L^1. Moreover, $\|f\|_{h^1} \sim \inf(\sum |\lambda_j| + \sum |\mu_j|)$, where the infimum is taken over all possible decompositions of f.

As in [14], we define $H_q^1 = H^1 + L^q$ and $h_q^1 = h^1 + L^q$ with the usual infimum norm for $q \in (1, \infty)$:

$$\|f\|_{H_q^1} = \inf_{f=f+g} (\|h\|_{H^1} + \|g\|_{L^q}), \quad \|f\|_{h_q^1} = \inf_{f=f+g} (\|h\|_{h^1} + \|g\|_{L^q}).$$

It is easy to see that $h_q^1 = H_q^1$ and $\|f\|_{h_q^1} \sim \|f\|_{H_q^1}$ for $1 < q < \infty$.

For the sake of completeness we shall recall here the definitions and some properties of the spaces we shall use.

Definition 2.4 We say that the measurable and locally integrable function f belongs to BMO if the seminorm

$$\|f\|_* = \sup_B \frac{1}{|B|} \int_B |f(y) - f_B|\, dy$$

is finite. Here, B ranges over all parabolic balls in $\mathbb{R}^n$ with radius r and centered at some point x, and $f_B = \frac{1}{|B|} \int_B f(y)\, dy$. Then $\|f\|_*$ is a norm in BMO modulo constant functions under which BMO is a Banach space.

Definition 2.5 LMO is a subspace of BMO, equipped with the semi-norm

$$\|f\|_{\mathrm{LMO}} = \sup_{r<1} \frac{1 + |\ln r|}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\, dx + \sup_{r\ge 1} \frac{1}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\, dx.$$

Proposition 2.6 *If $\varphi \in \mathrm{LMO} \cap L^\infty$, $f \in h^1$, then $\varphi f \in h^1$ and*

$$\|\varphi f\|_{h^1} \le C(\|\varphi\|_{L^\infty} + \|\varphi\|_{\mathrm{LMO}})\|f\|_{h^1},$$

where C is a positive constant depending only on n.

Proof By the atomic decomposition of an h^1 function, we only need to show that $\|\varphi a\|_{h^1} \le C(\|\varphi\|_{L^\infty} + \|\varphi\|_{\mathrm{LMO}})$ for an h^1-atom a. Suppose a is supported in $B = B_r(x_0)$. If a is a type (b) atom, it is easy to see that $b = \|\varphi\|_{L^\infty}^{-1}\varphi a$ is also a type (b)-atom and $\|b\|_{h^1} \le 1$. Next suppose a is a type (a)-atom. By definition, we need to show that $\|m_\phi a\|_{L^1} \le C(\|\varphi\|_{L^\infty} + \|\varphi\|_{LMO})$, where ϕ is a fixed mollifier.

If $x \in 2B$, then by the L^2-boundedness of the parabolic Hardy–Littlewood maximal function M and the inequality $m_\phi f \le Mf$ we have

$$\begin{aligned}\int_{2B} m_\phi(\varphi a)(x)\,dx &\le C|B|^{1/2}\|m_\phi(\varphi a)\|_{L^2} \le |B|^{1/2}\|M(\varphi a)\|_{L^2}\\ &\le C|B|^{1/2}\|\varphi a\|_{L^2}\|\varphi\|_{L^\infty} \le C\|\varphi\|_{L^\infty}.\end{aligned}$$

If $x \notin 2B$, then for $y \in B$, $t < 1$, we have

$$\begin{aligned}|\phi_t(x - x_0) &\le C\rho(x - x_0)^{-n-2},\\ |\phi_t(x - y) - \phi_t(x - x_0)| &\le C\rho(y - x_0)\rho^{-n-3}.\end{aligned}$$

By the cancellation property and size condition for a, we get

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}\setminus 2B} m_\phi(\varphi a)(x)\,dx\\ &= \int_{\mathbb{R}^{n+1}\setminus 2B} \sup_{t<1}\Big|\int_B \phi_t(x-y)\varphi(y)a(y)\,dy\Big|\,dx\\ &\le \int_{\mathbb{R}^{n+1}\setminus 2B} \sup_{t<1}\int_B |\phi_t(x-y) - \phi_t(x-x_0)|\,|\varphi(y)a(y)|\,dydx\\ &\qquad + C\int_{\mathbb{R}^{n+1}\setminus 2B} \sup_{t<1}\Big|\int_B |\phi_t(x-y)(\varphi(y) - \varphi_B)a(y)\,dy\Big|\,dx\\ &\le C\int_{\mathbb{R}^{n+1}\setminus 2B} \rho(y-x_0)\rho(x-x_0)^{-n-3}|\varphi(y) - \varphi_B|\,|a(y)|\,dydx\\ &\qquad + C\int_{\{2r\le\rho(x-x_0)<1\}} \rho(x-x_0)^{-n-2}\int_B |\varphi(y) - \varphi_B||a(y)|\,dydx\\ &\le Cr\int_{\mathbb{R}^{n+1}\setminus 2B} \rho(x-x_0)^{-n-3}\,dx\int_B |a(y)|\,dy\|\varphi(y)\|_{L^\infty}\\ &\qquad + C\frac{|\ln 2r|}{|B|}\int_B |\varphi(y) - \varphi_B|\,dy\\ &\le C(\|\varphi\|_{L^\infty} + \|\varphi\|_{\mathrm{LMO}}).\end{aligned}$$

■

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As a corollary, we have the following two lemmas.

Lemma 2.7 *If* $f \in h^1_q$, $1 < q < \infty$, $\varphi \in L^\infty \cap \mathrm{LMO}$, *then* $\varphi f \in h^1_q$ *and*

$$\|\varphi f\|_{h^1_q} \le C(\|\varphi\|_{L^\infty} + \|\varphi\|_{\mathrm{LMO}})\|f\|_{h^1_q}.$$

Lemma 2.8 *If* $f \in \mathrm{LMO}$ *and* $f \ge c > 0$, *then* $f^{-1}, f^{1/2} \in \mathrm{LMO}$, *and*

$$\|f^{-1}\|_{\mathrm{LMO}} \le 2c^{-2}\|f\|_{\mathrm{LMO}}, \ \|f^{1/2}\|_{\mathrm{LMO}} \le c^{-1/2}\|f\|_{\mathrm{LMO}}.$$

Moreover, if $f,\ g \in L^\infty \cap \mathrm{LMO}$, *then*

$$\|fg\|_{\mathrm{LMO}} \le 2(\|f\|_{L^\infty}\|g\|_{\mathrm{LMO}} + \|g\|_{L^\infty}\|f\|_{\mathrm{LMO}}).$$

The proof of Lemma 2.2 can be found in [5, 14].

3 H^1_q-Estimates for Variable Singular Integrals

In this section, we will establish h^1_q-estimates for parabolic singular integral commutator with variable kernels. For the purpose, we first establish H^1_q-estimates for parabolic singular integral commutator.

Definition 3.1 A function $k(x)$ is said to be a parabolic Calderón-Zygmund (PCZ) kernel in the space $\mathbb{R}^{n+1}$ if

(i) k is smooth on $\mathbb{R}^{n+1} \setminus \{0\}$,
(ii) $k(rx', r^2t) = r^{-(n+2)}k(x', t)$ for each $r > 0$,
(iii) $\int_{\rho(x)=r} k(x)\, d\sigma_x = 0$ for each $r > 0$.

Lemma 3.2 ([1]) *Let k be a PCZ kernel. Then for any parabolic ball* B_0 *of center* x_0 *one has*

$$|k(x-y) - k(x_0 - y)| \le C\frac{\rho(x_0 - x)}{\rho(x_0 - y)^{n+3}}$$

for any $x \in B_0$ *and* $y \notin 2B_0$.

A parabolic integral operator T defined by $Tf(x) = \mathrm{p.v.}\int k(x-y)f(y)dy$. It is well known that such a parabolic singular integral can be extended to a bounded operator on L^q for $1 < q < \infty$; see [6].

For a locally integrable function b, define $[T, b]f = T(bf) - bTf$. We call $[T, b]$ the commutator generated by the singular integral operators T and function b. M. Bramanti and M. C. Cerutti [1] proved that $[T, b]$ is bounded in L^p for $1 < p < \infty$ if $b \in \mathrm{BMO}$. Moreover, there exists a constant C depending on n, p, k such that

$$\|[T, b]f\|_{L^p} \le C\|b\|_*\|f\|_{L^p}.$$

Next we give H^1_q-estimates for $[T, b]$.

Proposition 3.3 *If $b \in$ LMO and $1 < q < \infty$, then $[T, b]$ is a bounded operator from H^1_q to H^1_q. Moreover, there is a positive constant C such that*

$$\|[T, b]f\|_{H^1_q} \le C\|b\|_{\mathrm{LMO}}\|f\|_{H^1_q}.$$

Proof Adapting the same arguments of Theorem 3.1 in [14], we give an outline of the proof.

Noting the L^q boundedness of $[T, b]$ and $H^1_q = h^1_q$, to prove the proposition it suffices to prove that for any an H^1-atom a such that $\|[T, b]a\|_{h^1_q} \le C\|b\|_{\mathrm{LMO}}$. Suppose a is supported in $B = B(x_0, r)$. If $r \ge 1/8$, we have

$$\|[T, b]a\|_{L^q} \le C\|b\|_*\|a\|_{L^q} \le C\|b\|_{\mathrm{LMO}}.$$

Next we assume $r < 1/8$. Let $B_0 = (x_0, 1)$ and $B_0^C = \mathbb{R}^{n+1} \setminus B_0$. We decompose $[T, b]$ as follows:

$$\begin{aligned}
[T, b] &= (B(y) - b_B)Ta(y) - T((b - b_B)a)(y)\\
&= (b(y) - b_B)Ta(y)\chi_{4B}(y) + (b(y) - b_B)Ta(y)\chi_{B_0\setminus 4B}(y)\\
&\qquad + (b(y) - b_B)Ta(y)\chi_{B_0^C}(y)\\
&= I_1 + I_2 + I_3 + I_4,
\end{aligned}$$

where χ_E is the the characteristic function of the set E and $kB = B(x_0, kr)$ for $k \in \mathbb{N}$.

We first consider the term I_1. By the L^4 boundendess of T, we have

$$\begin{aligned}
\int_{8B} m_\phi I_1(x)\,dx &\le C|B|^{1/2}\|m_\phi I_1\|_{L^2} \le C|B|^{1/2}\|I_1\|_{L^2}\\
&\le C|B|^{1/2}\Big(\int_{4B}|b(y) - b_B|^4\,dy\Big)^{1/4}\Big(\int_B |a(y)|^4\,dy\Big)^{1/4}\\
&\le C\|b\|_* \le C\|b\|_{\mathrm{LMO}}.
\end{aligned}$$

Note that for $x \notin 2B_0$, $\phi_t * I_1(x) = 0$ and $t < 1$, we get

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}\setminus 8B} m_\phi I_1(x)\,dx &= \int_{2B_0\setminus 8B} m_\phi I_1(x)\,dx\\
&\le C\int_{2B_0\setminus 8B}\rho(x - x_0)^{-(n+2)}\,dx\int_{4B}|(b(y) - b_B)Ta(y)|\,dy\\
&\le C(1 + |\ln 8r|)\int_{4B}|(b(y) - b_B)Ta(y)|\,dy\\
&\le C\|b\|_{\mathrm{LMO}}.
\end{aligned}$$

Hence, we obtain $\|I_1\|_{h^1} \le C\|b\|_{\mathrm{LMO}}$. For the term I_2, note that if $x \in 2B$ and $y \notin 4B$, then $\rho(y-x) \ge 2r$ and $\rho(y-x_0) \ge 4r$. By the property of kernel k of T (see Lemma 3.2), we then have

$$\begin{aligned}|\phi_t * I_2(x)| &\le C\int_{4r\le\rho(x_0-y)<1}\rho(x-y)^{-(n+2)}|b(y)-b_B||Ta(y)|\,dy\\ &\le Cr^{-(n+2)}r\int_{4r\le\rho(x_0-y)}\rho(x_0-y)^{-(n+3)}|b(y)-b_B|\,dy\\ &\le Cr^{-(n+2)}\|b\|_*.\end{aligned}$$

Thus,

$$\int_{2B}|\phi_t * I_2(x)|\,dx \le Cr^{-(n+2)}\|b\|_*|B| \le C\|b\|_{\mathrm{LMO}}.$$

If $x \notin 2B$, we have

$$\begin{aligned}|\phi_t * I_2(x)| &\le \Big|\int_{(B_0\setminus 4B)\cap B(x,r)}\phi_t(x-y)(b(y)-b_B)Ta(y)\,dy\Big|\\ &\quad+\Big|\int_{(B_0\setminus 4B)\cap B^C(x,r)}\phi_t(x-y)(b(y)-b_B)Ta(y)\,dy\Big|\\ &= I_{21}+I_{22}.\end{aligned}$$

Let $B_j = 2^jB$. Then

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}\setminus 2B} I_{21}(x)\,dx\\ &\quad=\sum_{j=1}^{\infty}\int_{B_{j+1}\setminus B_j}I_{21}(x)dx\\ &\quad\le C\sum_{j=1}^{\infty}\int_{B_{j+1}\setminus B_j}r\int_{B(x,r)}\phi_t(x-y)\rho(x_0-y)^{-(n+3)}|b(y)-b_B|\,dydx\\ &\quad\le C\sum_{j=1}^{\infty}\int_{B_{j+1}\setminus B_j}r\rho(x_0-x)^{-(n+3)}m_\phi(|b(y)-b_B|\chi_{B_{j+2}})(x)\,dx\\ &\quad\le Cr\sum_{j=1}^{\infty}(2^jr)^{-(n+3)}\int_{B_{j+1}\setminus B_j}m_\phi(|b(y)-b_B|\chi_{B_{j+2}})(x)\,dx\\ &\quad\le C\|b\|_* \le C\|b\|_{\mathrm{LMO}}.\end{aligned}$$

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Note that $\phi_t(x-y)=0$ for $\rho(x-y)>1$. We then have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}\setminus 2B} & I_{22}(x)\,dx \\ &\leq C\int_{\rho(y-x_0)\geq 4r}\int_{r\leq\rho(y-x)\leq 1}\rho(x-y)^{-(n+2)}|b(y)-b_B||Ta(y)|\,dy \\ &\leq C(1+|\ln r|)r\int_{\rho(y-x_0)\geq 4r}\rho(x_0-y)^{-(n+3)}|b(y)-b_B|\,dy \\ &\leq C\|b\|_{\mathrm{LMO}}.\end{aligned}$$

Thus, $\|I_2\|_{h^1}\leq C\|b\|_{\mathrm{LMO}}$. We now estimate the term I_3. For $1<q<\infty$, we have

$$\begin{aligned}\int_{\mathbb{R}^{n+1}}|I_3(x)|^q\,dx &\leq Cr^q\int_{\rho(y-x_0)\geq 1}\rho(x-y)^{-(n+3)q}|b(y)-b_B|^q\,dy \\ &\leq Cr^q\int_{\rho(y-x_0)\geq r}\rho(x-y)^{-(n+2)-q}|b(y)-b_B|^q\,dy \\ &\leq C\|b\|_*^q\leq C\|b\|_{\mathrm{LMO}}^q.\end{aligned}$$

Finally, we estimate the term I_4. Similar to the proof of I_1, we have

$$\|(b-b_B)a\|_{h^1}\leq C\|b\|_{\mathrm{LMO}}.$$

Note that T is bounded on H^1 and L^q, respectively, so T is bounded on H^1_q. Hence,

$$\|T((b-b_B)a\|_{h^1_q}\leq C\|T((b-b_B)a\|_{H^1_q}\leq C\|(b-b_B)a\|_{h^1_q}\leq C\|b\|_{\mathrm{LMO}}.$$

Thus, the proof of Proposition 3.1 is complete. ■

In order to apply our results to the interior estimates of a parabolic operator, we need to generalize Proposition 3.3 to commutators of parabolic singular integrals with variable kernels. For this goal we shall exploit a well-known technique based on an expansion into spherical harmonics [1].

Any homogeneous polynomial $p(x), x\in\mathbb{R}^N$, of degree m, that is a solution of Laplace's equation $\triangle u=0$ is called an N-dimensional solid harmonic of degree m. Its restriction to the unit sphere Σ_N is called an N-dimensional spherical harmonic of degree m.

Denote by Y_m the space of $(n+1)$-dimensional spherical harmonics of degree m. It is a finite-dimensional space with $\dim Y_m=g_m$, where

$$g_m=C^n_{m+n}-C^n_{m+n-2}\leq C(n)m^{n-1}$$

and the second binomial coefficient is equal to 0 when $m=0,1$, *i.e.*, $g_0=1$, $g_1=n+1$. Further, let $\{Y_{sm}(x)\}_{s=1}^{g_m}$ be an orthonormal basis of Y_m. Then $\{Y_{sm}(x)\}_{s=1,m=0}^{g_m,\infty}$ is a complete orthonormal basis in $L^2(S^{n+1})$ and

$$\sup_{x\in S^{n+1}}\left|\left(\frac{\partial}{\partial x}\right)^\beta Y_{sm}(x)\right|\leq C(n)m^{|\beta|+(n-1)/2},\ m=1,2,\dots.$$

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If, for instance, $\varphi \in C^\infty(S^{n+1})$, then $\varphi \sim \sum_{s,m} Y_{sm}(x)$ is the Fourier series expansion of φ with respect to $\{Y_{sm}(x)\}$, where

$$a_{sm} = \int_{S^{n+1}} \varphi(y) Y_{sm}(y)\, d\sigma,\ |a_{sm}| \le C(l) m^{-2l} \sup_{|\beta|=2l, y\in S^{n+1}} \Big| \Big(\frac{\partial}{\partial x}\Big)^\beta \varphi(y)\Big| \tag{3.1}$$

for every $l > 1$, where $\sum_{s,m}$ stands for $\sum_{m=0}^{\infty} \sum_{s=1}^{g_m}$.

Let k be a real-valued function defined on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \setminus \{0\}$ such that $k(x,\,\cdot\,)$ is a PCZ kernel for a.e. $x \in \mathbb{R}^{n+1}$. For $f \in C_0^\infty(\mathbb{R}^{n+1})$ define

$$Sf(x) = \lim_{\epsilon\to 0} \int_{\rho(x-y)>\epsilon} k(x, x-y) f(y)\, dy.$$

The operator S is called a parabolic singular integral with variable kernel k. For a locally integrable function b, the commutator generated by S and b is defined as

$$[S,b]f(x) = bSf(x) - S(bf)(x) = \lim_{\epsilon\to 0} \int_{\rho(x-y)>\epsilon} k(x,x-y)[b(x) - b(y)] f(y)\, dy.$$

The following results about the L^q boundedness of S and $[S,b]$ can be found in [1].

Theorem 3.4 ([1]) *If S is a parabolic singular integral with variable kernel k satisfying*

$$\max_{|\beta|\le 2(n+1)} \Big\| \Big(\frac{\partial}{\partial x}\Big)^\beta k(x,y) \Big\|_{L^\infty(\mathbb{R}^n \times S^{n+1})} \le M_1, \tag{3.2}$$

then S can be extend to a bounded operator on $L^q(1 < q < \infty)$ such that

$$\|Sf\|_{L^q} \le C(n,q,M_1)\|f\|_{L^q}.$$

If $b \in$ BMO, then $[S,b]$ can be extend to a bounded operator on $L^q(1 < q < \infty)$ such that $\|[S,b]f\|_{L^q} \le C(n,q,M_1)\|f\|_{L^q}$.

Our main result in this section is the corresponding h_q^1 estimate for the commutator of a parabolic integral with variable kernel.

Theorem 3.5 *Let S and k be the same as in the above theorem. Assume that for every fixed $y \in S^{n+1}$, we have $k(\,\cdot\,, y) \in$ LMO and*

$$\max_{|\beta|\le 2(n+1)} \|\Big(\frac{\partial}{\partial x}\Big)^\beta k(x,y)\|_{L^\infty(\mathrm{LMO}(\mathbb{R}^n), S^{n+1})} \le M_2, \tag{3.3}$$

where

$$\|h(x,y)\|_{L^\infty(\mathrm{LMO}(\mathbb{R}^n),S^{n+1})} = \sup_{y\in S^{n+1}} \|h(\cdot,y)\|_{\mathrm{LMO}(\mathbb{R}^n)}.$$

If $f \in h_q^1$ for some $1 < q < \infty$, then there exists a constant C such that

$$\|Sf\|_{h_q^1} \le C\|f\|_{h_q^1}.$$

If $b \in$ LMO, then $\|[S,b]f\|_{h_q^1} \le C\|b\|_{\mathrm{LMO}}\|f\|_{h_q^1}$, where C is a constant depending only on n, q, M_1, and M_2.

Proof By the density arguments it is enough to prove the theorem for $f \in C_0^\infty(\mathbb{R}^{n+1})$. Let $x, y \in \mathbb{R}^{n+1}$ and $\bar{y} = \frac{y}{\rho(y)} \in S^{n+1}$. Keeping in mind the homogeneity properties of the variable PCZ kernel, we can write

$$\rho(y)^{n+2}k(x,y) = k(x,\bar{y}) = \sum_{s,m} a_{sm}(x)Y_{sm}(\bar{y}).$$

Hence $k(x,y) = \rho(y)^{-(n+2)} \sum_{s,m} a_{sm}(x)Y_{sm}(\bar{y})$. From (3.1), (3.2), and (3.3) it follows that

$$\|a_{sm}\|_{L^\infty} + \|a_{sm}\|_{\mathrm{LMO}} \le C(n, M_1, M_2)m^{-2(n+1)}.$$

By the argument in [1] we have

$$Sf = \sum_{m=1}^{\infty}\sum_{k=1}^{g_m} a_{km}(x)R_{km}f(x), \quad [S,b]f = \sum_{m=1}^{\infty}\sum_{k=1}^{g_m} a_{km}(x)[R_{km},b]f(x),$$

where

$$R_{km}f(x) = \lim_{\epsilon\to 0}\int_{\rho(x-y)>\epsilon} \frac{Y_{km}(x-y)}{\rho(x-y)^{n+2}} f(y)\,dy$$

are parabolic singular integrals. Using Proposition 3.3 and Lemma 2.7 we obtain

$$\begin{aligned}
\|Sf\|_{h^1_q} &\le \sum_{m=1}^{\infty}\sum_{k=1}^{g_m} \|a_{km}R_{km}f\|_{h^1_q}\\
&\le C\sum_{m=1}^{\infty}\sum_{k=1}^{g_m}(\|a_{sm}\|_{L^\infty} + \|a_{sm}\|_{\mathrm{LMO}})\|R_{km}f\|_{h^1_q}\\
&\le C\sum_{m=1}^{\infty}\sum_{k=1}^{g_m} m^{-2(n+1)}\|f\|_{h^1_q} \le C\sum_{m=1}^{\infty} m^{n-1}m^{-2(n+1)}\|f\|_{h^1_q}\\
&\le C\|f\|_{h^1_q}.
\end{aligned}$$

The argument for $[S,b]$ is similar; by Proposition 3.3 we get

$$\begin{aligned}
\|[S,b]f\|_{h^1_q} &\le \sum_{m=1}^{\infty}\sum_{k=1}^{g_m} a_{km}[R_{km},b]f\|_{h^1_q}\\
&\le C\sum_{m=1}^{\infty}\sum_{k=1}^{g_m} \|(\|a_{sm}\|_{L^\infty} + \|a_{sm}\|_{\mathrm{LMO}})\|[R_{km},b]f\|_{h^1_q}\\
&\le C\sum_{m=1}^{\infty}\sum_{k=1}^{g_m} m^{-2(n+1)}\|b\|_{\mathrm{LMO}}\|f\|_{h^1_q}\\
&\le C\|b\|_{\mathrm{LMO}}\sum_{m=1}^{\infty} m^{n-1}m^{-2(n+1)}\|f\|_{h^1_q}\\
&\le C\|b\|_{\mathrm{LMO}}\|f\|_{h^1_q}.
\end{aligned}$$

■

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4 Main Results

In this section we give the interior Hardy type estimates for a second order parabolic operator. For this purpose, we first introduce some Hardy type space on a domain by restriction. Let $\Omega_T = \Omega \times [0, T]$ with a fixed $T > 0$ and Ω a domain in $\mathbb{R}^n$.

A measure function $f \in L^1(\Omega_T)$ is said to be in $h^1(\Omega_T)$ if it is a restriction to Ω_T of a function $F \in h^1(\mathbb{R}^{n+1})$, *i.e.*,

$$h^1(\Omega_T) = \{f \in L^1(\Omega_T) : \text{ there exists } F \in h^1(\mathbb{R}^{n+1}) \text{ such that } F|_{\Omega_T} = f\},$$

equipped with the norm

$$\|f\|_{h^1(\Omega_T)} = \inf\{\|f\|_{h^1} : F \in h^1(\mathbb{R}^{n+1}), F|_{\Omega_T} = f\}.$$

We say that the function u lies in the Hardy–Sobolev spaces $h^{2,1}(\Omega_T)$, if it is weakly differentiable and belongs to $h^1(\Omega_T)$ along with all its derivatives $D_t^2 D_x^s u$, $0 \le 2r+s \le 2$. Then the following norm is finite

$$\|u\|_{h^{2,1}(\Omega_T)} = \|u\|_{h^1(\Omega_T)} + \|D^2 u\|_{h^1(\Omega_T)} + \|D_t u\|_{h^1(\Omega_T)}.$$

We sometimes use the same notation to represent a function defined in $\mathbb{R}^{n+1}$ and its restriction to a domain.

Next we state some lemmas.

Lemma 4.1 *Let Ω_T be a bounded domain in $\mathbb{R}^{n+1}$. If $u \in h^1(\mathbb{R}^{n+1})$ and* supp $u \subset \Omega_T$, *then $\|u\|_{h^1} \le C\|u\|_{h^1(\Omega_T)}$, with C depending on Ω_T. Hence $\|u\|_{h^1} \sim \|u\|_{h^1(\Omega_T)}$.*

Lemma 4.2 *The restriction of h_q^1 to a bounded domain Ω_T is $h^1(\Omega_T)$ for any $q \in (1, \infty)$. In other words, if $f \in h_q^1$, then $f|_{\Omega_T} \in h^1(\Omega_T)$ with $\|f\|_{h^1(\Omega_T)} \le C\|f\|_{h_q^1}$.*

Now we introduce the space vanishing LMO.

Definition 4.3 We say a function $f \in$ LMO is in the space vanishing LMO, denoted by LMO_0, if $\lim_{t\to 0} \eta_f(\rho) = 0$, where

$$\eta_f(\rho) = \sup_{r<\rho} \frac{1 + |\ln r|}{|B_r|} \int_{B_r} |f(x) - f_{B_r}|\, dx, \quad \rho < 1.$$

Lemma 4.4 LMO_0 *is a closed subspace of* LMO *and it coincides with the closure of* $C^\infty \cap \text{LMO}_0$ *in* LMO. *Moreover, if $f \in L^\infty \cap \text{LMO}_0$, one can choose a sequence of functions $\{f_k\}_{k=1}^\infty \subset C^\infty \cap L^\infty$ such that $f_k \to f$ in* LMO *and $\|f_k\|_{L^\infty} \le \|f\|_{L^\infty}$.*

The proofs of Lemmas 4.1, 4.2, and 4.4 are simple. We will omit the details; see also [14].

We consider the uniformly parabolic operator

$$\mathcal{P}u = u_t - \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u,$$

with a_{ij} defined a.e in $\mathbb{R}^{n+1}$. We assume that there exists a constant $\lambda \in (0, 1]$, such that

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(A1) $a_{ij}(x) = a_{ji}(x),\ \lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda^{-1}|\xi|^2, \forall \xi \in \mathbb{R}^n$, a.e, $x \in \mathbb{R}^{n+1}$;
(A2) $a_{ij} \in \mathrm{LMO} \cap L^\infty$, with $\max_{i,j\le n}(\|a_{ij}\|_{\mathrm{LMO}} + \|a_{ij}\|_{L^\infty}) \le \lambda^{-1}$.

Denote by $\mathcal{P}_0$ a linear parabolic operator with constant coefficients a_0^{ij} that satisfy (A1). It is well known from linear theory (see [10]) that the fundamental solution of the operator $\mathcal{P}_0$ is given by the formula

$$\Gamma^0(y) = \begin{cases} \dfrac{1}{(4\pi\tau)^{n/2}\sqrt{\det a_0}} \exp\Big\{ -\dfrac{\sum_{i,j=1}^n A_0^{ij} y_i y_j}{4\tau} \Big\} & \text{if } \tau > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $a_0 = \{a_0^{ij}\}$ is the matrix of the coefficients of of $\mathcal{P}_0$ and $A_0 = \{A_0^{ij}\} = a_0^{-1}$ is its inverse matrix. Hereafter, we denote by Γ_i^0 and Γ_{ij}^0 the derivatives $\partial\Gamma^0/\partial y_i$ and $\partial^2\Gamma^0/\partial y_j \partial y_i$. In the problem under consideration, the coefficients of the operator $\mathcal{P}$ depend on x. To express this dependence in the fundamental solution we define

$$\Gamma(x, y) = \begin{cases} \dfrac{1}{(4\pi\tau)^{n/2}\sqrt{\det a(x)}} \exp\Big\{ -\dfrac{\sum_{i,j=1}^n A^{ij}(x) y_i y_j}{4\tau} \Big\} & \text{if } \tau > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $a(x) = \{a^{ij}(x)\}$ is the matrix of the coefficients of $\mathcal{P}$ and $A(x) = \{A^{ij}\} = a^{-1}(x)$ is its inverse matrix. The derivatives Γ_i and Γ_{ij} are taken with respect to the second variable y.

It is well known that $\Gamma_{ij}(x, y)$ are PCZ variable kernels; see [6].

Lemma 4.5 *If a_{ij} satisfies* (A1) *and* (A2)*, then $k(x, y) = \Gamma_{ij}(x, y)$ satisfies* (3.2) *and* (3.3) *in Theorems* 3.4 *and* 3.5*, with constants M_1 and M_2 depending on n and λ.*

Proof A direct calculation shows that

$$\begin{aligned} \Gamma_{ij}(x, y) &= \frac{1}{(4\pi\tau)^{n/2}\sqrt{\det a(x)}} \frac{\partial^2}{\partial y_i \partial y_j} \exp\Big\{ -\frac{\sum_{i,j=1}^n A^{ij}(x) y_i y_j}{4\tau} \Big\} \\ &= \frac{1}{(4\pi\tau)^{n/2}\sqrt{\det a(x)}} \exp\Big\{ -\frac{\sum_{i,j=1}^n A^{ij}(x) y_i y_j}{4\tau} \Big\} \\ &\quad \times \Big[-\frac{A^{ij}(x)}{2\tau} + \frac{\big(\sum_{l=1}^n A^{il}(x) y_l + \sum_{k=1}^n A^{kj}(x) y_k\big)^2}{16\tau^2} \Big]. \end{aligned}$$

Using Lemma 2.8, we know that $\det a(x)^{-1/2}, A^{kl} \in L^\infty \cap \mathrm{LMO}$, with norms depending only on λ and n. In addition, it is easy to see that $A^{kj}(x) y_k$ and $A^{il}(x) y_l$ belong to $L^\infty \cap \mathrm{LMO}$ with norms independent of $y \in \sum$. Finally, by the positivity of a and A, we can obtain the desired result. ■

Now let us state the main result.

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Theorem 4.6 *Let Ω be a bounded domain in $\mathbb{R}^{n+1}$, $u \in h^{2,1}(\Omega_T)$. If $a_{ij} \in \mathrm{LMO}_0$, then for any $\Omega' \subset\subset \Omega$,*

$$\|D^2u\|_{h^1(\Omega'_T)} + \|D_t u\|_{h^1(\Omega'_T)} \le C(\|u\|_{h^1(\Omega_T)} + \|Du\|_{h^1(\Omega_T)} + \|\mathcal{P}u\|_{h^1(\Omega_T)},$$

where C depends only on $\lambda, n, \Omega'_T, \Omega_T$, and $\eta_{a_{ij}} (1 \le i, j \le n)$ as defined in Definition 4.3.

To prove Theorem 4.6 we first establish the following proposition.

Proposition 4.7 *Suppose $u \in h^{2,1}(\Omega_T)$ with* supp $u \subset B \subset \Omega_T$, *where $B = B_r(x_0)$ is a ball with $r \le 1$. There exists $r_0 < 1$ such that if $r < r_0$, then*

$$\|D^2u\|_{h^1(\Omega_T)} + \|D_t u\|_{h^1(\Omega_T)} \le C\|\mathcal{P}u\|_{h^1(\Omega_T)}.$$

Proof We first introduce some notation. Let

$$T_{ij} f(x) = \text{p. v.} \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x, x - y) f(y)\, dy,$$
$$[T_{ij}, b] f(x) = \text{p. v.} \int_{\mathbb{R}^{n+1}} \Gamma_{ij}(x, x - y)(b(x) - b(y)) f(y)\, dy.$$

Since u has compact support, the zero extension of u(which we still denote by u) is in $h^{2,1}(\mathbb{R}^{n+1})$. For $x \in$ supp u the following interior representation formula holds (see [1]).

$$\begin{aligned} D_{ij}u(x) &= \sum_{k,l=1}^{n} [T_{ij}, a_{kl}] D_{kl}(x) + T_{ij}(\mathcal{P}u)(x) + (\mathcal{P}u)(x) \int_{S^{n+1}} \Gamma_i(x, y) n_j d\sigma_y \\ &= I(x) + II(x) + III(x), \end{aligned}$$

where n_i is the i-th component of the outer normal of the surface S^{n+1}. It is easy to see that $m(x) = \int_{S^{n+1}} \Gamma_i(x, y) n_j\, d\sigma_y \in L^\infty \cap \mathrm{LMO}$ and $\|m\|_{L^\infty} + \|m\|_{\mathrm{LMO}} \le C(n, \lambda)$. From Lemmas 4.1 and 4.2, we have

$$\begin{aligned} \|III\|_{h^1(\Omega_T)} &= \|m\mathcal{P}u\|_{h^1(\Omega_T)} \le \|m\mathcal{P}u\|_{h^1} \\ &\le C(\|m\|_{L^\infty} + \|m\|_{\mathrm{LMO}}))\|\mathcal{P}u\|_{h^1} \\ &\le C\|\mathcal{P}u\|_{h^1(\Omega_T)}. \end{aligned} \tag{4.1}$$

Using Theorem 3.5 and Lemmas 4.1 and 4.2, we have

$$\|II\|_{h^1(\Omega_T)} \le C\|II\|_{h^1_2} \le C\|\mathcal{P}u\|_{h^1_2} \le C\|\mathcal{P}u\|_{h^1} \le C\|\mathcal{P}u\|_{h^1(\Omega_T)}. \tag{4.2}$$

It remains to estimate I. For this goal, let a be one of a_{kl}; we consider B_r center at x_0 and of radius r and $B_r^C = \mathbb{R}^{n+1} \setminus B_r$. From the properties of LMO and LMO_0 functions (see Lemma 4.4), it follows that for any $\epsilon > 0$ there exists a number r_0 and a smooth

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function g such that $\|a-g\|_{\mathrm{LMO}} < \epsilon/2$. Fix a ball $B_r \subset \Omega$, such that $r \in (0, r_0)$ and construct a function

$$h(x) = \begin{cases} g(x) & \text{for } x \in B_r, \\ g\Big(x_0' + r\dfrac{x'-x_0'}{\rho(x-x_0)}, t_0 + r^2\dfrac{t-t_0}{\rho(x-x_0)^2}\Big) & \text{for } x \in B_r^C. \end{cases}$$

Obviously, h is a Lipschitz function, and $\omega_h = \sup_{x,y\in\mathbb{R}^{n+1}} |h(x)-h(y)|$ is equal to $\omega_h(r) = \sup_{x,y\in B} |h(x)-h(y)|$. Now we estimate the LMO norm of h. Let B_s be any ball in $\mathbb{R}^{n+1}$ with radius $s \le 1$. If $s \le r$, then

$$\frac{1+|\ln r|}{|B_s|}\int_{B_s} |h(x)-h_{B_s}|\,dx \le 2s(1+|\ln s|)\|h\|_{Lip_1} \le 2r(1+|\ln r|)\|Dh\|_{L^\infty(\bar{\Omega}_T)},$$

since the function $t(1+|\ln t|)$ is increasing in $t \in (0,1)$.

If $r < s$, then

$$\begin{aligned} \frac{1+|\ln r|}{|B_s|}\int_{B_s} |h(x)-h_{B_s}|\,dx &\le 2(1+|\ln s|)\omega_h \le (1+|\ln r|)\omega_h(r) \\ &\le 2r(1+|\ln r|)\omega_h(r)\|Dh\|_{L^\infty(\bar{\Omega}_T)}. \end{aligned}$$

Thus, there exists a constant C independent of r such that $\|h\|_{\mathrm{LMO}} \le Cr(1+|\ln r|)$. By Theorem 3.5, and Lemmas 4.1 and 4.2, we then get

$$\begin{aligned} \|I\|_{h^1(\Omega_T)} &\le \sum_{k,l=1}^{n} \|[T_{ij}, a_{kl}-g_{kl}]D_{kl}u\|_{h_2^1} + \sum_{k,l=1}^{n} \|[T_{ij}, g_{kl}]D_{kl}u\|_{h_2^1} \\ &\le C\sum_{k,l=1}^{n} (\|a_{kl}-g_{kl}\|_{\mathrm{LMO}} + \|g_{kl}\|_{\mathrm{LMO}})\|D_{kl}u\|_{h_2^1} \\ &\le C\epsilon\sum_{k,l=1}^{n} \|D_{kl}u\|_{h^1} \le C\epsilon\|D^2u\|_{h^1(\Omega_T)}, \end{aligned} \tag{4.3}$$

if we take $r < r_0$ such that $r_0(1+|\ln r_0|) < \epsilon/2$.

Finally, combining estimates (4.1), (4.2), and (4.3), we have

$$\|D^2u\|_{h^1(\Omega_T)} \le C\epsilon\|D^2u\|_{h^1(\Omega_T)} + C\|\mathcal{P}u\|_{h^1(\Omega_T)}.$$

By taking $\epsilon = 1/(2C)$, we have $\|D^2u\|_{h^1(\Omega_T)} \le C\|\mathcal{P}u\|_{h^1(\Omega_T)}$. To estimate u_t, we employ the equation $u_t = a_{ij}(x)D_{ij}u + \mathcal{P}u(x)$. Hence, by Lemma 2.7, we have

$$\begin{aligned} \|u_t\|_{h^1(\Omega_T)} &\le C\sum_{i,j=1}^{n} (\|a_{ij}\|_{L^\infty} + \|a_{ij}\|_{\mathrm{LMO}})\|D^2u\|_{h^1(\Omega_T)} + \|\mathcal{P}u\|_{h^1(\Omega_T)} \\ &\le C\|\mathcal{P}u\|_{h^1(\Omega_T)}. \end{aligned}$$

■

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Proof of Theorem 4.6 Let $\{\varphi_k\}_{k=1}^N$ be the partition of unity subordinate to $\bar{\Omega}'$ with $\operatorname{supp}\varphi_k \subset B_k \subset \Omega$ such that the radius of every B_k (in $\mathbb{R}^n$) is less than $r_0/4$. Take $T_l = [t_{l-1}, t_l]$ with $t_l = (Tl)/(r_0^2/16), l = 1, \dots, L-1 = [T/(r_0^2/16)]$ and $T_L = [t_{L-1}, T]$. Let $\eta_l = \chi_{T_l}(t)$ and $\varphi_{kl} = \varphi_k\eta_l$. Then $\{\varphi_{kl}\}_{k=1,l=1}^{N,L}$ is the partition of unity subordinate to $\bar{\Omega}'_T$ with $\operatorname{supp}\varphi_{ki} \subset B_{kl} \subset \Omega_T$ such that the radius of every ball B_{kl} (in $\mathbb{R}^{n+1}$) is less than r_0. Let $u_{kl} = \varphi_{kl}u$. Then

$$\mathcal{P}u_{kl} = \varphi_{kl}\mathcal{P}u + 2\sum_{i,j=1}^n a_{ij}D_i\varphi_{kl}D_ju + u\mathcal{P}\varphi_{kl}.$$

Since supp $u_{kl} \subset B_{kl} \subset \Omega_T$, by Proposition 4.7, we obtain

$$\begin{aligned}\|D_tu_{kl}\|_{h^1(\Omega_T)} + \|D^2u_{kl}\|_{h^1(\Omega_T)} &\le C\|\mathcal{P}u\|_{h^1(\Omega_T)}\\ &\le C\Big(\|\varphi_{kl}\mathcal{P}u\|_{h^1(\Omega_T)} + \sum_{i,j=1}^n \|a_{ij}D_i\varphi_{kl}D_ju\|_{h^1(\Omega_T)}\\ &\qquad + \|u\mathcal{P}\varphi_{kl}\|_{h^1(\Omega_T)}\Big)\\ &\le C(\|u\|_{h^1(\Omega_T)} + \|Du\|_{h^1(\Omega_T)} + \|\mathcal{P}u\|_{h^1(\Omega_T)}.\end{aligned}$$

Summing over k from 1 to N and l from 1 to L we obtain the estimate for

$$\|D_tu\|_{h^1(\Omega'_T)} + \|D^2u\|_{h^1(\Omega'_T)}. \qquad \blacksquare$$

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