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DENSENESS OF OPERATORS WHICH ATTAIN THEIR NUMERICAL RADIUS

WANG JIA-PING AND YU XIN TAI

We show that a bounded linear operator on a dual Banach space X may be perturbed by a compact operator of arbitrarily small norm to yield an operator which attains its numerical radius provided the weak star and norm topologies coincide on the unit sphere of X.

Let X be a Banach space, X^* its dual space and $L(X)$ the algebra of bounded linear operators on X. Let

$$\pi(X, X^*) = \{(x, x^*) \in X \times X^*; x^*(x) = \|x\| = \|x^*\| = 1\},$$
$$\pi(X^*, X) = \{(x^*, x) \in X^* \times X; x^*(x) = \|x\| = \|x^*\| = 1\}.$$

Define the numerical radius of $T \in L(X)$ by

$$\nu(T) = \sup\{|x^*(Tx)|\,; (x, x^*) \in \pi(X, X^*)\}$$

and say T attains its numerical radius if there exists $(x_0, x_0^*) \in \pi(X, X^*)$ with $\nu(T) = |x_0^*(Tx_0)|$. Denote

$$NRA(X) = \{T \in L(X)\,; T \text{ attains its numerical radius}\}.$$

Say a Banach space X has property (P) if every $T \in L(X)$ may be perturbed by a compact operator of arbitrarily small norm to obtain an operator in $NRA(X)$ in $L(X)$.

In [**2**], Berg and Sims proved that uniformly convex Banach spaces have property (P) and asked how far this result may be extended. In [**1**], Acosta and Paya proved that every reflexive Banach space has property (P). Recall that a dual Banach space X has property $(**)$ if it satisfies: whenever (x_α) is a net in X, x_α converges to x in the weak topology and $\|x_\alpha\|$ converges to $\|x\|$, then x_α converges to x in norm. We show that every dual Banach space with property $(**)$ has property (P).

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Theorem. *For X a dual Banach space with property $(**)$, given $T \in L(X)$ and $\varepsilon > 0$, there exists a compact operator C with $\|C\| < \varepsilon$ such that $T + C$ belongs to $NRA(X)$.*

Proof: Assume $X = Y^*$. For $\varepsilon > 0$, define $a_n = \varepsilon/8^n$ and $b_n = \varepsilon/2^n$. Given an arbitrary $T \in L(Y^*)$, we shall construct an operator $T_\infty \in B(Y^*)$ such that $T_\infty \in NRA(Y^*)$, $T_\infty - T$ is compact and $\|T_\infty - T\| < \varepsilon$. This will complete our proof.

For this purpose, we need

Lemma. *There exist $(T_n)_{n=1}^\infty$ in $L(Y^*)$ with $T_1 = T$, $(y_n^*, y_n) \in \pi(Y^*, Y)$ such that*

(i) $|(T_n y_n^*)(y_n)| > \nu(T_n) - a_n$, $n = 1, 2, \dots$
(ii) $\|T_n - T_{n-1}\| \leqslant b_{n-1}$ *and* $T_n - T_{n-1}$ *is a rank one operator,* $n = 2, \dots$
(iii) $|y_n^*(y_m)| > 1 - 1/2^k$, *where* $k = \min(n, m)$, $n, m = 1, 2, \dots$.

We first show how the proof of the theorem can be completed by using the above lemma.

Since $\sum_{n=2}^{\infty} \|T_n - T_{n-1}\| \leqslant \sum_{n=2}^{\infty} b_{n-1} = \varepsilon$, T_n converges to $T_\infty \in L(Y^*)$ in norm and $\|T_\infty - T\| = \|T_\infty - T_1\| \leqslant \sum_{n=2}^{\infty} \|T_n - T_{n-1}\| \leqslant \varepsilon$. Moreover $T_\infty - T$ is compact as $T_n - T_1$ is of finite rank for every n. Now it remains to show $T_\infty \in NRA(Y^*)$.

Since Y has property $(**)$, Y has the Radon-Nikodym property by [6], which implies that Y is weak sequentially compact and $\ell_1 \not\hookrightarrow Y$ (see [4]). So we may assume that y_n^* converges weakly* to y^* in Y^* and y_n converges weakly to y^{**} in Y^{**}.

Now

$$|y^{**}(y^*)| \geqslant \varliminf_m |y^*(y_m)| \geqslant \varliminf_m \varliminf_n |y_n^*(y_m)| \geqslant \varliminf_m \varliminf_n \{1 - \frac{1}{2^k}\} = 1,$$

where $k = \min(n, m)$.

Since $\|y^{**}\| \leqslant \varliminf_n \|y_n\| = 1$ and $\|y^*\| \leqslant \varliminf_n \|y_n^*\| = 1$, $\|y^*\| = \|y^{**}\| = 1$, and $(y^*, \lambda y^{**}) \in \pi(Y^*, Y^{**})$, where $\lambda = \overline{y^{**}(y^*)}$.

It now follows that y_n^* converges weakly* to y^* and $\|y_n^*\| = \|y^*\| = 1$, so y_n^* converges to y^* in norm by the property $(**)$ of Y.

Now

$$\begin{aligned}
\nu(T_\infty) &= \lim_n \nu(T_n) \text{ because } T_n \text{ converges to } T \text{ in norm} \\
&= \lim_n |(T_n y_n)(y_n)| \text{ by the lemma} \\
&= \lim_n |(T_\infty y^*)(y_n)| \text{ because } y_n^* \text{ converges to } y^* \text{ in norm} \\
&= |y^{**}(T_\infty y^*)| \text{ because } y_n \text{ converges to } y^{**} \text{ weakly}^* \\
&= |\lambda y^{**}(T_\infty y^*)|.
\end{aligned}$$

Hence T_∞ attains its numerical radius, that is $T_\infty \in NRA(Y^*)$. □

Now we turn to the proof of the lemma. First we quote a theorem from [**3**] as a sublemma.

SUBLEMMA. [**3**, p.84, Theorem 5] *$\nu(A) = \sup\{|Ay^*(y)|; (y^*, y) \in \pi(Y^*, Y)\}$ for every $A \in L(Y^*)$.*

Let $T_1 = T$, choose $(y_1^*, y_1) \in \pi(Y^*, Y)$ by the sublemma such that $|(T_1 y_1^*)(y_1)| > \nu(T_1) - a_1$. Define $T_2 y^* = T_1 y^* + b_1 e^{i\theta_1} y^*(y_1) y_1^*$, where $\theta_1 = \arg(T_1 y_1^*)(y_1)$.

Obviously, $T_2 - T_1$ is of rank one and $\|T_2 - T_1\| \leqslant b_1$.

Further,

$$\begin{aligned}\nu(T_2) &\geqslant |(T_2 y_1)(y_1)| \\ &= \left|(T_1 y_1^*)(y_1) + b_1 e^{i\theta_1}\right| \\ &= |(T_1 y_1^*)(y_1)| + b_1 > \nu(T_1) - a_1 + b_1,\end{aligned}$$

or equivalently, $\nu(T_2) - \nu(T_1) > b_1 - a_1$.

Now choose $(y_2^*, y_2) \in \pi(Y^*, Y)$, again by the sublemma, with $|(T_2 y_2^*)(y_2)| > \nu(T_2) - a_2$. Similarly, define $T_3 y^* = T_2 y^* + b_2 e^{i\theta_2} y^*(y_2) y_2^*$, where $\theta_2 = \arg(T_2 y_2^*)(y_2)$ and again $T_3 - T_2$ is of rank one, $\|T_3 - T_2\| \leqslant b_2$, $\nu(T_3) - \nu(T_2) > b_2 - a_2$. Inductively, we may construct (T_n) and $(y_n^*, y_n) \in \pi(Y^*, Y)$ with

(1) $\|T_n - T_{n-1}\| \leqslant b_{n-1}$;
(2) $T_n y^* = T_{n-1} y^* + b_{n-1} e^{i\theta_{n-1}} y^*(y_{n-1}) y_{n-1}^*$;
(3) $\nu(T_n) - \nu(T_{n-1}) > b_{n-1} - a_{n-1}$;
(4) $|(T_n y_n^*)(y_n)| > \nu(T_n) - a_n$.

CLAIM. (T_n) and (y_n^*, y_n) satisfy (i), (ii) and (iii) of the lemma. In fact, (1) and (2) imply (ii) and (4) is (i). It remains to establish (iii).

Since $\nu(T_n) - \nu(T_{n-1}) > b_{n-1} - a_{n-1}$,

$$\nu(T_n) - \nu(T_m) > b_{n-1} + \ldots + b_m - a_{n-1} - \ldots - a_m (n > m).$$

Now

$$\begin{aligned}T_n y^* &= T_{n-1} y^* + b_{n-1} e^{i\theta_{n-1}} y^*(y_{n-1}) y_{n-1}^* = \ldots \\ &= T_m y^* + b_m e^{i\theta_m} y^*(y_m) y_m^* + b_{m+1} e^{i\theta_{m+1}} y^*(y_{m+1}) y_{m+1}^* + \ldots \\ &\qquad + b_{n-1} e^{i\theta_{n-1}} y^*(y_{n-1}) y_{n-1}^*,\end{aligned}$$

and

$$\begin{aligned}\nu(T_n) - a_n < |(T_n y_n^*)(y_n)| &= |(T_m y_n^*)(y_n)| + b_m e^{i\theta_m} y_n^*(y_m) y_m^*(y_n) \\ &\quad + b_{m+1} e^{i\theta_{m+1}} y_n^*(y_{m+1}) y_{m+1}^*(y_n) + \ldots + b_{n-1} e^{i\theta_{n-1}} y_n^*(y_{n-1}) y_{n-1}^*(y_n) \\ &\leqslant \nu(T_m) + b_m |y_n^*(y_m) y_m^*(y_n)| + b_{m+1} + \ldots + b_{n-1},\end{aligned}$$

so

$$|y_n^*(y_m)y_m^*(y_n)| > (\nu(T_n) - \nu(T_m) - a_n - b_{m+1} - \ldots - b_{n-1})/b_m$$
$$> (b_m - a_m - a_{m+1} - \ldots - a_n)/b_m > 1 - \frac{1}{2^m}.$$

Hence $|y_n^*(y_m)| > 1 - 1/2^m$ and $|y_m^*(y_n)| > 1 - 1/2^m$, which implies (iii) of the lemma. This establishes the lemma and all the proof is completed. ▯

From the above proof, we can eaily deduce the following

COROLLARY 1. *If X^* has property (**), then X^{**} has property (P). In particular, $L(H)$ and ℓ_∞ have property (P), where H is a Hilbert space.*

COROLLARY 2. *$\mathcal{T}_1(H)$, the trace class on a Hilbert space H with the trace norm, has property (P) and $NRA(\mathcal{T}_1(H))$ is norm dense in $L(\mathcal{T}_1(H))$.*

PROOF: (H) has property (**) [5]. ▯

REMARK. Paya has given two examples of Banach spaces for which the numerical radius attaining operator is not dense.

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Department of Mathematics
East China Normal University
Shanghai 200062
China