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COMPRESSIBLE MATRIX RINGS

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Let $Z(K)$ denote the center of a ring K. A ring R is compressible if $Z(eRe) = eZ(R)$ for each idempotent e of R. In response to a question of S. Berberian, G. Bergman has constructed a (non-commutative) integral domain, satisfying a polynomial identity, for which the 2×2 matrix ring over the domain is not compressible. In contrast to Bergman's example, we show that the ring of n×n matrices over any commutative ring is always compressible.

The example constructed by Bergman [3] eliminates a large class of rings, prime Goldie rings, for which compressibility is a Morita invariant property. In a positive vein, A. Page has shown in unpublished work that compressibility is preserved in matrix rings over (compressible) von Neumann regular rings.

In contrast to Bergman's P.I. domain we have been able to show that R_n is always compressible whenever R is a commutative integral domain. In fact, a much more general result holds.

THEOREM 1. *If R is a commutative ring then R_n is compressible for all $n \geq 1$.*

Proof. Let $A = R_n$ and $\varepsilon \in A$ with $\varepsilon^2 = \varepsilon$. From Passman [4, Lemma 4.5, p. 253] we have $A\varepsilon A \cap \ell_A(A\varepsilon A) = 0$. We claim that

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$A = A\varepsilon A \oplus \ell_A(A\varepsilon A)$. Indeed $\varepsilon(I_n-\varepsilon) = 0$ so we have $\varepsilon(I_n-\varepsilon)\text{adj}(I_n-\varepsilon) = 0$. However $(I_n-\varepsilon)\text{adj}(I_n-\varepsilon) = \det(I_n-\varepsilon)I_n$. Hence $\det(I_n-\varepsilon)\varepsilon=\varepsilon\cdot\det(I_n-\varepsilon)I_n=0$. Let U be the ideal of R generated by the entries of ε. Then $\det(I_n-\varepsilon) = 1 - x$ where $x \in U$. Since $(1-x)\varepsilon = 0$ and $(1-x)I_n \in Z(A)$ we have $(1-x)I_n \in \ell_A(A\varepsilon A)$. Thus $I_n - xI_n \in \ell_A(A\varepsilon A)$. Because $A\varepsilon A$ contains all matrices yI_n with $y \in U$ we have $xI_n \in A\varepsilon A$ and so $I_n \in A\varepsilon A + \ell_A(A\varepsilon A)$. Accordingly $A = A\varepsilon A \oplus \ell_A(A\varepsilon A)$ as we claimed. However, both $A\varepsilon A$ and $\ell_A(A\varepsilon A)$ are ideals of A so $A\varepsilon A = A\varphi$ for some central idempotent $\varphi \in A$. Then $\varepsilon = \varepsilon\varphi$ so $\varepsilon A\varepsilon = \varepsilon A\varphi\varepsilon$; hence we may assume that $A = A\varepsilon A$. Now suppose $x \in Z(\varepsilon A\varepsilon)$. The mapping $\theta : A \to A$ given by $\theta(\Sigma\alpha_i\varepsilon\beta_i) = \Sigma\alpha_i x\beta_i$ is well-defined. Indeed, if $\Sigma\alpha_i\varepsilon\beta_i = 0$ then for all $\alpha,\beta \in A$ we have $(\sum_i\alpha_i x\beta_i)\alpha\varepsilon\beta = \sum_i\alpha_i x(\varepsilon\beta_i\alpha\varepsilon)\beta = \sum_i\alpha_i\varepsilon\beta_i\alpha x\beta = (\sum_i\alpha_i\varepsilon\beta_i)\alpha x\beta = 0$; therefore $\sum\alpha_i x\beta_i \in \ell_A(A) = 0$ showing that θ is well-defined. Then because $x \in Z(\varepsilon A\varepsilon)$, θ is an A-A-bimodule mapping. Thus θ is given by a right multiplication by a central element z of A. In particular, $x = \theta(\varepsilon) = \varepsilon z$ and so $x \in \varepsilon Z(A)$. Since we always have $\varepsilon Z(A) \subseteq Z(\varepsilon A\varepsilon)$ the proof is complete.

An immediate consequence is the

COROLLARY. *If M is a finitely generated projective R-module over a commutative ring R then $\text{Hom}_R(M,M)$ is a compressible ring.*

Recall that an algebra A over a commutative ring R is a *separable R-algebra* in case A is a projective A^e-module, where $A^e = A \otimes_R A^{op}$ is the enveloping algebra of A. The separable R-algebra A is *central separable*, or an *Azumaya algebra* if $R = Z(A)$. (See [3] for properties of separable algebras.)

THEOREM 2. *An Azumaya algebra A over a commutative ring R is compressible.*

Proof. Since A is Azumaya we have $A^e \cong \text{Hom}_R(A,A)$ and ${}_RA$ is a progenerator. By the corollary, A^e is a compressible ring. Furthermore R is both a left and right R-module direct summand of A. Suppose f is

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an idempotent in A. Then $f \otimes 1$ is an idempotent in A^e and $Z((f \otimes 1)A^e(f \otimes 1)) = (f \otimes 1)Z(A^e)$. Also $Z(fAf) \otimes_R R \subseteq fAf \otimes_R R$. Since R is a direct summand of ${}_RA$ we also have that R is a direct summand of ${}_RA^{op}$ and so $fAf \otimes_R R \subseteq fAf \otimes_R A^{op}$. However every element of $Z(fAf) \otimes_R R$ is central in $fAf \otimes_R A^{op}$ so that, in fact,

$$Z(fAf) \otimes_R R \subseteq Z(fAf \otimes_R A^{op})$$

$$= Z((f \otimes 1)A^e(f \otimes 1)) = (f \otimes 1)Z(A^e).$$

If $t \in Z(fAf)$ then $t \otimes 1 = (f \otimes 1)(\Sigma a_i \otimes b_i) = \Sigma f a_i \otimes b_i$, where $\Sigma a_i \otimes b_i \in Z(A^e)$. Now the isomorphism $\theta : A^e \to \mathrm{Hom}_R(A,A)$ is given by $\theta(\Sigma c_i \otimes d_i) : a \to \Sigma c_i a d_i$. Thus $\theta(\Sigma a_i \otimes b_i) \in Z(\mathrm{Hom}_R(A,A))$ hence $\theta(\Sigma a_i \otimes b_i)$ commutes with all left and right multiplications by elements of A. Let a_λ, a_ρ denote left and right multiplication by $a \in A$, respectively. Then we have $(\theta(\Sigma a_i \otimes b_i) \circ a_\lambda)(1) = \theta(\Sigma a_i \otimes b_i)(a) = \Sigma a_i a b_i$ and $(a_\lambda \circ \theta(\Sigma a_i \otimes b_i))(1) = a\Sigma a_i b_i$; thus $\Sigma a_i a b_i = a(\Sigma a_o b_i)$. Using a_ρ we also have $\Sigma a_i a b_i = (\Sigma a_i b_i)a$. Combining these we get $a(\Sigma a_i b_i) = (\Sigma a_i b_i)a$ for all $a \in A$. Therefore $\Sigma a_i b_i \in Z(A) = R$. From $t \otimes 1 = \Sigma f a_i \otimes b_i$ with $\Sigma a_i \otimes b_i \in Z(A^e)$, we have $\theta(t \otimes 1)(x) = tx$ and $\theta(t \otimes 1)(x) = \theta(\Sigma f a_i \otimes b_i)(x) = \Sigma f a_i x b_i$ for all $x \in A$. In particular for $x = 1$ we obtain $t = f(\Sigma a_i b_i) \in fZ(A)$. We conclude that A is compressible.

By [3, Theorem 3.8], A is a separable algebra over a commutative ring R if and only if A is central separable over $Z(A)$ and $Z(A)$ is a separable algebra over R. In particular A is an Azumaya algebra over its center which yields:

COROLLARY. *If R is a commutative ring then any separable R-algebra is compressible.*

A particular example of separability occurs when G is a finite group whose order is a unit in the commutative ring R. Thus we have

COROLLARY. *If R is a commutative ring, G a finite group whose order is invertible in R then the group ring RG is compressible.*

In a personal correspondence to the second author, K. Motose has pointed out that the group algebra $Z_2[S_4]$ is not a compressible ring. This example, coupled with Bergman's example, demonstrates the limitations of compressibility.

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