BISIMPLE INVERSE SEMIGROUPS AS SEMIGROUPS OF ORDERED TRIPLES

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Introduction. In **(8)** and **(13)** it has been shown that certain bisimple inverse semigroups, called bisimple ω-semigroups and bisimple Z-semigroups, can be represented as semigroups of ordered triples. In these cases, two of the components of each triple are integers, and the third is drawn from a fixed group. This representation is analogous to that given by the theorem of Rees **(1**, p. 94**)** concerning completely simple semigroups, and shares the same advantages.

In the present paper, it is shown that any bisimple inverse semigroup has a representation by ordered triples for each congruence ρ contained in Green's equivalence $\mathscr{H}$ **(1**, p. 48**)**, in which one of the components is drawn from a group. In general, this representation has the defect that distinct triples may correspond to the same element of the semigroup, but it is one-to-one in the case when $\mathscr{H}$ itself is a congruence and $\rho = \mathscr{H}$.

In the case of a bisimple inverse semigroup S with identity for which $\mathscr{H}$ is a congruence, R. J. Warne **(10)** found, by using Rédei's theory **(6)** of Schreier extensions of a group by a semigroup, a representation of S by quadruples from which the triples representation can be derived in a few lines. We shall follow the same procedure in the general case (§4). We can dispense with the requirement that S have an identity by means of RP-systems **(9)**. For this we find it necessary to formulate a theory of Schreier extensions of a group by an RP-system (§3).

For a bisimple inverse semigroup S with identity, Warne **(12)** has shown that there exists a one-to-one correspondence between the idempotent separating congruences on S and the normal subgroups V of the unit group of S satisfying $aV \subseteq Va$ for every right unit a of S. We extend this result to bisimple inverse semigroups without identity in §2.

1. Preliminary results. We adopt the notation and terminology of **(1)**. In particular, two elements of a semigroup S are said to be $\mathscr{L}$- ($\mathscr{R}$-)equivalent if they generate the same principal left (right) ideal of S. We write

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}$$

and

$$\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}.$$

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We denote by $L_a(R_a, H_a)$ the $\mathscr{L}$- ($\mathscr{R}$-, $\mathscr{H}$-)class of S containing the element a of S. S is said to be *bisimple* if it contains only one $\mathscr{D}$-class.

The elementary properties of inverse semigroups will be found in **(1**, §1.9**)**. The *inverse* a^{-1} of an element a of an inverse semigroup S is characterized by $aa^{-1}a = a$, $a^{-1}aa^{-1} = a^{-1}$. The idempotent element $aa^{-1}(a^{-1}a)$ is called the *left* (*right*) *unit* of a. An *inverse subsemigroup* of an inverse semigroup S is a subsemigroup T of S such that the inverse of every element of T also belongs to T.

Let S be a semigroup with an identity 1. If u and v are elements of S such that $uv = 1$, then we call u a *right unit* and v a *left unit* of S. An element which is both a left unit and a right unit is call a *unit* and the set of all units of S is a subgroup U of S, called the *unit group* of S. The set of all right units of S is a subsemigroup P of S, called the *right unit subsemigroup* of S. We note that $U = H_1$ and $P = R_1$. If, for a right unit u of S, there exists a right unit v of S such that $uv = 1$, then u is a unit of S. Hence the unit group of P is just U **(1**, p. 21**)**.

The following lemma is almost immediate from Lemma 1.2 of **(11)**.

LEMMA 1.1. *Let e be any idempotent of an inverse semigroup S. Then eSe is an inverse subsemigroup of S with identity e, which is bisimple if S is bisimple. Let P_e be the right unit subsemigroup of eSe. Then*

$$P_e = R_e \cap eSe = \{a \in R_e : ae = a\}.$$

Moreover, the unit group of P_e is just H_e.

For any idempotent e of an inverse semigroup S, we shall denote by P_e the right unit subsemigroup of eSe.

The following definition is the left-right dual of that given by Rees **(7)**. Let P be a right cancellative semigroup with an identity. Then a subgroup V of the unit group of P is called a *left normal divisor of P* if $aV \subseteq Va$ for all a in P. If S is an inverse semigroup, then we shall call a subgroup V of S a left normal divisor of S if V is a left normal divisor, in the above sense, of P_e, where e is the identity of V.

We define a *right partial semigroup* R to be a set R together with a partial binary operation on R satisfying the following condition:

(A) if, for elements a, b, c of R, $a(bc)$ is defined, then so also is $(ab)c$ defined, and then $a(bc) = (ab)c$.

A right partial semigroup S is said to be isomorphic with a right partial semigroup T if there exists a bijection ϕ of S onto T such that ab is defined in S if and only if $(a\phi)(b\phi)$ is defined in T, and then $(ab)\phi = (a\phi)(b\phi)$.

We define an *RP-system* (R, P) to be a right partial semigroup R together with a subsemigroup P of R such that:

(P1) ab is defined $(a, b \in R)$ if and only if $a \in P$;

(P2) R has a left identity contained in P;

(P3) $ac = bc$ $(a, b \in P; c \in R)$ implies $a = b$;

(P4) for every a, b in R, there exists c in R such that $Pa \cap Pb = Pc$.

It follows from (P1) and (P3) that any left identity of R contained in P is a two-sided identity for P, and so is unique. We describe (P3) by saying that R is right cancellative.

For the remainder of this section, let (R, P) be an RP-system. From **(9**, Lemma 2.1**)** we have the following.

LEMMA 1.2. *The relation $\mathscr{L}'$ defined on R by*

$$\mathscr{L}' = \{(a, b) \in R \times R\colon Pa = Pb\}$$

is an equivalence relation on R, and $(a, b) \in \mathscr{L}'$ if and only if $a = ub$ for some unit u of P.

We denote the $\mathscr{L}'$-class of R containing the element a of R by L'_a, and partially order the set $P(\mathscr{L}')$ of $\mathscr{L}'$-classes by writing $L'_a < L'_b$ if and only if $Pa \subset Pb$. Then $P(\mathscr{L}')$ is, by (P4), a semilattice. Select and keep fixed a representative from each $\mathscr{L}'$-class. If, for elements a, b of R, $Pa \cap Pb = Pc$, then let $a \vee b$ denote the representative from the $\mathscr{L}'$-class L'_c containing the element c. Since we lose nothing in the way of generality by doing so, we adopt the convention that the representative from the $\mathscr{L}'$-class L'_1 is 1, where 1 denotes the left identity of R. We call $\vee$ a *join operation* on R.

Define the operation $*$ on R by the rule that

$$(a * b)b = a \vee b \quad (\text{all } a, b \text{ in } R). \tag{1.1}$$

Then, for every pair of elements a, b of R, $a * b \in P$, and is, on account of (P3), uniquely determined.

We note the following immediate consequences of this definition. For any a in R, $(a * a)a = a \vee a = ua$, for some unit u in P, and so, by (P3), $a * a$ is a unit. By our convention above, $1 * 1 = 1 \vee 1 = 1$. We also have

$$(a * b)b = a \vee b = b \vee a = (b * a)a.$$

The following theorem **(9**, Theorem 2.2**)** is basic for our present objective.

THEOREM 1.3. *Let (R, P) be an RP-system, and let the operation $*$ be defined on R as above. Let $R^{-1} \circ R$ denote $R \times R$ under the multiplication*

$$(a, b)(c, d) = ((c * b)a, (b * c)d), \tag{1.2}$$

where we identify the pairs (a, b) and (a', b') if and only if $a = ua'$, $b = ub'$, for some unit u of P.

Then $R^{-1} \circ R$ is a bisimple inverse semigroup such that the semilattice of idempotents of $R^{-1} \circ R$ is isomorphic with $P(\mathscr{L}')$, and, for some $\mathscr{R}$-class R' of $R^{-1} \circ R$, R' is isomorphic with R as a right partial semigroup.

Conversely, if S is a bisimple inverse semigroup, then, for any idempotent e of S, (R_e, P_e) is an RP-system, and S is isomorphic with $R_e^{-1} \circ R_e$.

We list here some simple properties of $R^{-1} \circ R$, which follow directly from the definition of multiplication given in the theorem, and which we shall need later.

The idempotents of $R^{-1} \circ R$ are the "diagonal" elements (a, a) of $R^{-1} \circ R$, and

$$(1.3) \qquad (a, a)(b, b) = (a \vee b, a \vee b) \qquad \text{(all } a, b \text{ in } R).$$

For all a, b in R, we have

$$(1.4) \qquad (a, b)^{-1} = (b, a),$$

$$(1.5) \qquad (a, b)(a, b)^{-1} = (a, b)(b, a) = (a, a).$$

$$(1.6) \qquad (a, b)^{-1}(a, b) = (b, a)(a, b) = (b, b).$$

Thus (a, a) is the left unit, and (b, b) the right unit, of (a, b). Furthermore,

$$(1.7) \qquad (1, a)(1, b) = (1, ab) \qquad \text{(all } a \text{ in } P, b \text{ in } R),$$

and

$$(1.8) \qquad (a, 1)(1, b) = (a, b) \qquad \text{(all } a \text{ and } b \text{ in } R).$$

It is easy to see that $R_{(1,1)} = 1 \times R$ and $P_{(1,1)} = 1 \times P$, and then (1.7) shows that the mapping $a \to (1, a)$ is an isomorphism of R onto $R_{(1,1)}$ as partial semigroups; thus $R_{(1,1)}$ is the $\mathscr{R}$-class R' referred to in the theorem.

If $a, b \in R$ and u is a unit of P, then

$$ua \vee b = a \vee ub = a \vee b$$

from which it follows, using (P3), that

$$(1.9) \qquad ua * b = (a * ub)u = a * b.$$

Finally we note that $b * b$ is, for every b in R, a unit of P. Hence

$$(1.10) \qquad (a, b)(b, c) = ((b * b)a, (b * b)c) = (a, c).$$

If x and y are elements of any inverse semigroup, then $x \mathscr{R} y$ if and only if x and y have the same left unit, i.e., $xx^{-1} = yy^{-1}$. From (1.3) above, it follows that $(a, b) \mathscr{R} (c, d)$ if and only if $(a, a) = (c, c)$. From the definition of equality given in Theorem 1.3, this is the case if and only if $a = uc$ for some unit of P. Combining this remark with its left-right dual, we have

LEMMA 1.4 *Let (R, P) be an RP-system. Then, in $R^{-1} \circ R$,*

(i) *$(a, b) \mathscr{R} (c, d)$ if and only if $a = uc$ for some unit u of P, or, equivalently, if and only if $Pa = Pc$;*

(ii) *$(a, b) \mathscr{L} (c, d)$ if and only if $b = vd$ for some unit v of P, or equivalently, if and only if $Pb = Pd$;*

(iii) *$(a, b) \mathscr{H} (c, d)$ if and only if $a = uc$ and $b = vd$ for some units u, v of* **P**, *or, equivalently, if and only if $Pa = Pc$ and $Pb = Pd$.*

2. Idempotent separating congruences and left normal divisors. A congruence on a semigroup S is called idempotent separating if each congruence class contains at most one idempotent of S. It was essentially shown by

Preston **(4)** that a congruence ρ on an inverse semigroup S is idempotent separating if and only if $\rho \subseteq \mathcal{H}$.

It follows from a general result of Preston's **(5**, Lemma 1, p. 568**)** that, for any semigroup, there exists a maximum congruence μ contained in $\mathcal{H}$. Various formulations of μ when S is an inverse semigroup have been given by Howie **(2)**.

In **(12)**, Warne showed that there is a one-to-one correspondence between the idempotent separating congruences on a bisimple inverse semigroup S with identity, and the left normal divisors of the right unit subsemigroup of S. The purpose of the present section is to establish a corresponding result for an arbitrary bisimple inverse semigroup.

By a *congruence* on a right partial semigroup R we mean an equivalence relation σ such that if $a \, \sigma \, a'$, $b \, \sigma \, b'$, and ab is defined, then $a'b'$ is also defined, and $ab \, \sigma \, a'b'$. Let $a\sigma$ denote the σ-class containing a, and let R/σ denote the set of σ-classes of R. We define a partial product in R/σ by letting

$$(a\sigma)(b\sigma) = (ab)\sigma$$

if ab is defined in R; otherwise the product $(a\sigma)(b\sigma)$ is not defined. That this definition is independent of the choice of the representative element a of $a\sigma$ and b of $b\sigma$ follows from the defining property of the congruence σ. It is clear that R/σ becomes thereby a right partial semigroup.

LEMMA 2.1. *Let σ be a congruence on the right partial semigroup R of an RP-system (R, P). Then P is a union of σ-classes and P/σ is a subsemigroup of R/σ such that R/σ and P/σ satisfy conditions* (P1) *and* (P2) *for an RP-system.*

Proof. Let $a \, \sigma \, b$ with a in P and let $c \in R$. Then ac is defined, by (P1) for (R, P). Since σ is a congruence on R, bc is also defined, whence $b \in P$. Thus P is a union of σ-classes.

Properties (P1) and (P2) for $(R/\sigma, P/\sigma)$ are obvious.

Now let (R, P) be an RP-system and let V be a left normal divisor of P. Let $u \in V$ and $a \in P$. Since $aV \subseteq Va$, there exists u' in V such that $au = u'a$, and, by (P3), u' is uniquely determined by u and a. We denote it by u^a, so that the element u^a of V is defined by

$$(2.1) \qquad au = u^a a \qquad (\text{all } a \text{ in } P,\ u \text{ in } V).$$

Again using (P3), we see that the mapping $u \to u^a$ is, for each fixed element a of P, an endomorphism of V.

LEMMA 2.2. *Let (R, P) be an RP-system and V a left normal divisor of P. Then*

$$(2.2) \qquad \sigma_V = \{(a, b) \in R \times R \colon a = ub \text{ for some } u \text{ in } V\}$$

is a congruence on R such that $\sigma_V \subseteq \mathcal{L}'$ and R/σ_V is right cancellative.

Conversely, if σ is a congruence on R such that $\sigma \subseteq \mathcal{L}'$ and R/σ is right cancellative, then $\sigma = \sigma_V$ where $V = 1\sigma$.

Proof. It is trivial to verify that σ_V is an equivalence relation. To show that σ_V is a congruence on the partial semigroup R, let $(a, b) \in \sigma_V$, say $b = ua$ with u in V, and let $c \in R$. If ac is defined, then $a \in P$ by (P1) and $b = ua \in P$. From $bc = uac$ we conclude that $(ac, bc) \in \sigma_V$. If ca is defined, then $c \in P$, and so cb is also defined. From $cb = cua = u^c ca$, and $u^c \in V$, we conclude that $(ca, cb) \in \sigma_V$. Hence σ_V is a congruence on R.

Clearly $\sigma_V \subseteq \mathcal{L}'$. Now suppose that $a\sigma c\sigma = b\sigma c\sigma$. Then $(ac)\sigma = (bc)\sigma$ and so $ac = ubc$, for some $u \in V$, whence $a = ub$ and $a\sigma = b\sigma$.

Conversely, let σ be a congruence on R such that $\sigma \subseteq \mathcal{L}'$ and R/σ is right cancellative. Denote 1σ by V. Then $a\sigma = b\sigma$ implies that $a = ub$ for some unit u of P, as $\sigma \subseteq \mathcal{L}'$. Hence

$$1\sigma b\sigma = b\sigma = a\sigma = u\sigma b\sigma$$

and by cancellativity in R/σ, $u \in V$. Conversely, if $a = ub$, with $u \in V$, then $a\sigma = u\sigma b\sigma = 1\sigma b\sigma = b\sigma$.

Clearly V is a subgroup of the unit group of P and it only remains to be shown that V is a left normal divisor of P. Let $p \in P$, $u \in V$. Then

$$(pu)\sigma = p\sigma u\sigma = p\sigma 1\sigma = p\sigma.$$

Hence, $pu = u'p$ for some $u' \in V$, as required.

THEOREM 2.3. *Let (R, P) be an RP-system, V be a left normal divisor of P, and σ_V be defined as in* (2.2). *Then $(R/\sigma_V, P/\sigma_V)$ is an RP-system. Moreover, if $\vee$ is a join operation in R, then we can define a join operation $\vee$ in R/σ_V by*

(2.3) $$a\sigma_V \vee b\sigma_V = (a \vee b)\sigma_V \qquad (\text{all } a, b \text{ in } R).$$

It then follows that

(2.4) $$a\sigma_V * b\sigma_V = (a * b)\sigma_V$$

and the mapping θ defined by

(2.5) $$(a, b)\theta = (a\sigma_V, b\sigma_V)$$

is a homomorphism of $R^{-1} \circ R$ onto $(R/\sigma_V)^{-1} \circ (R/\sigma_V)$.

Proof. We know from Lemmas 2.1 and 2.2 that $(R/\sigma_V, P/\sigma_V)$ satisfies conditions (P1), (P2), and (P3) for an RP-system.

We establish both (P4) and the legitimacy of the definition (2.3) by showing hat

(2.6) $$\bar{P}(a\sigma_V) \cap \bar{P}(b\sigma_V) = \bar{P}(a \vee b)\sigma_V,$$

where we have written $\bar{P}$ for P/σ_V.

From $Pa \cap Pb = P(a \vee b)$ we have $a \vee b = pa$ with p in P. Hence $(a \vee b)\sigma_V = (p\sigma_V)(a\sigma_V)$, and so $(a \vee b)\sigma_V \in \bar{P}(a\sigma_V)$. Similarly

$$(a \vee b)\sigma_V \in \bar{P}(b\sigma_V).$$

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Conversely, let $d\sigma_V \in \bar{P}(a\sigma_V) \cap \bar{P}(b\sigma_V)$, say

$$d\sigma_V = (p_1\,\sigma_V)(a\sigma_V) = (p_2\,\sigma_V)(b\sigma_V).$$

Consequently $d \in (p_1\,a)\sigma_V = (p_2\,b)\sigma_V$, which implies that $d = up_1\,a = vp_2\,b$, where u and v belong to V. Hence, $d \in Pa \cap Pb$ and so $d = p_3(a \vee b)$. This implies that $d\sigma_V \in \bar{P}(a \vee b)\sigma_V$, which establishes (2.6).

Equation (2.4) is immediate from (1.1), (2.3), and (P3) for R/σ_V. To see that θ is single valued, we note first that if u is a unit of P, then $u\sigma_V$ is a unit of P/σ_V. Thus $(a, b) = (c, d)$ implies that $a = uc, b = ud$ for some unit u of P, and then

$$(a\sigma_V, b\sigma_V) = ((u\sigma_V)(c\sigma_V), (u\sigma_V)(d\sigma_V)) = (c\sigma_V, d\sigma_V).$$

That θ is a homomorphism then follows from (1.2) and (2.4), and it is clearly onto.

THEOREM 2.4. *Let (R, P) be an RP-system and let V be a left normal divisor of P. Define the relation ρ_V on $R^{-1} \circ R$ as follows:*

(2.8) $\quad (a, b)\,\rho_V\,(c, d) \Leftrightarrow$ *there exist units u and v in P such that $a = uc$, $b = vd$, and $u^{-1}v \in V$.*

Then ρ_V is a congruence on $R^{-1} \circ R$ such that $\rho_V \subseteq \mathscr{H}$. If V_1 and V_2 are left normal divisors of P, then $\rho_{V_1} \subseteq \rho_{V_2}$ if and only if $V_1 \subseteq V_2$. Conversely, if ρ is any congruence on $R^{-1} \circ R$ such that $\rho \subseteq \mathscr{H}$, then there exists a left normal divisor V of P such that $\rho = \rho_V$.

The restriction of ρ_V to $R_{(1,1)}$ is essentially σ_V, as defined in (2.2), *in the sense that* $(1, a)\,\rho_V\,(1, b)$ *if and only if $a\,\sigma_V\,b$ (a, b in R). The mapping θ defined by* (2.5) *is a homomorphism of $R^{-1} \circ R$ onto $(R/\sigma_V)^{-1} \circ (R/\sigma_V)$ with kernel ρ_V, and hence*

$$(R^{-1} \circ R)/\rho_V \cong (R/\sigma_V)^{-1} \circ (R/\sigma_V). \tag{2.9}$$

Before proving Theorem 2.4, we give an immediate corollary. By the left-right dual of Lemma 2.12 of (7), there is a unique maximum left normal divisor of P, namely

$$M = \{u \in U\colon au \in Ua, \text{ for all } a \in P\},$$

where U is the group of units of P.

COROLLARY 2.5. *Under the hypothesis of Theorem* 2.4, *$\rho_V = \mu$, the maximum idempotent separating congruence on $R^{-1} \circ R$, if and only if V is the maximum left normal divisor M of P. Moreover, $\mu = \mathscr{H}$ if and only if the unit group U of P is left normal in P.*

The last assertion of this corollary is due to Munn (verbal communication, May 1965).

Proof. Let ρ_V be defined by (2.8). Clearly ρ_V is reflexive and symmetric. Let $(a, b)\,\rho_V\,(c, d)$ and $(c, d)\,\rho_V\,(e, f)$, so that $a = uc$, $b = vd$, $c = we$, $d = xf$, with u, v, w, x units of P such that $u^{-1}v, w^{-1}x \in V$. Then $a = uwe$ and $b = vxf$ with

$$(uw)^{-1}(vx) = w^{-1}(u^{-1}v)x = (u^{-1}v)^{w^{-1}}w^{-1}x,$$

which belongs to V since $u^{-1}v$ and $w^{-1}x$ belong to V. Thus ρ_V is an equivalence relation.

Again let $(a, b) \rho_V (c, d)$, where $a = uc$, $b = vd$, and $u^{-1}v \in V$. Let (x, y) be any element of $R^{-1} \circ R$. Then

$$(a, b)(x, y) = ((x * b)a, (b * x)y),$$
$$(c, d)(x, y) = ((x * d)c, (d * x)y).$$

Using (1.9), we have

$$(x * b)a = (x * vd)uc = (x * d)v^{-1}uc = w(x * d)c,$$

where $w = (v^{-1}u)^{x*d} \in V$; and also

$$b * x = vd * x = d * x.$$

Hence

$$(a, b)(x, y) = (w(x * d)c, (d * x)y),$$

from which we conclude that $(a, b)(x, y) \rho_V (c, d)(x, y)$, since $w \in V$. Similarly we can show that $(x, y)(a, b) \rho_V (x, y)(c, d)$, and thus ρ_V is a congruence.

By Lemma 1.4, $\rho_V \subseteq \mathscr{H}$. Let V_1 and V_2 be left normal divisors of P. It is immediate from the definition of ρ_{V_1} and ρ_{V_2} that $V_1 \subseteq V_2$ implies $\rho_{V_1} \subseteq \rho_{V_2}$. The converse is immediate from the fact that V is the set of all units u of P such that $(1, u) \rho_V (1, 1)$.

Now let ρ be a congruence on $R^{-1} \circ R$ such that $\rho \subseteq \mathscr{H}$. Then the ρ-class V' containing $(1, 1)$ is contained in $H_{(1,1)}$, and so every element of V' is expressible in the form $(1, u)$ with u a unit of P. Let

$$V = \{u \in P: (1, u) \in V'\} = \{u \in P: u \text{ is a unit of } P \text{ and } (1, u) \rho (1, 1)\}.$$

Clearly V' is a normal subgroup of $H_{(1,1)}$, and, by (1.7), V is a normal subgroup of the group of units of P (isomorphic with V'). We proceed to show that V is left normal in P.

Let $u \in V$, $a \in P$. From $(1, u) \rho (1, 1)$ we have

$$(1, a)(1, u) \rho (1, a)(1, 1),$$

or, by (1.7),

$$(1, au) \rho (1, a).$$

Multiplying on the right by $(a, 1)$, and using (1.5), we have

$$(1, au)(a, 1) \rho (1, 1).$$

Thus $(1, au)(a, 1) \in V'$, and so has the form $(1, v)$ with $v \in V$. Multiplying on the right by $(1, a)$, we obtain

$$(1, va) = (1, au)(a, a) = (a * au, (au * a)a)$$
$$= (a * au, au \vee a) = (a * au, (a * au)au).$$

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By the definition of equality in $R^{-1} \circ R$, there exists a unit w of P such that

$$a * au = w1, \qquad (a * au)au = w(va).$$

This implies that $au = va$, which shows that V is left normal in P.

We proceed to show that $\rho = \rho_V$. First let $(a, b)\ \rho_V\ (c, d)$. Then $a = uc$ and $b = vd$ for some units u, v of P such that $u^{-1}v \in V$. Hence $(1, u^{-1}v)\ \rho\ (1, 1)$, and, by (1.7),

$$(1, u^{-1}vd)\ \rho\ (1, d).$$

Multiplying on the left by $(c, 1)$, and using (1.8), we have

$$(c, u^{-1}vd)\ \rho\ (c, d).$$

But $(c, u^{-1}vd) = (uc, vd) = (a, b)$. Hence $\rho_V \subseteq \rho$.

Conversely, let $(a, b)\ \rho\ (c, d)$. By Lemma 1.4 and the hypothesis $\rho \subseteq \mathscr{H}$, we have $a = uc$, $b = vd$, for some units u, v of P. Since

$$(a, b) = (uc, vd) = (c, u^{-1}vd),$$

we have

$$(1, c)(c, u^{-1}vd)(d, 1)\ \rho\ (1, c)(c, d)(d, 1).$$

By (1.10) the left member is equal to

$$(1, u^{-1}vd)(d, 1) = (1, u^{-1}vd)(u^{-1}vd, u^{-1}v) = (1, u^{-1}v),$$

and the right member is equal to $(1, 1)$. Hence $u^{-1}v \in V$, and we conclude that $(a, b)\ \rho_V\ (c, d)$. Hence $\rho \subseteq \rho_V$ and so $\rho = \rho_V$.

That $(1, a)\ \rho_V\ (1, b)$ if and only if $a\ \sigma_V\ b$ $(a, b$ in $R)$ is evident. That θ defined by (2.5) is a homomorphism onto follows from Theorem 2.3, and all that remains is to show that the kernel $\theta \circ \theta^{-1}$ of θ is ρ_V.

Let $((a, b), (c, d)) \in \theta \circ \theta^{-1}$, that is $(a, b)\theta = (c, d)\theta$. Then

$$(a\sigma_V, b\sigma_V) = (c\sigma_V, d\sigma_V),$$

and hence there exists a unit $w\sigma_V$ of P/σ_V such that

$$a\sigma_V = (w\sigma_V)(c\sigma_V) \quad \text{and} \quad b\sigma_V = (w\sigma_V)(d\sigma_V).$$

Thus $a\ \sigma_V\ wc$ and $b\ \sigma_V\ wd$, so $a = xwc$, $b = ywd$, with x, y in V. Since $\sigma_V \subseteq \mathscr{L}'$, w must be a unit of P, and hence $u = xw$ and $v = yw$ are units of P. Since $u^{-1}v = w^{-1}(x^{-1}y)w \in V$, and $a = uc$, $b = vd$, we conclude that $(a, b)\ \rho_V\ (c, d)$. Hence $\theta \circ \theta^{-1} \subseteq \rho_V$.

Conversely, assume that $(a, b)\ \rho_V\ (c, d)$, so that $a = uc$, $b = vd$, with u and v units of P such that $w = u^{-1}v \in V$. Since $(wd)\sigma_V = d\sigma_V$, we have $a\sigma_V = (u\sigma_V)(c\sigma_V)$ and $b\sigma_V = (uwd)\sigma_V = (u\sigma_V)(d\sigma_V)$, whence

$$(a\sigma_V, b\sigma_V) = (c\sigma_V, d\sigma_V), \ (a, b)\theta = (c, d)\theta,$$

and $\rho_V \subseteq \theta \circ \theta^{-1}$.

3. Rédei–Schreier extension theorem for RP-systems. Let (R, P) be an RP-system, and let V be a left normal divisor of P. Denote the identity element of P (and V) by e. By Lemma 2.2, σ_V is a congruence on R, and $(\bar{R}, \bar{P})$ is an RP-system, where $\bar{R} = R/\sigma_V$ and $\bar{P} = P/\sigma_V$. We shall denote the elements of $\bar{R}$ by Greek letters $\alpha, \beta, \gamma, \ldots$, and its identity element by 1.

For each element α of $\bar{R}$ pick an element r_α of R such that $r_\alpha \sigma_V = \alpha$. In particular, choose $r_1 = e$. By (P3) and the definition of σ_V, every element of R is uniquely expressible in the form ur_α with u in V and α in $\bar{R}$.

By (P3) and the hypothesis that V is left normal in P, the rule

$$r_\alpha u = u^\alpha r_\alpha \tag{3.1}$$

defines an endomorphism $u \to u^\alpha$ of V. Since $r_\alpha r_\beta \in r_{\alpha\beta} \sigma_V$, there exists, for each α in $\bar{P}$ and β in $\bar{R}$, an element $f_{\alpha,\beta}$ of V, unique by (P3), such that

$$r_\alpha r_\beta = f_{\alpha,\beta} r_{\alpha\beta}. \tag{3.2}$$

Exactly as in the classical theory of group extensions (see, for example, Kurosh (**3**, Chapter 12)), we find that the system of endomorphisms $u \to u^\alpha$ and the factor set $f_{\alpha,\beta}$ together satisfy the following conditions:

$$f_{\alpha,\beta} u^{\alpha\beta} = (u^\beta)^\alpha f_{\alpha,\beta} \qquad (\alpha, \beta \in \bar{P}; u \in V), \tag{3.3}$$

$$f_{\alpha,\beta} f_{\alpha\beta,\gamma} = f_{\beta,\gamma}^\alpha f_{\alpha,\beta\gamma} \qquad (\alpha, \beta \in \bar{P}; \gamma \in \bar{R}), \tag{3.4}$$

$$u^1 = u \qquad (u \in V), \tag{3.5}$$

$$f_{\alpha,1} = f_{1,\beta} = e \qquad (\alpha \in \bar{P}; \beta \in \bar{R}). \tag{3.6}$$

(3.3) and (3.4) arise from the associativity conditions $(r_\alpha r_\beta)u = r_\alpha(r_\beta u)$ and $(r_\alpha r_\beta)r_\gamma = r_\alpha(r_\beta r_\gamma)$, respectively, making vital use of (P3). These products exist if and only if α and β are restricted to $\bar{P}$. (3.5) and (3.6) arise from the normalization condition $r_1 = e$.

If $ur_\alpha \in P$ and $vr_\beta \in R$, then by (3.1) and (3.2),

$$ur_\alpha vr_\beta = uv^\alpha r_\alpha r_\beta = uv^\alpha f_{\alpha,\beta} r_{\alpha\beta} \qquad (u, v \in V; \alpha \in \bar{P}; \beta \in \bar{R}). \tag{3.7}$$

If we represent the element ur_β of R by the pair (u, β) in $V \times \bar{R}$, then (3.7) becomes

$$(u, \alpha)(v, \beta) = (uv^\alpha f_{\alpha,\beta}, \quad \alpha\beta) \qquad (u, v \in V; \alpha \in \bar{P}; \beta \in \bar{R}). \tag{3.8}$$

THEOREM 3.1. *Let $(\bar{R}, \bar{P})$ be an RP-system, and let V be a group. Let 1 be the identity of $\bar{P}$, and e that of V. For each α in $\bar{P}$, let $u \to u^\alpha$ $(u \in V)$ be an endomorphism of V, and for each α in $\bar{P}$ and β in $\bar{R}$, let $f_{\alpha,\beta}$ be an element of V, such that the conditions* (3.3)–(3.6) *hold. Define a partial product in $R = V \times \bar{R}$ by* (3.8). *Then R becomes a partial semigroup with a subsemigroup $P = V \times \bar{P}$ such that (R, P) is an RP-system. The unit group of P is $U = V \times \bar{U}$, where $\bar{U}$ is the unit group of $\bar{P}$. Moreover, $V \times 1$ is a left normal divisor of P isomorphic with V, and $R/\sigma_{V\times 1} \cong \bar{R}$. If a join operation $\vee$ has been defined in $\bar{R}$, then we can define $\vee$ in R by*

$$(u, \alpha) \vee (v, \beta) = (1, \alpha \vee \beta) \qquad (u, v \in V; \alpha, \beta \in \bar{R}). \tag{3.9}$$

Conversely, let (R, P) *be an RP-system, and let* V *be a left normal divisor of* P. *Let* $\bar{R} = R/\sigma_V$ *and* $\bar{P} = P/\sigma_V$. *If we select a system of representatives* r_α $(\alpha \in \bar{R})$ *from the* σ_V*-classes of* R, *then* (3.1) *and* (3.2) *define a system of endomorphisms* $u \rightarrow u^\alpha$ *of* V $(u \in V, \alpha \in \bar{P})$ *and a factor system* $f_{\alpha,\beta}$ $(\alpha \in \bar{P}, \beta \in \bar{R})$ *satisfying* (3.3)–(3.6), *and* $R \cong V \times \bar{R}$, *with product in* $V \times \bar{R}$ *defined by* (3.8).

Proof. Except for the routine details of verification of (3.3)–(3.6), which we omit, the converse part has already been shown. Comparison of (3.7) and (3.8) shows that $ur_\alpha \rightarrow (u, \alpha)$ is an isomorphism of R onto $V \times \bar{R}$ under which P is mapped onto $V \times \bar{P}$.

Turning to the direct part, condition (P1) for the pair (R, P) is immediate from (3.8), and the associativity condition (A) then follows, as in the classical case of group extensions, from (3.3) and (3.4). From (3.5) and (3.6) we see that $(e, 1)$ is a left identity of R, so that (P2) holds. The proof of (P3) is also mechanical, using (P3) for $\bar{R}$ and cancellation in the group V. We shall establish both (P4) and (3.9) by showing that

$$P(u, \alpha) \cap P(v, \beta) = P(1, \alpha \vee \beta) \qquad (u, v \in V; \alpha, \beta \in \bar{R}). \tag{3.10}$$

It is clear from (3.8) that $P(u, \alpha) = V \times P\alpha$. From $P\alpha \cap P\beta = P(\alpha \vee \beta)$ we have

$$(V \times P\alpha) \cap (V \times P\beta) = V \times P(\alpha \vee \beta),$$

which is the same as (3.10).

An element (u, α) of P is a unit if and only if there exists (v, β) in P such that

$$(u, \alpha)(v, \beta) = (v, \beta)(u, \alpha) = (e, 1),$$

that is,

$$(uv^\alpha f_{\alpha,\beta}, \alpha\beta) = (vu^\beta f_{\beta,\alpha}, \beta\alpha) = (e, 1).$$

This requires that $\alpha\beta = \beta\alpha = 1$, so $\alpha \in \bar{U}$ and $\beta = \alpha^{-1}$. Hence $U \subseteq V \times \bar{U}$. Conversely, if $\alpha \in \bar{U}$, then we may solve

$$vu^{\alpha^{-1}} f_{\alpha^{-1},\alpha} = e \tag{3.11}$$

for v in V, and then check as follows that

$$uv^\alpha f_{\alpha,\alpha^{-1}} = e. \tag{3.12}$$

Setting $\beta = \alpha^{-1}$, and $\gamma = \alpha$ in (3.4), and using (3.6), we obtain

$$f_{\alpha,\alpha^{-1}} = f^\alpha_{\alpha^{-1},\alpha}.$$

Setting $\beta = \alpha^{-1}$ in (3.3),

$$f_{\alpha,\alpha^{-1}}\, u = (u^{\alpha^{-1}})^\alpha f_{\alpha,\alpha^{-1}} = (u^{\alpha^{-1}})^\alpha f^\alpha_{\alpha^{-1},\alpha}.$$

Hence, from (3.11),

$$e = e^\alpha = v^\alpha (u^{\alpha^{-1}})^\alpha f^\alpha_{\alpha^{-1},\alpha} = v^\alpha f_{\alpha,\alpha^{-1}}\, u,$$

which implies (3.12). Hence $U = V \times \bar{U}$.

Finally, to show that $V \times 1$ is left normal in P, let $(u, \alpha) \in P$ and $(v, 1) \in V \times 1$. Then, since $f_{\alpha,1} = f_{1,\alpha} = e$,

$$(u, \alpha)(v, 1) = (uv^{\alpha}, \alpha) = (uv^{\alpha}u^{-1}, 1)(u, \alpha) \in V(u, \alpha).$$

That $R/\sigma_{V\times 1} \cong \bar{R}$ follows from the observation that the mapping $(u, \alpha) \rightarrow \alpha$ is a homomorphism of $R = V \times \bar{R}$ onto $\bar{R}$, the kernel of which is $\sigma_{V\times 1}$. This concludes the proof of Theorem 3.1.

For later purposes, we give a consequence of (3.9). From (1.1) and (3.9), we have

$$[(u, \alpha) * (v, \beta)](v, \beta) = (1, \alpha \vee \beta).$$

Since

$$(w, \alpha * \beta)(v, \beta) = (wv^{\alpha*\beta}f_{\alpha*\beta,\beta}, \alpha \vee \beta),$$

we conclude that

$$(3.13) \qquad (u, \alpha) * (v, \beta) = (f^{-1}_{\alpha*\beta,\beta}(v^{-1})^{\alpha*\beta}, \alpha * \beta) \qquad (u, v \in V; \alpha, \beta \in \bar{R}).$$

4. Representation by triples. Let S be a bisimple inverse semigroup, and let V be a left normal divisor of S. By definition (§1) this means that V is a left normal divisor of P_e, where e is the identity of V. We shall write P for P_e and R for R_e.

By Theorem 1.3, (R, P) is an RP-system, and $S \cong R^{-1} \circ R$; we shall identify S with $R^{-1} \circ R$. By Lemma 2.2, σ_V is a congruence on R, and $(\bar{R}, \bar{P})$ is an RP-system, where $\bar{R} = R/\sigma_V$ and $\bar{P} = P/\sigma_V$. By Theorem 3.1, $R \cong V \times \bar{R}$, where $V \times \bar{R}$ is provided with the partial product defined by (3.8), and we shall identify R with $V \times \bar{R}$. The unit group U of P is $V \times \bar{U}$, where $\bar{U}$ is the unit group of $\bar{P}$. Clearly, V is normal in U, and $U/V \cong \bar{U}$.

Putting these results together, each element of S is represented as a quadruple $((u, \alpha), (v, \beta))$, with u, v in V and α, β in $\bar{R}$. Now $(u^{-1}, 1) \in U$, and, by definition of equality in $R^{-1} \circ R$,

$$\begin{aligned} ((u, \alpha), (v, \beta)) &= ((u^{-1}, 1)(u, \alpha), (u^{-1}, 1)(v, \beta)) \\ &= ((1, \alpha), (u^{-1}v, \beta)). \end{aligned}$$

Let us write

$$(4.1) \qquad (\alpha; u; \beta) = ((1, \alpha), (u, \beta)).$$

Then

$$(4.2) \qquad ((u, \alpha), (v, \beta)) = (\alpha; u^{-1}v; \beta).$$

Moreover,

$$(\alpha; u; \beta) = (\alpha'; u'; \beta')$$

if and only if there exists (v, ϵ) with v in V and ϵ in $\bar{U}$ such that

$$(1, \alpha') = (v, \epsilon)(1, \alpha) = (vf_{\epsilon,\alpha}, \epsilon\alpha)$$

and

$$(u', \beta') = (v, \epsilon)(u, \beta) = (vu^{\epsilon}f_{\epsilon,\beta}, \epsilon\beta).$$

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The first of these implies that $v = f_{\epsilon,\alpha}^{-1}$, and we conclude that

$$(\alpha; u; \beta) = (\alpha'; u'; \beta')$$

if and only if there exists a unit ϵ in $\bar{P}$ such that

$$(4.3) \qquad \alpha' = \epsilon\alpha, \qquad \beta' = \epsilon\beta, \qquad u' = f_{\epsilon,\alpha}^{-1} u^{\epsilon} f_{\epsilon,\beta}.$$

Clearly this implies equality of triples if and only if $V = U$, which is possible if and only if $\mathscr{H}$ is a congruence on S (Corollary 2.5).

Using (4.1), (1.2), and (3.13), we have

$$\begin{aligned}(\alpha; u; \beta)(\gamma; v; \delta) &= ((1, \alpha), (u, \beta))((1, \gamma), (v, \delta)) \\ &= ([(1, \gamma) * (u, \beta)](1, \alpha), [(u, \beta) * (1, \gamma)](v, \delta)) \\ &= ((f_{\gamma*\beta,\beta}^{-1}(u^{-1})^{\gamma*\beta}, \gamma * \beta)(1, \alpha), (f_{\beta*\gamma,\gamma}^{-1}, \beta * \gamma)(v, \delta)) \\ &= ((f_{\gamma*\beta,\beta}^{-1}(u^{-1})^{\gamma*\beta} f_{\gamma*\beta,\alpha}, (\gamma * \beta)\alpha), (f_{\beta*\gamma,\gamma}^{-1} v^{\beta*\gamma} f_{\beta*\gamma,\delta}, (\beta * \gamma)\delta).\end{aligned}$$

Now, using (4.2), we conclude that

$$(4.4) \qquad (\alpha; u; \beta)(\gamma; v; \delta) = ((\gamma * \beta)\alpha; f_{\gamma*\beta,\alpha}^{-1} u^{\gamma*\beta} f_{\gamma*\beta,\beta} f_{\beta*\gamma,\gamma}^{-1} v^{\beta*\gamma} f_{\beta*\gamma,\delta}; (\beta * \gamma)\delta).$$

The expressions for equality (4.3) and product (4.4) of triples appear less forbidding if we introduce the notation

$$(4.5) \qquad u_{\beta,\gamma}^{\alpha} = f_{\alpha,\beta}^{-1} u^{\alpha} f_{\alpha,\gamma} \qquad (\alpha \in \bar{P}; \beta, \gamma \in \bar{R}; u \in V).$$

Then (4.3) becomes

$$(4.3') \qquad \alpha' = \epsilon\alpha, \qquad \beta' = \epsilon\beta, \qquad u' = u_{\alpha,\beta}^{\epsilon},$$

and (4.4) becomes

$$(4.4') \qquad (\alpha; u; \beta)(\gamma; v; \delta) = ((\gamma * \beta)\alpha; u_{\alpha,\beta}^{\gamma*\beta} v_{\gamma,\delta}^{\beta*\gamma}; (\beta * \gamma)\delta).$$

This brings us to the principal objective of this note.

THEOREM 4.1. *Let $(\bar{R}, \bar{P})$ be an RP-system and V a group. Let $u \to u^{\alpha}$ $(u \in V, \alpha \in \bar{P})$ and $f_{\alpha,\beta}$ $(\alpha \in \bar{P}, \beta \in \bar{R}, f_{\alpha,\beta} \in V)$ be a system of endomorphisms and factors satisfying* (3.3)–(3.6). *Define a binary operation on the set $T = \bar{R} \times V \times \bar{R}$ by* (4.4), *and a relation τ on T by $(\alpha; u; \beta)\ \tau\ (\alpha'; u'; \beta')$ if and only if* (4.3) *holds for some unit ϵ of $\bar{P}$. Then τ is a congruence on the groupoid T, and T/τ is a bisimple inverse semigroup isomorphic with $R^{-1} \circ R$, where $R = V \times \bar{R}$ with product defined by* (3.8). *We denote the semigroup T/τ by $\bar{R}^{-1} \circ V \circ \bar{R}$.*

Conversely, let S be a bisimple inverse semigroup, and let V be a left normal divisor of S. Let e be the identity of V, and let $R = R_e$ and $P = P_e$. Let σ_V be the congruence on R defined by (2.2), *and let $\bar{R} = R/\sigma_V$ and $\bar{P} = P/\sigma_V$. Then $(\bar{R}, \bar{P})$ is an RP-system, and there exists a system of endomorphisms $u \to u^{\alpha}$ and factors $f_{\alpha,\beta}$ satisfying* (3.3)–(3.6), *such that $S \cong \bar{R}^{-1} \circ V \circ \bar{R}$.*

Proof. $(R, P) = (V \times \bar{R}, V \times \bar{P})$ is an RP-system by Theorem 3.1, and hence $R^{-1} \circ R$ is a bisimple inverse semigroup, by Theorem 1.3. As noted above, every element of $R^{-1} \circ R$ can be represented, in at least one way, in the form $((1, \alpha), (u, \beta))$ with u in V and α, β in $\bar{R}$. Define θ: $T \to R^{-1} \circ R$ by

$$(\alpha; u; \beta)\theta = ((1, \alpha), (u, \beta)). \tag{4.5}$$

From the derivation of (4.4) when we were thinking of $(\alpha; u; \beta)$ as just another notation for $((1, \alpha), (u, \beta))$, and the fact that product in T is defined by (4.4), it follows that θ is a homomorphism of the groupoid T onto the semigroup $R^{-1} \circ R$. But from the derivation of (4.3) it is apparent that the kernel of θ is just τ, whence $T/\tau \cong R^{-1} \circ R$.

The converse follows from the first two paragraphs of this section, and the direct part of the theorem.

Our final theorem gives some elementary properties of the semigroup $\bar{R}^{-1} \circ V \circ \bar{R}$.

THEOREM 4.2. *Let $(\bar{R}, \bar{P})$ be an RP-system, and V a group satisfying the hypotheses of Theorem 4.1. Let $\bar{U}$ be the unit group of $\bar{P}$. Then the following assertions hold for the bisimple inverse semigroup*

$$S = \bar{R}^{-1} \circ V \circ \bar{R} \qquad (\alpha, \beta \in \bar{R}; u \in V):$$

(a) $(\alpha; u; \beta)^{-1} = (\beta; u^{-1}; \alpha)$.
(b) *The idempotents of S are the elements of the form* $(\alpha; e; \alpha)$.
(c) $R_{(\alpha;u;\beta)} = \bar{U}\alpha \times V \times \bar{R} = \alpha \times V \times \bar{R}$.
(d) $L_{(\alpha;u;\beta)} = \bar{R} \times V \times \bar{U}\beta = \bar{R} \times V \times \beta$.
(e) $H_{(\alpha;u;\beta)} = \bar{U}\alpha \times V \times \bar{U}\beta$.
(f) $P_{(\alpha;e;\alpha)} = \bar{U}\alpha \times V \times \bar{P}\alpha = \alpha \times V \times \bar{P}\alpha$.
(g) $V' = 1 \times V \times 1$ *is a left normal divisor of* $P_{(1;e;1)}$.

Proof. By (4.1), (1.4), and (4.2),

$$(\alpha; u; \beta)^{-1} = ((1, \alpha), (u, \beta))^{-1} = ((u, \beta), (1, \alpha)) = (\beta; u^{-1}; \alpha).$$

Hence (a) holds, and (b) is immediate from (a). By Lemma (1.4),

$$((1, \alpha), (u, \beta)) \mathscr{R} ((1, \gamma), (v, \delta))$$

if and only if there exists a unit (w, ϵ) in $U = V \times \bar{U}$ (Theorem 3.1) such that $(1, \gamma) = (w, \epsilon)(1, \alpha) = (wf_{\epsilon,\alpha}, \epsilon\alpha)$, hence if and only if there exists ϵ in $\bar{U}$ such that $\gamma = \epsilon\alpha$. Thus

$$\begin{aligned} R_{(\alpha;u;\beta)} &= \{(\epsilon\alpha; v; \delta) : \epsilon \in \bar{U}, v \in V, \delta \in \bar{R}\} \\ &= \{(\alpha; v'; \delta') : v' \in V, \delta' \in \bar{R}\} \end{aligned}$$

by (4.3). Hence (c) holds, and (d) is the left-right dual of (c). (e) follows from (c) and (d) since

$$H_{(\alpha;u;\beta)} = R_{(\alpha;u;\beta)} \cap L_{(\alpha;u;\beta)}.$$

From (4.4), recalling that $1 * \beta = 1$, and using (3.6), we have

$$(1; u; \beta)(1; v; \delta) = (1; uv^{\beta * 1}f_{\beta * 1, \delta}; (\beta * 1)\delta). \tag{4.6}$$

Now $P_{(\alpha; e; \alpha)}$ consists of all elements $(\alpha; u; \beta)$ of $R_{(\alpha; e; \alpha)}$ such that

$$(\alpha; u; \beta)(\alpha; e; \alpha) = (\alpha; u; \beta),$$

that is, by (4.4),

$$((\alpha * \beta)\alpha; f_{\alpha*\beta,\alpha}^{-1} u^{\alpha*\beta} f_{\alpha*\beta,\beta} f_{\beta*\alpha,\alpha}^{-1} e^{\beta*\alpha} f_{\beta*\alpha,\alpha}; (\beta * \alpha)\alpha) = (\alpha; u; \beta),$$

or

$$((\alpha * \beta)\alpha; f_{\alpha*\beta,\alpha}^{-1} u^{\alpha*\beta} f_{\alpha*\beta,\beta}; (\beta * \alpha)\alpha) = (\alpha; u; \beta). \tag{4.7}$$

By (4.3), this means that $\beta \vee \alpha = (\beta * \alpha)\alpha = \epsilon\beta$, for some unit ϵ of P so that $\beta \in P_\alpha$. Conversely, with $\beta \in P_\alpha$, $(\beta * \alpha)\alpha = \beta \vee \alpha = \epsilon\beta$ for some unit ϵ of P and then since $(\alpha * \beta)\beta = \alpha \vee \beta = \epsilon\beta$, $(\alpha * \beta) = \epsilon$ and, by (4.3), (4.7) holds. Thus $(\alpha; u; \beta) \in P_{(\alpha; e; \alpha)}$ if and only if $\beta \in P\alpha$ and (f) follows.

When $\beta \in \bar{P}$, (4.6) becomes

$$(1; u; \beta)(1; v; \delta) = (1; uv^\beta f_{\beta,\delta}; \beta\delta).$$

Comparing with (3.8), we see that the mapping $(u, \alpha) \rightarrow (1; u; \alpha)$ is an isomorphism of the partial semigroup $R = V \times \bar{R}$ onto $R_{(1; e; 1)}$. $P = V \times \bar{P}$ is mapped onto $P_{(1; e; 1)}$, and the subgroup $V \times 1$ of P is mapped onto the subgroup $V' = 1 \times V \times 1$ of $P_{(1; e; 1)}$. By Theorem 3.1, $V \times 1$ is a left normal divisor of R, whence (g) follows.

References

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7 (American Mathematical Society, Providence, R.I., 1961).

2. J. M. Howie, *The maximum idempotent-separating congruence on an inverse semigroup*, Proc. Edinburgh Math. Soc. (II), *14* (1964), 71–79.

3. A. G. Kurosh, *The theory of groups*, translated by K. A. Hirsch (New York, 1956).

4. G. B. Preston, *Inverse semi-groups*, J. London Math. Soc., *29* (1954), 396–403.

5. ——— *Congruences on completely* 0*-simple semigroups*, Proc. London Math. Soc. (3), *11* (1961), 557–576.

6. L. Rédei, *Die Verallgemeinerung der Schreierschen Erweiterungstheorie*, Acta Sci. Math. Szeged, *14* (1952), 252–273.

7. D. Rees, *On the ideal structure of a semigroup satisfying a cancellation law*, Quart. J. Math., Oxford Ser., *19* (1948), 101–108.

8. N. R. Reilly, *Bisimple ω-semigroups*, Proc. Glasgow Math. Assoc., *7* (1966), 160–167.

9. ——— *Bisimple inverse semigroups*, to appear.

10. R. J. Warne, *Homomorphisms of d-simple inverse semigroups with identity*, Pacific J. Math., *14* (1964), 1111–1122.

11. ——— *Regular d-classes whose idempotents obey certain conditions*, Duke J. Math., *33* (1966), 187–195.

12. ——— *The idempotent separating congruences of a bisimple inverse semigroup with identity*, to appear.

13. ——— *Bisimple Z-semigroups*, to appear.

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