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A VARIATIONAL TECHNIQUE FOR BOUNDED STARLIKE FUNCTIONS

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1. Introduction. Let $K_M = \{z : |z| < M\}$, $1 \leqq M < \infty$ and $K = K_1$. Let S denote the collection of functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ that are regular and univalent in K. We write, for $1 < M < \infty$,

$$S(M) = \{f : f \in S, f(K) \subset K_M\},$$
$$S^*(M) = \{f : f \in S(M), f(K) \text{ is starlike with respect to the origin}\}.$$

In this paper we develop a variational technique for slit domains and give some applications with respect to finding the

$$\max_{f \in S^*(M)} |a_3|, \quad \max_{f \in S^*(M)} |f'(z)| \; (|z| = r \text{ fixed}),$$

and the

$$\max_{f \in S^*(M)} \operatorname{Re}\{\Phi[\log f(z)/z]\}$$

for any nonconstant entire function $\Phi(w)$ and a given $z \in K$.

2. Variations within $S^*(M)$. Let $S_R^*(M)$ denote the set of functions in $S^*(M)$ whose Taylor coefficients are real. In this section we use the Löwner Theory on slit mappings to produce variations within $S_R^*(M)$.

Suppose that the boundary of the domain $f(K)$ is a piecewise analytic curve. Let $f(K)$ have a radial slit T_0 in CL(UHP) (the closure of the upper half plane). We denote by $\bar{T}_0$ the reflection of T_0 in the real axis (possibly $T_0 = \bar{T}_0$). Let $f(e^{i\theta_0})$ and $f(e^{-i\theta_0})$ be the interior endpoints (tips) of T_0 and $\bar{T}_0$ respectively. We shall construct from f a new function in $S_R^*(M)$ by introducing radial slits T_1 and $\bar{T}_1$ into $f(K)$ at $f(e^{i\theta_1})$ and $f(e^{-i\theta_1})$, respectively, while shortening T_0 and $\bar{T}_0$ to preserve the mapping radius (and hence the normalization of the corresponding function). We shall use the well-known [**2**, Chapter 4] continuity and monotonicity properties of the mapping radius.

Let γ and $\bar{\gamma}$ be the arcs in K, with interior endpoints ω_1 and ω_2, respectively, such that $f(\gamma) = T_1$ and $f(\bar{\gamma}) = \bar{T}_1$. Uniqueness of γ and $\bar{\gamma}$ follows from the univalence of f. Parametrize γ and $\bar{\gamma}$ by $\omega = \omega(t)$ $(0 \leqq t \leqq t_1)$, $\omega(0) = e^{i\theta_1}$, and $\bar{\omega} = \overline{\omega(t)}$, $\overline{\omega(0)} = e^{-i\theta_1}$, respectively. Denote by $g(z, t)$ the function that maps K onto K minus the parametrized arcs $\omega([0, t])$, $\overline{\omega([0, t])}$ with $g(0, t) = 0$

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and $g'(0, t) > 0$. It follows from [**8**] and [**1**, § 6.2] that the function $g(z, t) = e^{-t}z + \ldots$ with $g(z, 0) = z$, satisfies Löwner's differential equation in the form

$$(1) \quad \frac{\partial g(z,t)}{\partial t} = -z\frac{\partial g}{\partial z}\frac{1-z^2}{1-2z\cos\theta_1(t)+z^2}.$$

The function $\theta_1(t)$ is a continuous function of t and $g(e^{i\theta_1(t)}, t) = \omega(t)$ and $g(e^{-i\theta_1(t)}, t) = \overline{\omega(t)}$ represent the endpoints of the lengthened arcs respectively.

Consider the composition $f[g(z, t)] = e^{-t}z + \ldots$. This function does not belong to $S_R^*(M)$, because its first coefficient is different from 1. We perform an additional variation. Parametrize T_0 and $\bar{T}_0$ by $w = w(s)$, $(0 \leqq s \leqq s_0)$, $w(0) = f(e^{i\theta_0})$ and $\bar{w} = \overline{w(s)}, \overline{w(0)} = f(e^{-i\theta_0})$, respectively. Denote by $\hat{f}(z, s)$ the function that has $\hat{f}(0, s) = 0$ and $\hat{f}'(0, s) > 0$ and maps K onto $f[g(K, t)]$ except that the slits whose interior endpoints were at $w(0)$ and $\overline{w(0)}$ are now shortened so that these endpoints are at $w(s)$ and $w(s)$, respectively. It follows from [**4**] and [**8**] that under these conditions the function

$$w(z) = \hat{f}(z, s) = f_1[g(z, t), s] = e^s e^{-t}z + \ldots$$

satisfies the condition

$$(2) \quad \frac{\partial \hat{f}}{\partial s} = z\frac{1-z^2}{1-2z\cos\theta_0(s)+z^2}\frac{\partial \hat{f}}{\partial z},$$

with $f(z, 0) = f_1[g(z, t), 0] = f[g(z, t)]$. The function $\theta_0(s)$ is a continuous function of s and $e^{\pm i\theta_0}$ represent the preimages of the endpoints of the shortened slits. Given t, put s equal to t. Then the function

$$F(z, t) = f_1[g(z, t), s(t)] = z + \sum_{n=2}^{\infty} \hat{a}_n(t)z^n$$

is in $S_R^*(M)$ for each t sufficiently small.

The main result for this section is the following lemma.

LEMMA 1. *If* $f(z) = z + a_2z^2 + \ldots + a_nz^n + \ldots$, T_0, T_1, θ_0, θ_1, *and* $F(z, t) = z + \hat{a}_2(t)z^2 + \ldots + \hat{a}_n(t)z^n + \ldots$ *are defined as above and*

$$(3) \quad h_n(e^{i\theta}) = \sum_{v=1}^{n-1} (n-v)a_{n-v}e^{iv\theta},$$

then the inequality

$$(4) \quad \operatorname{Re}\{h_n(e^{i\theta_0})\} > \operatorname{Re}\{h_n(e^{i\theta_1})\}$$

implies the existence of a t_0 *such that*

$$(5) \quad \hat{a}_n(t_0) > \hat{a}_n(0) = a_n.$$

Proof. Because the coefficients $\hat{a}_n(t)$ are continuous, it suffices to show that

$$(6)\qquad \left.\frac{\partial a_n(t)}{\partial t}\right|_{t=0} > 0.$$

It follows from the definition of $F(z, t)$ that

$$\frac{\partial F(z,t)}{\partial t} = \sum_{n=2}^{\infty} \frac{\partial \hat{a}_n(t)}{\partial t} z^n,$$

where the term by term differentiation with respect to t can be justified by expressing the coefficients as contour integrals in terms of $f_1[g(z, t), s(t)]$. We obtain from (1), (2) and the previous paragraphs the following:

$$\begin{aligned}
\left.\frac{\partial F(z,t)}{\partial t}\right|_{t=0} &= \left.\left[\frac{\partial f_1}{\partial g}\frac{\partial g}{\partial t} + \frac{\partial f_1}{\partial t}\right]\right|_{t=0} \\
&= \left.\left[\frac{\partial \hat{f}}{\partial z}\left(\frac{\partial g}{\partial z}\right)^{-1}\frac{\partial g}{\partial t} + \frac{\partial f_1}{\partial s}\frac{\partial s}{\partial t}\right]\right|_{t=0} \\
&= -z\frac{\partial f(z)}{\partial z}\frac{1-z^2}{1-2z\cos\theta_1+z^2} + z\frac{1-z^2}{1-2z\cos\theta_0+z^2}\frac{\partial f}{\partial z} \\
&= zf'(z)2\left[\sum_{n=1}^{\infty}(\cos n\theta_0)z^n - \sum_{n=1}^{\infty}(\cos n\theta_1)z^n\right] \\
&= 2\left(\sum_{v=1}^{\infty} v a_v z^v\right)\sum_{n=1}^{\infty}(\cos n\theta_0 - \cos n\theta_1)z^n \\
&= 2\sum_{n=2}^{\infty} c_n z^n, \\
c_n &= \sum_{v=1}^{n-1}(n-v)a_{n-v}(\cos v\theta_0 - \cos v\theta_1).
\end{aligned}$$

Thus, by comparing coefficients, we obtain

$$(9)\qquad \left.\frac{\partial \hat{a}_n(t)}{\partial t}\right|_{t=0} = \operatorname{Re}\left\{\sum_{v=1}^{n-1}(n-v)a_{n-v}e^{iv\theta_0} - \sum_{v=1}^{n-1}(n-v)a_{n-v}e^{iv\theta_1}\right\}.$$

The right hand of (9) is positive when (4) holds. This proves Lemma 1.

Remark. Because of its geometric nature, the technique developed in this section can be used, with straightforward adjustments, to define similar variations within many of the standard subclasses of S, e.g. close to convex functions, functions convex in one direction, $S(M)$, S, and the corresponding subclasses with real coefficients.

3. Applications. We shall use the technique developed in the last section to solve some extremal problems in $S^*(M)$. Let $\Phi(w)$ be an entire function.

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We show that the extremal functions which occur when finding the maximum value for all $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ in $S^*(M)$ of the functionals $|a_2|$, $|a_3|$, $|f'(z)|$ ($|z| = r$ fixed), and $|\Phi\{\log[f(z)/z]\}|$ (z fixed in K) are those functions that map K onto K_M minus at most two radial slits. A general form for an $f^*(z)$ satisfying this mapping property can be obtained by using the boundary behavior of $zf'^*(z)/f^*(z)$ as in [**6**]. Let

$$0 \leqq \alpha_0 \leqq \beta_0 \leqq \gamma_0 \leqq \gamma_1 \leqq \beta_1 \leqq \alpha_1 \leqq 2\pi.$$

Consider

$$(10) \quad w = F(z) = \frac{zf^{*\prime}(z)}{f^*(z)} = \left[\frac{(z - e^{i\beta_0})^2(z - e^{i\beta_1})^2}{(z - e^{i\alpha_0})(z - e^{i\alpha_1})(z - e^{i\gamma_0})(z - e^{i\gamma_1})}\right]^{1/2}, \qquad F(0) = 1.$$

Since, for δ real, $e^{i\theta} - e^{i\delta} = e^{i[(\theta+\delta)/2]}2i \sin[(\theta - \delta)/2]$, then

$$F^2(e^{i\theta}) = F(0) \sin^2\left(\frac{\theta - \beta_0}{2}\right) \sin^2\left(\frac{\theta - \beta_1}{2}\right) \Big/ \sin\left(\frac{\theta - \alpha_0}{2}\right) \sin\left(\frac{\theta - \alpha_1}{2}\right) \times \sin\left(\frac{\theta - \gamma_0}{2}\right) \sin\left(\frac{\theta - \gamma_1}{2}\right)$$

is a real valued function of θ which is continuous for $\theta \in [0, 2\pi]\backslash\{\alpha_0, \alpha_1, \gamma_0, \gamma_1\}$. It follows that $F^2(z)$ maps K onto a domain whose boundary consists of 2 rays: one containing the negative reals, the other contained in the positive reals. Hence $\partial F(K)$ consists of the imaginary axis and a line segment and a ray that are contained in the positive reals with $\operatorname{Re}\{F(z)\} > 0$ for $z \in K$. Thus $f^* \in S^*$. $F(e^{i\theta})$ has either its real or imaginary part equal to zero when $\theta \neq \alpha_0, \alpha_1, \gamma_0, \gamma_1$. Since

$$F(e^{i\theta}) = \frac{\partial \arg f^*(e^{i\theta})}{\partial\theta} - i\frac{\partial \log |f^*(e^{i\theta})|}{\partial\theta}$$

(any branch of $\arg f^*(e^{i\theta})$, $0 < \theta < 2\pi$ can be chosen), it follows that the boundary of the image domain of $f^*(z)$ defined by

$$(11) \quad \log [f^*(z)/z] = \int_0^z \frac{F(w) - 1}{w}\, dw$$

consists of the arcs of two circles centered at the origin and four radial sides. If we assume that

$$(12) \quad \begin{cases} f^*(e^{i\gamma_0}) = f^*(e^{i\gamma_1}) = Me^{i\phi}, & \alpha_0 = 2\pi - \alpha_1 = 0 \\ f^*(e^{i\alpha_0}) = f^*(e^{i\gamma_0}) = Me^{i\phi}, & \text{otherwise} \end{cases}$$

for some real ϕ, then the two circles coincide and have radius M. Thus $f^*(z)$ defined by (11) and (12) is in $S^*(M)$ and maps K onto K_M minus at most two radial slits with $f^*(e^{i\beta_0})$ and $f^*(e^{i\beta_1})$ the interior endpoints of the slits (possibly coincident).

For notational purposes we begin by using the technique of section 2 to maximize $|a_3|$ in $S^*(M)$.

THEOREM 1. *If* $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ *is in* $S^*(M)$, *then* $|a_3| \leqq a_3^*$ *where* $f^*(z) = z + a_2^*z^2 + a_3^*z^3 + \ldots$ *will be of the form defined by* (11) *and* (12) *with* $\alpha_0 = 2\pi - \alpha_1$, $\gamma_0 = 2\pi - \gamma_1$ *and* $\beta_0 = 2\pi - \beta_1$ *in* (10), *i.e.*, $f^*(K)$ *is* K_M *minus at most two radial slits symmetric with respect to the real axis.*

Proof. We first show that we may assume real coefficients as was done in section 2.

LEMMA 2. *There exists a function* $f(z) = z + a_2z^2 + a_3z^3 + \ldots \in S^*(M)$ *with* $a_2 \geqq 0$, $a_3 > 0$ *and* a_k *real* $(k = 4, 5, \ldots)$ *that maximizes* $|a_3|$ *for all* $f(z) \in S^*(M)$.

Proof. Since $S^*(M)$ is a closed subclass of the compact class S the extremal problem has a solution in $S^*(M)$. Let $g(z) = z + a_2z^2 + a_3z^3 + \ldots$ be an extremal function. Because of the invariance of $S^*(M)$ under rotation we may assume a_3 is positive. Also it follows from the maximum principle that if f is in S^*, then f is in $S^*(M)$ if and only if $|f(z)/z| \leqq M$ $(z \in K)$. Define

$$\text{(13)} \quad \frac{zf'(z)}{f(z)} = \frac{1}{2}\frac{zg'(z)}{g(z)} + \frac{1}{2}\overline{\left(\frac{\bar{z}g'(\bar{z})}{g(\bar{z})}\right)}.$$

Then

$$f(z) = z\left(\frac{g(z)}{z}\right)^{1/2}\overline{\left(\frac{g(\bar{z})}{\bar{z}}\right)^{1/2}}$$

$$= z + \operatorname{Re}\{a_2\}z^2 + \left[\frac{1}{4}\left(|a_2|^2 - \operatorname{Re}\{a_2^2\}\right) + a_3\right]z^3 + \sum_{n=4}^{\infty} c_n z^n,$$

c_n real $n = 4, 5, \ldots$

The inequalities $\operatorname{Re}\{zf'(z)/f(z)\} > 0$, $(z \in K)$ and $|f(z)/z| = |g(z)/z|^{1/2} \overline{|g(\bar{z})/\bar{z}|^{1/2}} \leqq M (z \in K)$ imply that f is in $S^*(M)$. Since $g(z)$ is extremal and a_3 is positive it follows that $|a_2|^2 - \operatorname{Re}\{a_2^2\} = 0$ and hence a_2 is real. Thus, the resulting function f will solve the extremal problem and have real coefficients. Further, we can assume $a_2 \geqq 0$. Indeed, if $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ is in $S_R^*(M)$ then $-f(-z) = z - a_2z^2 + a_3z^3 - \ldots$ is in $S_R^*(M)$. This proves Lemma 2.

Let $B = \{f(K) : f \in S_R^*(M)\}$ and $B_n = \{D \in B : \partial D = \partial K_M$ and at most n radial slits$\}$. Clearly $S_R^*(M)$ and $\{f \in S : f(K) \in B_n\}$ are compact classes. Hence for each n there exists a domain $D_n^* = f_n^*(K) \in B$ where $f_n^*(z) = z + a_2^*z^2 + a_3^*z^3 + \ldots$ satisfies $a_3 \leqq a_3^*$ for all $f_n(K) \in B_n$ with $f_n(z) = z + a_2z^2 + a_3z^3 + \ldots$. A straightforward argument shows that the union of the B_n's is dense in B. Thus it follows from the Caratheodory convergence theorem that any convergent subsequence $\{D_{nk}^*\}_{k=1}^{\infty}$ of extremal domains in B_{n_k} converges to an extremal domain D^* in B (from the compactness of $S_R^*(M)$). Let $f^*(K) = D^*$.

We first show that if D_n^* is an extremal domain maximizing $|a_3|$ within B_n, then D_n^* is K_M minus at most four radial slits.

For suppose to the contrary, that D_n^* is K_M minus at least five radial slits. From symmetry at least three of these slits must lie in CL(UHP). We will assume that we are performing the variations in CL(UHP) while the corresponding variations are being made in CL(LHP) at the same time. For $0 \leqq \theta, \theta' \leqq \pi$ we consider the function

$$(14) \quad P(\theta, \theta') = \operatorname{Re}\{2a_2e^{i\theta} + e^{2i\theta}\} - \operatorname{Re}\{2a_2e^{i\theta'} + e^{2i\theta'}\} = \operatorname{Re}\{h(\theta)\} - \operatorname{Re}\{h(\theta')\}$$

with $0 \leqq a_2 < 2$, and $f(e^{i\theta})$ and $f(e^{i\theta'})$ will be chosen at the interior endpoints of the slits or at points on ∂K_M where a radial slit may be introduced.

We observe that $h(\theta)$ in (14) can be realized as a map of the unit circle onto a curve symmetric about the real axis with either one loop or a cusp at $h(\pi)$ so that any vertical line intersects the image curve in at most four points. This also follows since, for a fixed θ or θ', $P(\theta, \theta')$ is a quadratic in $\cos\theta$ or $\cos\theta'$. Thus $P(\theta, \theta')$ has at most two zeros for $\theta \in [0, \pi]$. With three points to choose from it follows that there exists a θ'' and a θ' such that $P(\theta'', \theta')$ is positive, i.e., we can shorten the slit ending at $f(e^{i\theta''})$ while extending the slit ending at $f(e^{i\theta'})$ so that (4) holds with $n = 3$. Since this would contradict the extremal property of D_n^*, D_n^* is K_M minus at most four radial slits.

It follows from this characterization of D_n^* and the properties of convergence of domains that D^* is K_M minus at most four radial slits. Indeed we prove that D^* is K_M minus at most two radial slits.

Suppose to the contrary that D^* is K_M minus at least three radial slits, two of which must lie in CL(UHP). From symmetry we need only describe the variations being made in CL(UHP) as before. Then, we shorten one slit while either extending another slit or introducing a new radial slit into one of the arcs on ∂K_M, i.e., we show that we can choose a θ and θ' such that $P(\theta, \theta')$ is positive. Let $e^{i\theta_0}$ and $e^{i\theta_1}$ be the two points corresponding to the interior endpoints of the slits. When $\operatorname{Re}\{h(\theta_0)\} \neq \operatorname{Re}\{h(\theta_1)\}$ we can assume $\operatorname{Re}\{h(\theta_0)\} > \operatorname{Re}\{h(\theta_1)\}$. Then $P(\theta_0, \theta_1)$ would be positive, which would imply a contradiction. Now consider the case when $\operatorname{Re}\{h(\theta_0)\} = \operatorname{Re}\{h(\theta_1)\}$. Since $0 \leqq a_2 < 2$, $\operatorname{Re}\{h(\theta)\}$ has at most one point where an absolute minimum occurs for $\theta \in [0, \pi]$. Hence $\operatorname{Re}\{h(\theta_0)\} = \operatorname{Re}\{h(\theta_1)\}$ is not an absolute minimum. Thus a θ' can be chosen such that a slit can be introduced at $f(e^{i\theta'})$ and the slit ending at $f(e^{i\theta_0})$, say, shortened so that $P(\theta_0, \theta')$ is positive. This would again imply a contradiction. Therefore D^* is K_M minus at most two radial slits.

The proof of Theorem 1 thus reduces to showing that, in the set of domains K_M minus one or two radial slits along the real axis, those domains having two radial slits of unequal length are not extremal. Assume to the contrary that the extremal domain has exactly two radial slits of unequal length. From the symmetry of the domains these slits must have their endpoints corresponding to $\theta = 0$ and π. A straightforward calculation of the mapping function shows that $a_2 \neq 0$ in this case. So for $P(\theta, \theta')$ as defined in (14) we have either $P(0, \pi)$ or $P(\pi, 0)$ is positive. This would imply a contradiction. Thus, Theorem 1 follows.

In [7] O. Tammi found the extremal domain maximizing $|a_3|$ for f in $S(M)$ to be K_M minus two vertical slits of equal length when $1 \leqq M \leqq e$. While for $e \leqq M < \infty$, the extremal domain is K_M minus a forked slit symmetrical about the real axis. At $M = e$ the formula for the two parameter family of extremal domains with two non-radial symmetrical slits was given by V. Singh [4]. Since the extremal domain maximizing $|a_3|$ for f in $S(M)$ is starlike for $1 < M < e$ the extremal domain maximizing $|a_3|$ for f in $S^*(M)$ is given for $1 < M < e$ by letting $\beta_0 = 2\pi - \beta_1 = \pi/2$, $\alpha_0 = \pi - \gamma_0 = \gamma_1 - \pi = 2\pi - \alpha_1$ in (10).

Note. Since the author has been unable to find a workable closed form for the function $f^*(z)$ defined by (11) and (12) the question as to the exact upper bounds for $|a_3|$ in $S^*(M)$, $M > e$, remains open. However, the author's work suggests the conjecture that the only functions $f^*(z)$ defined by (11) and (12) that are extremal for maximizing $|a_3|$ within $S^*(M)$ are those having as an image domain either K_M minus two vertical slits for $1 < M \leqq 3$, or K_M minus one slit when $3 \leqq M < \infty$. One fact that suggests the conjecture is that the one and two slit domains just defined have the same third coefficient, i.e., $3 - 8/M + 5/M^2 = 1 - M^{-2} = 8/9$ when $M = 3$. The author has verified the conjecture in the case when $M \geqq 5$ in a forthcoming joint paper with J. L. Lewis.

THEOREM 2. *Let $r_0 = 2 - \sqrt{3}$ be the radius of convexity for S^*. The extremal function maximizing $|f'(z)|$, $|z| = r$ fixed, for all $f \in S^*(M)$ has as its image domain K_M minus one radial slit for $0 < r < r_0$. For $r_0 \leqq r < 1$ the extremal function has the general form defined by* (11) *and* (12), *i.e., its image domain is K_M minus at most two radial slits.*

Remark. The upper bound for the range of r's occurring in Theorem 2 that have the associated extremal domain with only one slit can not be made arbitrarily close to one for all M. This follows because for $3 < M < \infty$ the domain K_M minus one slit is not extremal for maximizing $|f'(z)|$ over $S^*(M)$ for fixed r sufficiently close to one. To show this let $w_1(z)$ be the function mapping K onto K_M minus one slit along the negative reals and let $w_2(z)$ be the function mapping K onto K_M minus two slits of equal length along the real axis. Then $w_1(z)$ and $w_2(z)$ are defined by $w_1(z)[1 - w_1(z)/M]^{-2} = z(1 - z)^{-2}$ and $w_2(z)[1 + (w_2(z)/M)^2]^{-1} = z(1 + z^2)^{-1}$. Let $z_1 = e^{i\phi_1}$ and $z_2 = e^{i\phi_2}$ be defined by $w_1(z_1) = w_2(z_2) = -M$ where we choose ϕ_1 and ϕ_2 in $(0, \pi)$ for $1 < M < \infty$. Then $z_1 = [M - 2 + 2(1 - M)^{1/2}]/M$ and $z_2 = [-1 + (1 - M^2)^{1/2}]/M$. A straightforward calculation shows the follow-

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ing rather unexpected result:

$$(15)\quad \left|\frac{w_1'(z_1)}{w_2'(z_2)}\right|^2 = \lim_{r\to 1}\left|\frac{w_1'(re^{i\phi_1})}{w_2'(re^{i\phi_2})}\frac{1+w_1(re^{i\phi_1})/M}{1+w_2(re^{i\phi_2})/M}\right|$$

$$= 4\left|\frac{(1+z_1)(1-z_2^{\,2})}{(1-z_1)^3(1+z_2^{\,2})^2}\right| = \frac{2}{(1+M)^{1/2}}.$$

It follows from (15), using a standard continuity argument, that

$$\max_{\phi} |w_1'(re^{i\phi})| < \max_{\phi} |w_2'(re^{i\phi})|$$

for a fixed r sufficiently close to one and $3 < M < \infty$.

Proof of Theorem 2. Although the proof closely follows the outline of the proof of Theorem 1, the transformation (13) can not be performed while preserving $|f'(r)|$. Hence, since symmetry can not be assumed, we use the standard form of the Löwner differential equation [**1**]. In this case we construct another function in $S^*(M)$ from a function $f \in S^*(M)$ by introducing or lengthening one radial slit T_1 at the point $f(e^{i\theta_1})$ while shortening a radial slit T_0 whose interior endpoint is at $f(e^{i\theta_0})$. We make the obvious changes in the first part of section 2 with $g(z, t)$ and $\hat{f}(z, s)$ replaced by $g_2(z, t)$ and $\hat{f}_2(z, s) = f_2[g_2(z, t), s] = e^s e^{-t} z + \ldots$, and

$$(16)\quad \frac{\partial g_2(z,t)}{\partial t} = -zg'(z,t)\,\frac{1+e^{-i\theta_1}z^{(t)}}{1-e^{-i\theta_1}z^{(t)}}$$

$$(17)\quad \frac{\partial \hat{f}_2}{\partial s} = z\,\frac{1+e^{-i\theta_0(s)}z}{1-e^{-i\theta_0(s)}z}\cdot\frac{\partial \hat{f}_2}{\partial z}.$$

When we put s equal to t in f_2 this produces the function

$$F_2(z, t) = f_2[g_2(z, t), t] = z + \ldots$$

that is in $S^*(M)$ for each t sufficiently small.

LEMMA 3. *Let* $f(z)$, T_0, T_1, θ_0, θ_1, *and* $F_2(z, t)$ *be defined as above and let*

$$(18)\quad g(\theta) = \frac{2e^{-i\theta}z}{(1-e^{-i\theta}z)^2} + A(z)\left[\frac{1+e^{-i\theta}z}{1-e^{-i\theta}z}\right]$$

for

$$(19)\quad A(z) = 1 + zf''(z)/f'(z).$$

Then the inequality

$$(20)\quad \mathrm{Re}\{g(\theta_0) - g(\theta_1)\} > 0$$

implies the existence of a $t_0 > 0$ *such that* $|F_2'(z, t_0)| > |f'(z)|$ *for a fixed* $z \in K$.

Proof. Because of the continuity of the partials it suffices to show that

$$(21)\quad \left.\frac{\partial \log |F_2'(z,t)|}{\partial t}\right|_{t=0} > 0.$$

To obtain an expression for the left hand side of (21) we use (16), (17), and the continuity of the partials at $t = 0$. If we let $'$ represent $\partial/\partial z$ then

$$\begin{aligned}\left.\frac{\partial \log |F_2'(z,t)|}{\partial t}\right|_{t=0} &= \operatorname{Re}\left.\left\{\frac{1}{F_2'(z,t)}\left(\frac{\partial F_2(z,t)}{\partial t}\right)'\right\}\right|_{t=0} \\ &= \operatorname{Re}\left.\left\{\frac{1}{F_2'}\left(\frac{\partial f_2}{\partial g_2}\frac{\partial g_2}{\partial t} + \frac{\partial f_2}{\partial t}\right)'\right\}\right|_{t=0} \\ &= \operatorname{Re}\left\{\frac{1}{f'(z)}\left[-zf'\frac{1+e^{-i\theta_1}z}{1-e^{-i\theta_1}z} + zf'\frac{1+e^{-i\theta_0}z}{1-e^{-i\theta_0}z}\right]'\right\} \\ &= \operatorname{Re}\left\{\frac{1}{f'(z)}\left[zf'\left(\frac{2e^{-i\theta_0}}{(1-e^{-i\theta_0}z)^2} - \frac{2e^{-i\theta_1}}{(1-e^{-i\theta_1}z)^2}\right)\right.\right. \\ &\qquad \left.\left. + \left(\frac{1+e^{-i\theta_0}z}{1-e^{-i\theta_0}z} - \frac{1+e^{-i\theta_1}z}{1-e^{-i\theta_1}z}\right)(zf''+f')\right]\right\} \\ &= \operatorname{Re}\{g(\theta_0) - g(\theta_1)\}.\end{aligned}$$

Thus when (20) holds (21) follows. This proves Lemma 3.

Let z_0, $|z_0| = r$, be such that $|f'(z)| \leqq |f'(z_0)|$ for all z, $|z| = r$. Then

$$0 = \left.\frac{\partial}{\partial \theta}\log |f'(z)|\right|_{z=z_0} = \operatorname{Im} A(z_0)$$

implies $A(z_0)$ is real. Thus from Lemma 3 there exists a $t_0 > 0$ such that

$$(22)\quad |F'(z_0, t_0)| > |f'(z_0)|$$

when (20) holds with $z = z_0$ in (18).

Now form B and B_n as in Theorem 1 with D_n^* the extremal domains in B_n that have the corresponding mapping functions $f_n^*(z)$ such that $|f_n'(z)| \leqq |f_n^{*\prime}(z_0)|$ for all f_n such that $f_n(K) \in B_n$. As before this gives the corresponding extremal domain D^* in B.

Recall the definition of $g(\theta)$ in (18). If we let $x = e^{-i\theta}z$ and $A = A(z_0)$, we consider

$$g_1(x) = 2x(1-x)^{-2} + A(1+x)(1-x)^{-1} = A + 2(1+A)x + \dots.$$

Clearly, for a fixed θ or θ'

$$\begin{aligned}P_1(\theta,\theta') &= \operatorname{Re}\{g_1(e^{i\theta})\} - \operatorname{Re}\{g_1(e^{i\theta'})\} \\ &= 2(1+\mathrm{A})\operatorname{Re}\{[e^{i\theta} - Ar^2e^{2i\theta}/(1+A)](1-re^{i\theta})^{-2} \\ &\qquad -[e^{i\theta'} - Ar^2e^{2i\theta'}/(1+A)](1-re^{i\theta'})^{-2}\}\end{aligned}$$

is a quadratic in $\cos\theta$ (or θ') and so has at most four zeros in $[0, 2\pi)$. Thus, arguing as in Theorem 1, the extremal function has as its image domain K_M minus at most two slits and has the general form given by (11) and (12) for any r, $0 < r < 1$. However, for $0 < r < r_0 = 2 - \sqrt{3}$ we can show that the extremal domain is K_M minus exactly one slit. To do this we use the fact that since $f \in S^*$, [**2**, inequality 1.6, p. 5] implies that

$$(23)\quad A = A(z_0) \geqq (1 - 4|z_0| + |z_0|^2)/(1 - |z_0|^2).$$

We claim that as a function of x, $|x| < 1$,

$$g_1(x) = A + 2(1 + A)[x - Ax^2/(1 + A)]/(1 - x)^2$$

is univalent for $A > 0$. Indeed, if we let $c(x) = (1 + x)/(1 - x)$, then $g_3(x) = [g_1(x) - A]/2(1 + A)$ has the property that $g_3'(x)/c'(x)$ has the positive real part for $A > 0$. Since $c(x)$ is convex, this implies from a definition of close to convexity that $g_3(x)$ is close to convex. Thus $g_1(x)$ is univalent for $A > 0$. It follows from (23) that $A > 0$ whenever $0 < r < r_0$. Also since $g_3(x) \in S$ it follows from [**2**, § 6.11] that $g_3(|x| \leqq r)$ is convex for $0 < r < r_0$. Thus any vertical line would intersect $g_1(|x| = r)$, $0 < r < r_0$ in at most two points. Hence for $0 < r < r_0$ and fixed θ or θ' $P_1(\theta, \theta')$ has at most two zeros in $[0, 2\pi)$. An argument similar to that used in the proof of Theorem 1 shows that the extremal domain is K_M minus exactly one slit for this range of r. This proves Theorem 2.

Similar methods can be used to prove the following theorem noting that the image curves in the crucial function corresponding to $g(\theta)$ in (18) are just circles, hence convex.

THEOREM 3. *Let $\Phi(w)$ be a nonconstant entire function and z a given point in K. The extremal function maximizing* $\mathrm{Re}\{\Phi(\log[f(z)/z])\}$ *for all $f \in S^*(M)$, $1 < M < \infty$, has as its image domain K_M minus one radial slit.*

We note that with the appropriate choice of $\Phi(w)$ in Theorem 3 the solutions to some of the classical extremal problems follow for the class $S^*(M)$, e.g., K_M minus one slit is an extremal domain for the functionals $|f(z)|$ and $|a_2|$, as is well-known. Also, we obtain that K_M minus one slit is an extremal domain for the functional $|\arg[f(z)/z]|$ in $S^*(M)$.

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