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ON JOINT RUIN PROBABILITIES OF A TWO-DIMENSIONAL RISK MODEL WITH CONSTANT INTEREST RATE

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Abstract

In this note we consider the two-dimensional risk model introduced in Avram, Palmowski and Pistorius (2008) with constant interest rate. We derive the integral-differential equations of the Laplace transforms, and asymptotic expressions for the finite-time ruin probabilities with respect to the joint ruin times $T_{\max}(u_1, u_2)$ and $T_{\min}(u_1, u_2)$, respectively.

Keywords: Two-dimensional risk model; constant interest rate; joint ruin probability; integral-differential equation; asymptotic expression

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1. Introduction and preliminaries

Ruin theory for the univariate risk model has been studied extensively; see [1], [10], and many recent papers. However, there is limited research on multivariate risk models. Chan *et al.* [5] studied the two-dimensional risk model

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t - \sum_{j=1}^{N(t)} \begin{pmatrix} X_{1j} \\ X_{2j} \end{pmatrix},$$

where, for fixed $i = 1$ or 2, $\{X_{ij},\ j = 1, 2, \ldots\}$ are independent and identically distributed (i.i.d.) claim size random variables, and $\{X_{1j},\ j = 1, 2, \ldots\}$ and $\{X_{2j},\ j = 1, 2, \ldots\}$ are independent, and also independent of the Poisson process $N(t)$.

Cai and Li [3] studied the multivariate risk model

$$\begin{pmatrix} U_1(t) \\ \vdots \\ U_s(t) \end{pmatrix} = \begin{pmatrix} u_1 + p_1 t - \sum_{n=1}^{N(t)} X_{1,n} \\ \vdots \\ u_s + p_s t - \sum_{n=1}^{N(t)} X_{s,n} \end{pmatrix}, \tag{1.1}$$

where $\{(X_{1,n}, \ldots, X_{s,n}),\ n \geq 1\}$ is a sequence of i.i.d. nonnegative random vectors, and independent of the Poisson process $N(t)$. Model (1.1) was further studied by Cai and Li [4].

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Yuen *et al.* [11] discussed the bivariate compound Poisson model

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t - \begin{pmatrix} \sum_{i=1}^{M_1(t)+M(t)} X_i \\ \sum_{i=1}^{M_2(t)+M(t)} Y_i \end{pmatrix},$$

where $M_1(t)$, $M_2(t)$, and $M(t)$ are three independent Poisson processes, and $\{X_i,\ i \geq 1\}$ and $\{Y_i,\ i \geq 1\}$ are i.i.d. claim size random variables, independent of each other and the three Poisson processes.

Li *et al.* [7] discussed the bidimemsional perturbed risk model

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t - \sum_{j=1}^{N(t)} \begin{pmatrix} X_{1j} \\ X_{2j} \end{pmatrix} + \begin{pmatrix} \sigma_1 B_1(t) \\ \sigma_2 B_2(t) \end{pmatrix},$$

where $N(t)$ is a Poisson process, $\{(X_{1j}, X_{2j}),\ j \geq 1\}$ is a sequence of i.i.d. random vectors, $(B_1(t), B_2(t))$ is a standard bidimensional Brownian motion, and the three processes are mutually independent.

Avram *et al.* [2] studied the two-dimensional risk model

$$\begin{pmatrix} U_1(t) \\ U_2(t) \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} t - \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} S(t), \tag{1.2}$$

where $S(t)$ is a Lévy process with only upward jumps that represents the cumulative amount of claims up to time t, and focused on the classic Cramér–Lundberg model, i.e. $S(t)$ is a compound Poisson process.

In this note we discuss the two-dimensional risk model (1.2) with constant interest rate. For univariate ruin models with investment income, a lot of research has been carried out; see the recent survey paper [8] and the references therein.

Now we introduce our model. Let r be a nonnegative constant, which represents the interest rate. Then our model can be expressed as

$$U_i(t) = \mathrm{e}^{rt} u_i + c_i \int_0^t \mathrm{e}^{r(t-v)}\,\mathrm{d}v - \delta_i \int_0^t \mathrm{e}^{r(t-v)}\,\mathrm{d}S_v, \qquad i = 1, 2, \tag{1.3}$$

where the u_i are the initial reserves, the c_i are the premium rates, and $0 < \delta_1, \delta_2 < 1$ with $\delta_1 + \delta_2 = 1$. Here S_t is taken to be a compound Poisson process, i.e. $S_t = \sum_{k=1}^{N(t)} \sigma_k,\ t \geq 0$, where $N(t)$ is a Poisson process with intensity $\lambda > 0$ and $\{\sigma_k,\ k \geq 1\}$ is a sequence of i.i.d. random variables independent of $N(t)$. Denote by F the distribution function and by f the probability density function of σ_k. Let θ_k be the arrival time of the kth claim. Then we can rewrite (1.3) as

$$U_i(t) = \mathrm{e}^{rt} u_i + \frac{c_i}{r}(\mathrm{e}^{rt} - 1) - \delta_i \sum_{k=1}^{N(t)} \mathrm{e}^{r(t-\theta_k)} \sigma_k, \qquad i = 1, 2. \tag{1.4}$$

For $k = 1, 2, \ldots,$ denote by T_k the intertime between the $(k-1)$th claim and the kth claim. Then $\{T_k,\ k \geq 1\}$ is a sequence of i.i.d. random variables that has exponential distribution with parameter λ, and $\theta_k = \sum_{i=1}^k T_i$.

Define two joint ruin times by

$$T_{\min}(u_1, u_2) := \inf\{t \geq 0 \mid \min\{U_1(t), U_2(t)\} < 0\},$$
$$T_{\max}(u_1, u_2) := \inf\{t \geq 0 \mid \max\{U_1(t), U_2(t)\} < 0\},$$

and the corresponding ruin probabilities by

$$\psi_{\min}(u_1, u_2) := \mathbb{P}\{T_{\min}(u_1, u_2) < \infty\},$$
$$\psi_{\max}(u_1, u_2) := \mathbb{P}\{T_{\max}(u_1, u_2) < \infty\}.$$

As in [2], we assume that $c_1/\delta_1 > c_2/\delta_2$. Then if $u_1/\delta_1 > u_2/\delta_2$, the above two joint ruin probabilities degenerate into one-dimensional ruin probabilities as follows:

$$\psi_{\min}(u_1, u_2) = \psi_2(u_2) := \mathbb{P}\{\text{there exists } t < \infty \text{ such that } U_2(t) < 0\},$$
$$\psi_{\max}(u_1, u_2) = \psi_1(u_1) := \mathbb{P}\{\text{there exist } t < \infty \text{ such that } U_1(t) < 0\}.$$

We refer the reader to [2] for the deduction. Throughout the rest of this note, we assume that $c_1/\delta_1 > c_2/\delta_2$ and $u_1/\delta_1 \le u_2/\delta_2$.

Remark 1.1. For each i, we know that

$$U_i(t) = \mathrm{e}^{rt} u_i + c_i \int_0^t \mathrm{e}^{r(t-v)}\,\mathrm{d}v - \delta_i \int_0^t \mathrm{e}^{r(t-v)}\,\mathrm{d}S_v = \mathrm{e}^{rt} u_i + \int_0^t \mathrm{e}^{r(t-v)}\,\mathrm{d}(c_i v - \delta_i S_v).$$

Define $\vec{U}(t) = (U_1(t), U_2(t))$, $\vec{u} = (u_1, u_2)$, and $\vec{Z}_v = (c_1 v - \delta_1 S_v, c_2 v - \delta_2 S_v)$. Then we have

$$\vec{U}(t) = \mathrm{e}^{rt}\vec{u} + \int_0^t \mathrm{e}^{r(t-v)}\,\mathrm{d}\vec{Z}_v = \mathrm{e}^{rt}\left(\vec{u} + \int_0^t \mathrm{e}^{-rv}\,\mathrm{d}\vec{Z}_v\right). \tag{1.5}$$

Differentiating both sides of (1.5) relative to t, we obtain

$$\mathrm{d}\vec{U}(t) = r\mathrm{e}^{rt}\left(\vec{u} + \int_0^t \mathrm{e}^{-rv}\,\mathrm{d}\vec{Z}_v\right)\mathrm{d}t + \mathrm{e}^{rt}\mathrm{e}^{-rt}\,\mathrm{d}\vec{Z}_t = r\vec{U}(t)\,\mathrm{d}t + \mathrm{d}\vec{Z}_t. \tag{1.6}$$

Integrating both sides of (1.6) relative to t, we obtain

$$\vec{U}(t) = \vec{U}(0) + r\int_0^t \vec{U}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}\vec{Z}_s. \tag{1.7}$$

By (1.7) and the fact that $(t, \vec{Z}_t) = (t, c_1 t - \delta_1 S(t), c_2 t - \delta_2 S(t))$ is a three-dimensional Lévy process, following Protter [9, Theorem 32], we know that $\vec{U}(t)$ is a two-dimensional homogeneous strong Markov process.

The rest of this note is organized as follows. In Section 2 we show the integral-differential equations of the Laplace transforms of the joint ruin times $T_{\min}(u_1, u_2)$ and $T_{\max}(u_1, u_2)$. In Section 3 we provide two asymptotic expressions for the finite-time ruin probabilities with respect to the joint ruin times $T_{\max}(u_1, u_2)$ and $T_{\min}(u_1, u_2)$.

2. Integral-differential equation

In this section we establish the integral-differential equations of the Laplace transforms of the joint ruin times $T_{\min}(u_1, u_2)$ and $T_{\max}(u_1, u_2)$.

2.1. The result for $T_{\min}(u_1, u_2)$

In this subsection we consider the joint ruin time $T_{\min}(u_1, u_2)$. For convenience, we denote $T_{\min}(u_1, u_2)$ by $\tau(u_1, u_2)$. Its Laplace transform is defined by

$$\Psi_{\min}(u_1, u_2, s) := \mathbb{E}[\mathrm{e}^{-s\tau(u_1,u_2)}] \quad \text{for } s > 0.$$

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Then

$$0 \le \Psi_{\min}(u_1, u_2, s) \le 1. \tag{2.1}$$

Now we have the following result.

Theorem 2.1. *For $u_1/\delta_1 \le u_2/\delta_2$ and $s > 0$, the function $\Psi_{\min}(\cdot, \cdot, s)$ satisfies the integral-differential equation*

$$\begin{aligned}&\bigg(u_1 + \frac{c_1}{r}\bigg)\frac{\partial \Psi_{\min}}{\partial u_1} + \bigg(u_2 + \frac{c_2}{r}\bigg)\frac{\partial \Psi_{\min}}{\partial u_2} - \frac{\lambda + s}{r}\Psi_{\min}\\ &+ \frac{\lambda}{r}\int_0^{\infty} \Psi_{\min}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) f(z)\,\mathrm{d}z\\ &\quad = 0\end{aligned} \tag{2.2}$$

with boundary condition

$$\Psi_{\min}\bigg(u_1, \frac{\delta_2}{\delta_1}u_1, s\bigg) = \mathbb{E}[\mathrm{e}^{-s\tau_2(\delta_2 u_1/\delta_1)}], \tag{2.3}$$

where $f(z)$ is the probability density function of σ_k and τ_2 is the ruin time of the risk process $U_2(t)$. Furthermore, $\Psi_{\min}$ is the unique solution of (2.2)–(2.3).

Proof. Existence. For any $h > 0$, by considering the occurrence time T_1 of the first claim, we have

$$\mathbb{E}[\mathrm{e}^{-s\tau(u_1,u_2)}] = \mathbb{E}[\mathrm{e}^{-s\tau(u_1,u_2)},\ T_1 > h] + \mathbb{E}[\mathrm{e}^{-s\tau(u_1,u_2)},\ T_1 \le h]. \tag{2.4}$$

For any $t \ge 0$, denote by $\mathcal{F}_t$ the information of the two-dimensional risk process $\{(U_1(s), U_2(s)) \colon s \ge 0\}$ up to time t, and by θ_t the shift operator of the sample path, i.e. $(\theta_t(\omega))_s = \omega_{s+t}$ for any sample path $\omega = (\omega_s,\ s \ge 0)$. By the properties of conditional expectation and the strong Markov property, we have

$$\begin{aligned}&\mathbb{E}[\mathrm{e}^{-s\tau(u_1,u_2)},\ T_1 > h]\\ &\quad = \mathbb{E}[\mathrm{e}^{-s\tau(u_1,u_2)}\mathbf{1}_{\{T_1 > h\}}]\\ &\quad = \mathbb{E}[\mathbb{E}[\mathrm{e}^{-s\tau(u_1,u_2)}\mathbf{1}_{\{T_1 > h\}} \mid \mathcal{F}_h]]\\ &\quad = \mathbb{E}[\mathbf{1}_{\{T_1 > h\}}\mathbb{E}[\mathrm{e}^{-s[h + \tau\circ\theta_h]} \mid \mathcal{F}_h]]\\ &\quad = \mathbb{E}[\mathbf{1}_{\{T_1 > h\}}\mathrm{e}^{-sh}\mathbb{E}_{(U_1(h), U_2(h))}[\mathrm{e}^{-s\tau}]]\\ &\quad = \int_h^{\infty} \mathrm{e}^{-sh}\Psi_{\min}\bigg(\mathrm{e}^{rh}u_1 + \frac{c_1}{r}(\mathrm{e}^{rh} - 1), \mathrm{e}^{rh}u_2 + \frac{c_2}{r}(\mathrm{e}^{rh} - 1), s\bigg)\lambda \mathrm{e}^{-\lambda u}\,\mathrm{d}u\\ &\quad = \mathrm{e}^{-(\lambda + s)h}\Psi_{\min}\bigg(\mathrm{e}^{rh}u_1 + \frac{c_1}{r}(\mathrm{e}^{rh} - 1), \mathrm{e}^{rh}u_2 + \frac{c_2}{r}(\mathrm{e}^{rh} - 1), s\bigg).\end{aligned} \tag{2.5}$$

For the second term on the right-hand side of (2.4), we have

$$\begin{aligned}&\mathbb{E}[\mathrm{e}^{-s\tau(u_1,u_2)},\ T_1 \le h]\\ &= \mathbb{E}\bigg[\mathrm{e}^{-s\tau(u_1,u_2)},\ T_1 \le h,\ \sigma_1 \le \frac{\mathrm{e}^{rT_1}u_1 + c_1 r^{-1}(\mathrm{e}^{rT_1} - 1)}{\delta_1} \wedge \frac{\mathrm{e}^{rT_1}u_2 + c_2 r^{-1}(\mathrm{e}^{rT_1} - 1)}{\delta_2}\bigg]\\ &\quad + \mathbb{E}\bigg[\mathrm{e}^{-s\tau},\ T_1 \le h,\ \sigma_1 > \frac{\mathrm{e}^{rT_1}u_1 + c_1 r^{-1}(\mathrm{e}^{rT_1} - 1)}{\delta_1} \wedge \frac{\mathrm{e}^{rT_1}u_2 + c_2 r^{-1}(\mathrm{e}^{rT_1} - 1)}{\delta_2}\bigg].\end{aligned} \tag{2.6}$$

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By the strong Markov property, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\mathrm{e}^{-s\tau(u_1,u_2)},\ T_1\le h,\ \sigma_1\le \frac{\mathrm{e}^{rT_1}u_1+c_1r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_1}\wedge\frac{\mathrm{e}^{rT_1}u_2+c_2r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_2}\bigg]\\
&\quad=\mathbb{E}\bigg[\mathrm{e}^{-sT_1}\mathbb{E}_{(U_1(T_1),U_2(T_1))}[\mathrm{e}^{-s\tau}],\ T_1\le h,\\
&\qquad\quad \sigma_1\le \frac{\mathrm{e}^{rT_1}u_1+c_1r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_1}\wedge\frac{\mathrm{e}^{rT_1}u_2+c_2r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_2}\bigg]\\
&\quad=\int_0^h\lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t\int_0^{(\mathrm{e}^{rt}u_1+c_1r^{-1}(\mathrm{e}^{rt}-1))/\delta_1\wedge(\mathrm{e}^{rt}u_2+c_2r^{-1}(\mathrm{e}^{rt}-1))/\delta_2}\mathrm{e}^{-st}\\
&\qquad\quad\times\Psi_{\min}\bigg(\mathrm{e}^{rt}u_1+\frac{c_1}{r}(\mathrm{e}^{rt}-1)-\delta_1z,\mathrm{e}^{rt}u_2+\frac{c_2}{r}(\mathrm{e}^{rt}-1)-\delta_2z,s\bigg)f(z)\,\mathrm{d}z.
\end{aligned}\tag{2.7}$$

On the other hand, if

$$\sigma_1>\frac{\mathrm{e}^{rT_1}u_1+c_1r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_1}\wedge\frac{\mathrm{e}^{rT_1}u_2+c_2r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_2}$$

then $\tau(u_1,u_2)=T_1$, and, thus,

$$\begin{aligned}
&\mathbb{E}\bigg[\mathrm{e}^{-s\tau(u_1,u_2)},\ T_1\le h,\ \sigma_1> \frac{\mathrm{e}^{rT_1}u_1+c_1r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_1}\wedge\frac{\mathrm{e}^{rT_1}u_2+c_2r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_2}\bigg]\\
&\quad=\mathbb{E}\bigg[\mathrm{e}^{-sT_1},T_1\le h,\ \sigma_1> \frac{\mathrm{e}^{rT_1}u_1+c_1r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_1}\wedge\frac{\mathrm{e}^{rT_1}u_2+c_2r^{-1}(\mathrm{e}^{rT_1}-1)}{\delta_2}\bigg]\\
&\quad=\int_0^h\lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t\int_{(\mathrm{e}^{rt}u_1+c_1r^{-1}(\mathrm{e}^{rt}-1))/\delta_1\wedge(\mathrm{e}^{rt}u_2+c_2r^{-1}(\mathrm{e}^{rt}-1))/\delta_2}^{\infty}\mathrm{e}^{-st}f(z)\,\mathrm{d}z.
\end{aligned}\tag{2.8}$$

By (2.4)–(2.8), we obtain

$$\begin{aligned}
&\Psi_{\min}(u_1,u_2,s)\\
&\quad=\mathrm{e}^{-(\lambda+s)h}\Psi_{\min}\bigg(\mathrm{e}^{rh}u_1+\frac{c_1}{r}(\mathrm{e}^{rh}-1),\mathrm{e}^{rh}u_2+\frac{c_2}{r}(\mathrm{e}^{rh}-1),s\bigg)\\
&\qquad+\int_0^h\lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t\int_0^{(\mathrm{e}^{rt}u_1+c_1r^{-1}(\mathrm{e}^{rt}-1))/\delta_1\wedge(\mathrm{e}^{rt}u_2+c_2r^{-1}(\mathrm{e}^{rt}-1))/\delta_2}\mathrm{e}^{-st}\\
&\qquad\qquad\times\Psi_{\min}\bigg(\mathrm{e}^{rt}u_1+\frac{c_1}{r}(\mathrm{e}^{rt}-1)-\delta_1z,\mathrm{e}^{rt}u_2+\frac{c_2}{r}(\mathrm{e}^{rt}-1)-\delta_2z,s\bigg)f(z)\,\mathrm{d}z\\
&\qquad+\int_0^h\lambda\mathrm{e}^{-\lambda t}\,\mathrm{d}t\int_{(\mathrm{e}^{rt}u_1+c_1r^{-1}(\mathrm{e}^{rt}-1))/\delta_1\wedge(\mathrm{e}^{rt}u_2+c_2r^{-1}(\mathrm{e}^{rt}-1))/\delta_2}^{\infty}\mathrm{e}^{-st}f(z)\,\mathrm{d}z.
\end{aligned}\tag{2.9}$$

By the definition of $\Psi_{\min}(\cdot,\cdot,\cdot)$, we know that if

$$z>\frac{\mathrm{e}^{rt}u_1+c_1r^{-1}(\mathrm{e}^{rt}-1)}{\delta_1}\wedge\frac{\mathrm{e}^{rt}u_2+c_2r^{-1}(\mathrm{e}^{rt}-1)}{\delta_2}$$

then $\Psi_{\min}(\mathrm{e}^{rt}u_1+c_1r^{-1}(\mathrm{e}^{rt}-1)-\delta_1z,\mathrm{e}^{rt}u_2+c_2r^{-1}(\mathrm{e}^{rt}-1)-\delta_2z,s)=1$. By virtue of this fact and letting $y:=\mathrm{e}^{rh}-1$, $q_1:=u_1+c_1r^{-1}$, and $q_2:=u_2+c_2r^{-1}$, we can rewrite

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(2.9) as

$$
\begin{aligned}
&\Psi_{\min}(u_1, u_2, s) \\
&\quad = e^{-(\lambda+s)h}\Psi_{\min}(u_1 + q_1 y, u_2 + q_2 y, s) \\
&\qquad + \int_0^h \lambda e^{-\lambda t}\, dt \int_0^\infty e^{-st} \\
&\qquad\qquad \times \Psi_{\min}\bigg(e^{rt}u_1 + \frac{c_1}{r}(e^{rt} - 1) - \delta_1 z, e^{rt}u_2 + \frac{c_2}{r}(e^{rt} - 1) - \delta_2 z, s\bigg) f(z)\, dz.
\end{aligned} \tag{2.10}
$$

It is easy to check that $y \uparrow 0$ if and only if $h \downarrow 0$. Hence, by (2.10), we have

$$
\lim_{y\uparrow 0} \Psi_{\min}(u_1 + q_1 y, u_2 + q_2 y, s) = \Psi_{\min}(u_1, u_2, s). \tag{2.11}
$$

By (2.10), for any $h > 0$ and $y = e^{rh} - 1$, we have

$$
\begin{aligned}
0 &= \frac{\Psi_{\min}(u_1 + q_1 y, u_2 + q_2 y, s) - \Psi_{\min}(u_1, u_2, s)}{y} \\
&\quad + \frac{e^{-(\lambda+s)h} - 1}{y}\Psi_{\min}(u_1 + q_1 y, u_2 + q_2 y, s) \\
&\quad + \frac{1}{y}\int_0^h \lambda e^{-\lambda t}\, dt \int_0^\infty e^{-st} \\
&\qquad \times \Psi_{\min}\bigg(e^{rt}u_1 + \frac{c_1}{r}(e^{rt} - 1) - \delta_1 z, e^{rt}u_2 + \frac{c_2}{r}(e^{rt} - 1) - \delta_2 z, s\bigg) f(z)\, dz \\
&= \frac{\Psi_{\min}(u_1 + q_1 y, u_2 + q_2 y, s) - \Psi_{\min}(u_1, u_2, s)}{y} \\
&\quad + \frac{e^{-(\lambda+s)h} - 1}{e^{rh} - 1}\Psi_{\min}(u_1 + q_1 y, u_2 + q_2 y, s) \\
&\quad + \frac{1}{e^{rh} - 1}\int_0^h \lambda e^{-\lambda t}\, dt \int_0^\infty e^{-st} \\
&\qquad \times \Psi_{\min}\bigg(e^{rt}u_1 + \frac{c_1}{r}(e^{rt} - 1) - \delta_1 z, e^{rt}u_2 + \frac{c_2}{r}(e^{rt} - 1) - \delta_2 z, s\bigg) f(z)\, dz.
\end{aligned}
$$

By (2.11), letting $y \uparrow 0$ and $h \downarrow 0$ in the above formula and noting that (2.1) assures the interchange of the limit and integration, we obtain

$$
q_1 \frac{\partial \Psi_{\min}}{\partial u_1} + q_2 \frac{\partial \Psi_{\min}}{\partial u_2} - \frac{\lambda + s}{r}\Psi_{\min} + \frac{\lambda}{r}\int_0^\infty \Psi_{\min}(u_1 - \delta_1 z, u_2 - \delta_2 z, s) f(z)\, dz = 0. \tag{2.12}
$$

Replacing q_1 and q_2 in (2.12) by $u_1 + c_1 r^{-1}$ and $u_2 + c_2 r^{-1}$, respectively, we obtain the integral-differential equation. When $u_1/\delta_1 = u_2/\delta_2$, the joint ruin model degenerates into a univariate model, and then, by the analysis in [2], we obtain the boundary condition.

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Uniqueness. By using similar arguments as in [6] and noting (2.10), we define an operator $\mathcal{T}$ by

$$\mathcal{T}g(u_1,u_2,s) = \mathrm{e}^{-(\lambda+s)h} g\Big(u_1+q_1y, u_2+q_2y, s\Big) + \int_0^h \lambda \mathrm{e}^{-\lambda t}\,\mathrm{d}t \int_0^\infty \mathrm{e}^{-st} g\Big(\mathrm{e}^{rt}u_1 + \frac{c_1}{r}(\mathrm{e}^{rt}-1) - \delta_1 z, \mathrm{e}^{rt}u_2 + \frac{c_2}{r}(\mathrm{e}^{rt}-1) - \delta_2 z, s\Big) f(z)\,\mathrm{d}z$$

for any $h > 0$. It can be easily seen that $\Psi_{\min}$ is a fixed point of the operator $\mathcal{T}$, as $\mathcal{T}\Psi_{\min} = \Psi_{\min}$. Also, for two different functions g_1 and g_2, we have, for any $h > 0$ and $s > 0$,

$$\begin{aligned}
&|\mathcal{T}g_1 - \mathcal{T}g_2| \\
&\quad \le \mathrm{e}^{-(\lambda+s)h}|g_1(u_1+q_1y, u_2+q_2y, s) - g_2(u_1+q_1y, u_2+q_2y, s)| \\
&\qquad + \int_0^h \lambda \mathrm{e}^{-\lambda t}\,\mathrm{d}t \\
&\qquad \times \int_0^\infty \mathrm{e}^{-st}\Big|g_1\Big(\mathrm{e}^{rt}u_1 + \frac{c_1}{r}(\mathrm{e}^{rt}-1) - \delta_1 z, \mathrm{e}^{rt}u_2 + \frac{c_2}{r}(\mathrm{e}^{rt}-1) - \delta_2 z, s\Big) \\
&\qquad\qquad - g_2\Big(\mathrm{e}^{rt}u_1 + \frac{c_1}{r}(\mathrm{e}^{rt}-1) - \delta_1 z, \mathrm{e}^{rt}u_2 + \frac{c_2}{r}(\mathrm{e}^{rt}-1) - \delta_2 z, s\Big)\Big| f(z)\,\mathrm{d}z \\
&\quad \le \mathrm{e}^{-(\lambda+s)h}||g_1 - g_2||_\infty + \Big(\int_0^h \lambda \mathrm{e}^{-(\lambda+s)t}\,\mathrm{d}t\Big)||g_1 - g_2||_\infty \\
&\quad = \frac{\lambda + s\mathrm{e}^{-(\lambda+s)h}}{\lambda+s}||g_1 - g_2||_\infty,
\end{aligned}$$

where $||\cdot||_\infty$ is the supremum norm over $(u_1,u_2) \in \mathbb{R}^2$. Therefore, $\mathcal{T}$ is a contraction and, by Banach's fixed point theorem and (2.1), the solution of (2.2)–(2.3) is unique.

Remark 2.1. One way to obtain the Laplace transform $\Psi_{\min}$ of the joint ruin probability $T_{\min}(u_1,u_2)$ is to solve the above integral-differential equation (2.2)–(2.3) numerically. The following natural question arises.

- Can we give an analytical representation for the solution to (2.2)–(2.3) in some special cases such as exponential claim sizes?

Unfortunately, even in the case of exponential claim sizes, we have not found a way to solve (2.2)–(2.3).

2.2. The result for $T_{\max}(u_1,u_2)$

Define the Laplace transform of $T_{\max}(u_1,u_2)$ by

$$\Psi_{\max}(u_1,u_2,s) := \mathbb{E}[\mathrm{e}^{-sT_{\max}(u_1,u_2)}] \quad \text{for } s > 0.$$

Then we have the following result.

Theorem 2.2. *For $u_1/\delta_1 \le u_2/\delta_2$ and $s > 0$, the function $\Psi_{\max}(\cdot,\cdot,s)$ satisfies the same integral-differential equation (2.2) with boundary condition*

$$\Psi_{\max}\Big(u_1, \frac{\delta_2}{\delta_1}u_1, s\Big) = \mathbb{E}[\mathrm{e}^{-s\tau_1(u_1)}], \tag{2.13}$$

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where $f(z)$ is the probability density function of σ_k and τ_1 is the ruin time of the risk process $U_1(t)$. Furthermore, $\Psi_{\max}$ is the unique solution of (2.2) and (2.13).

Proof. The proof is almost the same as that of Theorem 2.1; we need only note the following three things.

1. In this case, (2.6) becomes

$$\begin{aligned}
&\mathbb{E}[e^{-s\tau(u_1,u_2)},\ T_1 \le h] \\
&= \mathbb{E}\bigg[e^{-s\tau(u_1,u_2)},\ T_1 \le h, \\
&\qquad \sigma_1 \le \frac{e^{rT_1}u_1 + c_1 r^{-1}(e^{rT_1}-1)}{\delta_1} \vee \frac{e^{rT_1}u_2 c_2 r^{-1}(e^{rT_1}-1)}{\delta_2}\bigg] \\
&\quad + \mathbb{E}\bigg[e^{-s\tau},\ T_1 \le h, \\
&\qquad \sigma_1 > \frac{e^{rT_1}u_1 + c_1 r^{-1}(e^{rT_1}-1)}{\delta_1} \vee \frac{e^{rT_1}u_2 + c_2 r^{-1}(e^{rT_1}-1)}{\delta_2}\bigg],
\end{aligned}$$

   where $\tau(u_1, u_2)$ stands for $T_{\max}(u_1, u_2)$.

2. $\tau(u_1, u_2) = T_1$ if $\sigma_1 > (e^{rT_1}u_1 + c_1 r^{-1}(e^{rT_1}-1))/\delta_1 \vee (e^{rT_1}u_2 + c_2 r^{-1}(e^{rT_1}-1))/\delta_2$.
3. If $z > (e^{rt}u_1 + c_1 r^{-1}(e^{rt}-1))/\delta_1 \vee (e^{rt}u_2 + c_2 r^{-1}(e^{rt}-1))/\delta_2$ then

$$\Psi_{\max}\left(e^{rt}u_1 + \frac{c_1}{r}(e^{rt}-1) - \delta_1 z, e^{rt}u_2 + \frac{c_2}{r}(e^{rt}-1) - \delta_2 z, s\right) = 1.$$

We omit the details.

3. Asymptotics for finite-time ruin probabilities

In this section we consider the finite-time ruin probability associated with $T_{\max}(u_1, u_2)$ and $T_{\min}(u_1, u_2)$. The original idea comes from [7, Section 4].

Define $X_i(t) := e^{-rt}U_i(t)/\delta_i$, $i = 1, 2$. Then $(X_1(t), X_2(t))$ has the same ruin times and probabilities with $(U_1(t), U_2(t))$. Define $x_i := u_i/\delta_i$ and $p_i := c_i/r\delta_i$, $i = 1, 2$. Then, by (1.4) and our assumptions, we have

$$X_i(t) = x_i + p_i(1 - e^{-rt}) - \sum_{k=1}^{N(t)} e^{-r\theta_k}\sigma_k, \qquad i = 1, 2,$$

where $p_1 > p_2$ and $x_1 \le x_2$.

For $T > 0$, define $\Psi_{\max}(x_1, x_2, T) := \mathbb{P}\{T_{\max}(\delta_1 x_1, \delta_2 x_2) \le T\}$. Then we have

$$\Psi_{\max}(x_1, x_2, T) = \mathbb{P}\{\text{there exists } t \le T \text{ such that } X_1(t) < 0 \text{ and } X_2(t) < 0\}. \qquad (3.1)$$

Alternatively, we can also define $\Psi_{\min}(x_1, x_2, T) := \mathbb{P}\{T_{\min}(\delta_1 x_1, \delta_2 x_2) \le T\}$ and get

$$\Psi_{\min}(x_1, x_2, T) = \mathbb{P}\{\text{there exists } t \le T \text{ such that } X_1(t) < 0 \text{ or } X_2(t) < 0\}. \qquad (3.2)$$

In the following, we will provide asymptotic results on both $\Psi_{\max}(x_1, x_2, T)$ and $\Psi_{\min}(x_1, x_2, T)$ under some condition.

3.1. Asymptotic result for $T_{\max}(u_1, u_2)$

Let $T > 0, n \in \mathbb{N}$, and $\{V_k,\ k = 1, 2, \dots, n\}$ be a sequence of i.i.d. random variables with the uniform distribution on $(0, T]$. Denote by $(V_1^*, \dots, V_n^*)$ the ordered statistic of $(V_1, \dots, V_n)$. It is well known that, conditioning on $\{N(t) = n\}$, the random vectors $(\theta_1, \dots, \theta_n)$ and $(V_1^*, \dots, V_n^*)$ have the same distribution. Assume that $\{V_k,\ k = 1, 2, \dots, n\}$ is independent of $\{\sigma_k,\ k \geq 1\}$. Define $F_T(x) = \mathbb{P}\{\mathrm{e}^{-rV_1}\sigma_1 \leq x\}$. Then we have

$$\begin{aligned}\mathbb{P}\bigg\{\sum_{k=1}^{n} \mathrm{e}^{-r\theta_k}\sigma_k > x \ \bigg|\ N(T) = n\bigg\} &= \mathbb{P}\bigg\{\sum_{k=1}^{n} \mathrm{e}^{-rV_k^*}\sigma_k > x\bigg\} \\ &= \mathbb{P}\bigg\{\sum_{k=1}^{n} \mathrm{e}^{-rV_k}\sigma_k > x\bigg\} \\ &= \overline{F_T{}^{*n}}(x),\end{aligned} \tag{3.3}$$

where $F_T{}^{*n}(x)$ stands for the n-multiple convolution of $F_T(x)$.

Theorem 3.1. *If σ_k has a regularly varying tail with $\mathbb{P}\{\sigma_k > x\} = L(x)/x^{\alpha}$, where L is continuous and slowly varying, $\lim_{x\to\infty} L(x) = \infty$, and $\alpha > 0$, then, for any $T > 0$, we have*

$$\lim_{x_2 \geq x_1 \to \infty} \frac{\Psi_{\max}(x_1, x_2, T)}{\lambda T\, \overline{F_T}(x_2)} = 1. \tag{3.4}$$

Before proving Theorem 3.1, we need the following lemma.

Lemma 3.1. *Suppose that σ_k satisfies the condition in Theorem 3.1. Then F_T has a regularly varying tail.*

Proof. By the independence of V_1 and σ_1, we have

$$\begin{aligned}\overline{F_T}(x) &= \mathbb{P}\{\mathrm{e}^{-rV_1}\sigma_1 > x\} \\ &= \int_0^T \mathbb{P}\{\mathrm{e}^{-ry}\sigma_1 > x\}\frac{1}{T}\,\mathrm{d}y \\ &= \frac{1}{T}\int_0^T \frac{L(\mathrm{e}^{ry}x)}{(\mathrm{e}^{ry}x)^{\alpha}}\,\mathrm{d}y \\ &:= \frac{S(x)}{x^{\alpha}},\end{aligned}$$

where

$$S(x) = \frac{1}{T}\int_0^T \frac{L(\mathrm{e}^{ry}x)}{(\mathrm{e}^{ry})^{\alpha}}\,\mathrm{d}y,$$

which together with the assumption that L is continuous and $\lim_{x\to\infty} L(x) = \infty$ implies that

$$\lim_{x\to\infty} S(x) = \infty. \tag{3.5}$$

By the change of variable, we obtain

$$S(x) = \frac{x^{\alpha}}{rT}\int_x^{\mathrm{e}^{rT}x} \frac{L(u)}{u^{\alpha+1}}\,\mathrm{d}u. \tag{3.6}$$

For any $t > 0$, by (3.5), (3.6), and the fact that L is a slowing varying function, we obtain

$$\begin{aligned}\lim_{x\to\infty}\frac{S(tx)}{S(x)} &= \lim_{x\to\infty}\frac{t^{\alpha}\int_{tx}^{\mathrm{e}^{rT}tx}(L(u)/u^{\alpha+1})\,\mathrm{d}u}{\int_{x}^{\mathrm{e}^{rT}x}(L(u)/u^{\alpha+1})\,\mathrm{d}u}\\ &= \lim_{x\to\infty}\frac{t^{\alpha}(L(\mathrm{e}^{rT}tx)\mathrm{e}^{rT}t/(\mathrm{e}^{rT}tx)^{\alpha+1} - L(tx)t/(tx)^{\alpha+1})}{L(\mathrm{e}^{rT}x)\mathrm{e}^{rT}/(\mathrm{e}^{rT}x)^{\alpha+1} - L(x)/x^{\alpha+1}}\\ &= \lim_{x\to\infty}\frac{L(\mathrm{e}^{rT}tx)/(\mathrm{e}^{rT})^{\alpha} - L(tx)}{L(\mathrm{e}^{rT}x)/(\mathrm{e}^{rT})^{\alpha} - L(x)}\\ &= \lim_{x\to\infty}\frac{L(\mathrm{e}^{rT}tx)/L(\mathrm{e}^{rT}x) - (\mathrm{e}^{rT})^{\alpha}L(tx)/L(\mathrm{e}^{rT}x)}{1 - (\mathrm{e}^{rT})^{\alpha}L(x)/L(\mathrm{e}^{rT}x)}\\ &= \frac{1-(\mathrm{e}^{rT})^{\alpha}}{1-(\mathrm{e}^{rT})^{\alpha}}\\ &= 1.\end{aligned}$$

Hence, F_T has a regularly varying tail.

Proof of Theorem 3.1. By Lemma 3.1 and [1, Proposition IX.1.4], we know that F_T is a subexponential distribution. By (3.1) and (3.3), we have

$$\begin{aligned}&\Psi_{\max}(x_1, x_2, T)\\ &\quad= \mathbb{P}\Bigg\{\sum_{k=1}^{N(t)}\mathrm{e}^{-r\theta_k}\sigma_k > x_i + p_i(1-\mathrm{e}^{-rt}),\ i=1,2,\ \text{for some } t\le T\Bigg\}\\ &\quad\ge \mathbb{P}\Bigg\{\sum_{k=1}^{N(T)}\mathrm{e}^{-r\theta_k}\sigma_k > x_i + p_i(1-\mathrm{e}^{-rT}),\ i=1,2\Bigg\}\\ &\quad= \sum_{n=0}^{\infty}\mathbb{P}\{N(T)=n\}\\ &\qquad\times\mathbb{P}\Bigg\{\sum_{k=1}^{n}\mathrm{e}^{-r\theta_k}\sigma_k > x_i + p_i(1-\mathrm{e}^{-rT}),\ i=1,2\ \Bigg|\ N(T)=n\Bigg\}.\end{aligned}\tag{3.7}$$

If $x_1 + p_1(1-\mathrm{e}^{-rT}) \ge x_2 + p_2(1-\mathrm{e}^{-rT})$ then, by (3.3) and the assumption that $x_2 \ge x_1$, we obtain

$$\begin{aligned}&\mathbb{P}\Bigg\{\sum_{k=1}^{n}\mathrm{e}^{-r\theta_k}\sigma_k > x_i + p_i(1-\mathrm{e}^{-rT}),\ i=1,2\ \Bigg|\ N(T)=n\Bigg\}\\ &\qquad= \mathbb{P}\Bigg\{\sum_{k=1}^{n}\mathrm{e}^{-r\theta_k}\sigma_k > x_1 + p_1(1-\mathrm{e}^{-rT})\ \Bigg|\ N(T)=n\Bigg\}\\ &\qquad= \overline{F_T}^{*n}(x_1 + p_1(1-\mathrm{e}^{-rT})),\end{aligned}\tag{3.8}$$

and $x_2 + p_1(1-\mathrm{e}^{-rT}) \ge x_1 + p_1(1-\mathrm{e}^{-rT}) \ge x_2 + p_2(1-\mathrm{e}^{-rT}) > x_2$, which implies that

$$\overline{F_T}^{*n}(x_2 + p_1(1-\mathrm{e}^{-rT})) \le \overline{F_T}^{*n}(x_1 + p_1(1-\mathrm{e}^{-rT})) \le \overline{F_T}^{*n}(x_2),$$

and, thus,

$$\frac{\overline{F_T{}^{*n}}(x_2 + p_1(1 - \mathrm{e}^{-rT}))}{\overline{F_T{}^{*n}}(x_2)} \le \frac{\overline{F_T{}^{*n}}(x_1 + p_1(1 - \mathrm{e}^{-rT}))}{\overline{F_T{}^{*n}}(x_2)} \le 1. \tag{3.9}$$

Since F_T is a subexponential distribution, by [1, Proposition IX.1.5] and (3.9), it holds that

$$\liminf_{x_1\to\infty} \frac{\overline{F_T{}^{*n}}(x_1 + p_1(1 - \mathrm{e}^{-rT}))}{\overline{F_T{}^{*n}}(x_2)} = 1. \tag{3.10}$$

By Fatou's lemma, (3.10), and [1, Proposition IX.1.7], we have

$$\begin{aligned}
&\liminf_{x_1\to\infty} \frac{\sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\}\overline{F_T{}^{*n}}(x_1 + p_1(1 - \mathrm{e}^{-rT}))}{\lambda T\overline{F_T}(x_2)} \\
&\quad = \liminf_{x_1\to\infty} \sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\} \frac{\overline{F_T{}^{*n}}(x_1 + p_1(1 - \mathrm{e}^{-rT}))}{\overline{F_T{}^{*n}}(x_2)} \frac{\overline{F_T{}^{*n}}(x_2)}{\lambda T\overline{F_T}(x_2)} \\
&\quad \ge \frac{1}{\lambda T} \sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\} \liminf_{x_1\to\infty} \frac{\overline{F_T{}^{*n}}(x_1 + p_1(1 - \mathrm{e}^{-rT}))}{\overline{F_T{}^{*n}}(x_2)} \liminf_{x_1\to\infty} \frac{\overline{F_T{}^{*n}}(x_2)}{\overline{F_T}(x_2)} \\
&\quad = \frac{1}{\lambda T} \sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\} \liminf_{x_1\to\infty} \frac{\overline{F_T{}^{*n}}(x_2)}{\overline{F_T}(x_2)} \\
&\quad = \frac{1}{\lambda T} \sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\} n \\
&\quad = \frac{1}{\lambda T} \mathbb{E}[N(t)] \\
&\quad = 1.
\end{aligned} \tag{3.11}$$

By (3.7), (3.8), and (3.11), and under the condition that $x_1 + p_1(1 - \mathrm{e}^{-rT}) \ge x_2 + p_2(1 - \mathrm{e}^{-rT})$, we have

$$\liminf_{x_1\to\infty} \frac{\Psi_{\max}(x_1, x_2, T)}{\lambda T\overline{F_T}(x_2)} \ge 1.$$

If $x_1 + p_1(1 - \mathrm{e}^{-rT}) < x_2 + p_2(1 - \mathrm{e}^{-rT})$ then

$$\begin{aligned}
&\mathbb{P}\bigg[\sum_{k=1}^{n} \mathrm{e}^{-r\theta_k}\sigma_k > x_i + p_i(1 - \mathrm{e}^{-rT}),\ i = 1, 2 \;\bigg|\; N(T) = n\bigg] \\
&\qquad = \mathbb{P}\bigg[\sum_{k=1}^{n} \mathrm{e}^{-r\theta_k}\sigma_k > x_2 + p_2(1 - \mathrm{e}^{-rT}) \;\bigg|\; N(T) = n\bigg] \\
&\qquad = \overline{F_T{}^{*n}}(x_2 + p_2(1 - \mathrm{e}^{-rT})).
\end{aligned} \tag{3.12}$$

Since F_T is a subexponential distribution and $x_2 \ge x_1$, by [1, Proposition IX.1.5] we have

$$\lim_{x_1\to\infty} \frac{\overline{F_T{}^{*n}}(x_2 + p_2(1 - \mathrm{e}^{-rT}))}{\overline{F_T{}^{*n}}(x_2)} = 1. \tag{3.13}$$

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Now, by (3.7), (3.12), and (3.13), similar to the arguments in (3.11), we find that, under the condition that $x_1 + p_1(1 - \mathrm{e}^{-rT}) < x_2 + p_2(1 - \mathrm{e}^{-rT})$,

$$\liminf_{x_1\to\infty} \frac{\Psi_{\max}(x_1, x_2, T)}{\lambda T \overline{F_T}(x_2)} \geq 1.$$

Hence, we always have

$$\liminf_{x_1\to\infty} \frac{\Psi_{\max}(x_1, x_2, T)}{\lambda T \overline{F_T}(x_2)} \geq 1. \tag{3.14}$$

On the other hand, by the assumption that $x_2 \geq x_1$, and (3.3), we have

$$\begin{aligned}
&\Psi_{\max}(x_1, x_2, T)\\
&\quad= \mathbb{P}\bigg\{\sum_{k=1}^{N(t)} \mathrm{e}^{-r\theta_k}\sigma_k > x_i + p_i(1 - \mathrm{e}^{-rt}),\ i = 1, 2,\ \text{for some } t \leq T\bigg\}\\
&\quad\leq \mathbb{P}\bigg\{\sum_{k=1}^{N(T)} \mathrm{e}^{-r\theta_k}\sigma_k > x_2\bigg\}\\
&\quad= \sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\}\mathbb{P}\bigg\{\sum_{k=1}^{n} \mathrm{e}^{-r\theta_k}\sigma_k > x_2 \ \bigg|\ N(T) = n\bigg\}\\
&\quad= \sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\}\overline{F_T{}^{*n}}(x_2).
\end{aligned}$$

By Fatou's lemma, the above formula, and [1, Proposition IX.1.7], we have

$$\begin{aligned}
\limsup_{x_1\to\infty} \frac{\Psi_{max}(x_1, x_2, T)}{\lambda T \overline{F_T}(x_2)} &\leq \limsup_{x_1\to\infty} \frac{\sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\}\overline{F_T{}^{*n}}(x_2)}{\lambda T \overline{F_T}(x_2)}\\
&\leq \frac{1}{\lambda T}\sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\} \limsup_{x_1\to\infty} \frac{\overline{F_T{}^{*n}}(x_2)}{\overline{F_T}(x_2)}\\
&\leq \frac{1}{\lambda T}\sum_{n=0}^{\infty} \mathbb{P}\{N(T) = n\}n\\
&= \frac{1}{\lambda T}\mathbb{E}[N(t)]\\
&= 1.
\end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that (3.4) holds.

3.2. Asymptotic result for $T_{\min}(u_1, u_2)$

By Theorem 3.1 we can easily obtain the asymptotic result for $\Psi_{\min}(x_1, x_2, T)$, which is formulated as follows.

Theorem 3.2. *If σ_k has a regularly varying tail with $\mathbb{P}\{\sigma_k > x\} = L(x)/x^{\alpha}$, where L is continuous and slowly varying, $\lim_{x\to\infty} L(x) = \infty$, and $\alpha > 0$, then, for any $T > 0$, we have*

$$\lim_{x_2\geq x_1\to\infty} \frac{\Psi_{\min}(x_1, x_2, T)}{\lambda T \overline{F_T}(x_1)} = 1.$$

Proof. First, for $i = 1, 2$, define

$$\psi_i(x_i, T) = \mathbb{P}\{\text{there exists } t \le T \text{ such that } X_i(t) < 0\},$$

i.e. $\psi_i(x_i, T),\ i = 1, 2$ represents the ruin probability of $X_i(t),\ i = 1, 2$ within finite time T.

Note the fact that

$$\begin{aligned}
&\mathbb{P}\{\text{there exists } t \le T \text{ such that } X_1(t) < 0 \text{ and } X_2(t) < 0\} \\
&\quad = \mathbb{P}\{\text{there exists } t \le T \text{ such that } X_1(t) < 0\} \\
&\qquad + \mathbb{P}\{\text{there exists } t \le T \text{ such that } X_2(t) < 0\} \\
&\qquad - \mathbb{P}\{\text{there exists } t \le T \text{ such that } X_1(t) < 0 \text{ or } X_2(t) < 0\}.
\end{aligned}$$

Then, by (3.1) and (3.2), we have

$$\Psi_{\max}(x_1, x_2, T) = \psi_1(x_1, T) + \psi_2(x_2, T) - \Psi_{\min}(x_1, x_2, T). \tag{3.16}$$

By Lemma 3.1, F_T is a subexponential distribution. Then, by [1, Proposition IX.1.5], for $i = 1, 2$, we have

$$\lim_{x_2 \ge x_1 \to \infty} \frac{\psi_i(x_i, T)}{\lambda T \overline{F_T}(x_i)} = 1. \tag{3.17}$$

By (3.16), (3.17), (3.4), and the fact that $x_2 \ge x_1$, we obtain

$$\begin{aligned}
&\left| \frac{\Psi_{\min}(x_1, x_2, T)}{\lambda T \overline{F_T}(x_1)} - 1 \right| \\
&\quad = \left| \frac{\psi_1(x_1, T) - \lambda T \overline{F_T}(x_1) + \psi_2(x_2, T) - \Psi_{\max}(x_1, x_2, T)}{\lambda T \overline{F_T}(x_1)} \right| \\
&\quad \le \left| \frac{\psi_1(x_1, T) - \lambda T \overline{F_T}(x_1)}{\lambda T \overline{F_T}(x_1)} \right| + \left| \frac{\psi_2(x_2, T) - \Psi_{\max}(x_1, x_2, T)}{\lambda T \overline{F_T}(x_2)} \right| \left| \frac{\overline{F_T}(x_2)}{\overline{F_T}(x_1)} \right| \\
&\quad \le \left| \frac{\psi_1(x_1, T) - \lambda T \overline{F_T}(x_1)}{\lambda T \overline{F_T}(x_1)} \right| + \left| \frac{\psi_2(x_2, T) - \Psi_{\max}(x_1, x_2, T)}{\lambda T \overline{F_T}(x_2)} \right| \\
&\quad \to 0 \quad \text{as } x_2 \ge x_1 \to \infty.
\end{aligned}$$

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