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ON CERTAIN FUNCTIONAL IDENTITIES IN E^N

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1. If f is a real-valued function possessing a Taylor series convergent in $(a - R, a + R)$, then it satisfies the following operational identity

$$f(a + r) + f(a - r) = 2\{\cosh rD f(u)\}_{u=a} \qquad (r < R) \tag{1.1}$$

in which $D^2 \equiv d^2/du^2$. Furthermore, when g is a solution of $y'' + \lambda^2 y = 0$ in $(a - R, a + R)$, then g is such a function and (1.1) specializes to

$$g(a + r) + g(a - r) = 2 \cos \lambda r\, g(a) \qquad (r < R). \tag{1.2}$$

In this note we generalize these results to the real Euclidean space E^N, our conclusions being Theorems 1 and 2 below. Clearly, (1.2) is a special case of (1.1) but in higher-dimensional space it is of interest to allow g, now a solution of

$$\nabla^2 y + \lambda^2 y = 0 \qquad (N \geqq 2) \tag{1.3}$$

to possess singularities at isolated points away from the origin. It is then necessary to consider not only a neighbourhood of the origin but annular regions also. The generalization of (1.2) for these annular regions cannot be deduced from the generalization of (1.1).

Of our two theorems, probably the second is the more interesting and many results can easily be deduced from it. For example, a special case of the case $\lambda = 0$, $N = 2$ is Jensen's theorem on the distribution of the zeros of an integral function.

We follow closely the notation of [**1**] and no loss of generality ensues if we take $a = 0$ in (1.1) and (1.2) or $\mathbf{a} = \mathbf{0}$ below.

2. Let $I_\nu(z)$ be the modified Bessel function of the first kind and write $G_\nu(z) \equiv \Gamma(\nu + 1)(2/z)^\nu I_\nu(z)$. $G_\nu(z)$ is a function of z^2 and we write $G_\nu(r\nabla)$ for the corresponding function of the operator ∇^2. Let S_r^{N-1} or simply $S(r)$ denote both the $(N - 1)$-dimensional manifold $||\mathbf{x}|| = r$ and its volume. The volume element in this manifold will be written as σ_r. We suppose that an orthonormal basis has been taken in E^N so that $\mathbf{x}$ has coordinates x_k $(k = 1, 2, \ldots, N)$ relative to it.

THEOREM 1. *Let $f(\mathbf{x})$ possess a Taylor series in the coordinates of $\mathbf{x}$, convergent when $||\mathbf{x}|| < R$. Then writing $\nu \equiv \frac{1}{2}N - 1$ we have*

$$\frac{1}{S(r)} \int_{S(r)} f(\mathbf{x})\sigma_r = \{G_\nu(r\nabla) f(\mathbf{u})\}_{\mathbf{u}=\mathbf{0}} \qquad (r < R). \tag{2.1}$$

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If $N \geqq 3$, then a solution of $\nabla^2 y = 0$ for $r = ||\mathbf{x}|| > 0$ is r^{2-N}. If for some $\epsilon > 0$ and some constant K the function g has the property that

$$g(\mathbf{x}) - K||\mathbf{x} - \mathbf{b}||^{2-N}$$

possesses continuous first partial derivatives in $||\mathbf{b}|| - \epsilon < ||\mathbf{x}|| < ||\mathbf{b}|| + \epsilon$, we shall say that g has a "harmonic singularity at $\mathbf{b}$ of strength K". If $N = 2$, we make a similar definition with $||\mathbf{x} - \mathbf{b}||^{2-N}$ replaced by $\log||\mathbf{x} - \mathbf{b}||$.

The following theorem allows g to possess such singularities at points $\mathbf{b}_m$ $(m = 1, 2, 3, \ldots)$. Writing b_m for $||\mathbf{b}_m||$ we suppose that

$$0 < b_1 < b_2 < b_3 < \ldots.$$

This last restriction could easily be relaxed. We shall find it convenient to write $\mathbf{b}_0 = \mathbf{0}$ but we shall always suppose that the origin is not a singularity. We also write ν for $\frac{1}{2}N - 1$ throughout.

THEOREM 2. *Let g be a solution of* (1.3) *except at the points* $\mathbf{b}_m$ $(m = 1, 2, 3, \ldots)$, *where g has harmonic singularities of strengths K_m. Then if $b_m < r < b_{m+1}$, we have*

$$\frac{1}{S(r)} \int_{S(r)} g(\mathbf{x})\sigma_r = \Gamma(\nu + 1)\left(\frac{2}{\lambda r}\right)^\nu J_\nu(\lambda r) g(\mathbf{0}) + \beta\pi \sum_{k=1}^{m} K_k\{J_\nu(\lambda r) Y_\nu(\lambda b_k) - J_\nu(\lambda b_k) Y_\nu(\lambda r)\} r^{-2\nu}.$$

Here, $\beta = \nu$ if $N \geqq 3$ and $\beta = -\frac{1}{2}$ if $N = 2$. In the case when $\lambda = 0$, the above right-hand side is to be replaced by

$$g(\mathbf{0}) + \sum_{k=1}^{m} K_k(r^{-2\nu} - b_k^{-2\nu}) \qquad (N \geqq 3)$$

or

$$g(\mathbf{0}) + \sum_{k=1}^{m} K_k(\log r - \log b_k) \qquad (N = 2).$$

It can be easily seen that the result for $\lambda = 0$ is the formal limit as $\lambda \to 0$ of the main result.

3. To prove Theorem 1, the following lemmas are needed.

LEMMA 1. *If $P_n(\mathbf{x})$ is a homogeneous polynomial of degree n in the coordinates of $\mathbf{x} \in E^N$, then*

$$\int_{S(r)} P_n(\mathbf{x})\sigma_r = \frac{N + n}{r} \int_{V(r)} P_n(\mathbf{x})\omega,$$

where $V(r)$ is the ball $||\mathbf{x}|| \leqq r$ with volume element ω.

Proof. $P_n(\mathbf{x})\sigma_r$ is an $N - 1$ form. In fact,

$$\sigma_r = \sum_{i=1}^{N} (-1)^{i-1} \frac{x_i}{r} dx_1 dx_2 \ldots \widehat{dx_i} \ldots dx_N.$$

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Hence by Stokes' theorem in E^N, we obtain

$$\int_{\dot{S}(r)} P_n\sigma_r = \int_{\partial V(r)} P_n\sigma_r = \int_{V(r)} d(P_n\sigma_r).$$

However, $d(P_n\sigma_r) = dP_n\sigma_r + P_n d\sigma_r$. The latter term is $(N/r)P_n\omega$ whilst the former reads

$$\left\{\sum_{i=1}^{N} \frac{\partial P_n}{\partial x_i} dx_i\right\}\left\{\sum_{i=1}^{N} (-1)^{i-1}\frac{x_i}{r} dx_1 dx_2 \ldots \widehat{dx_i} \ldots dx_N\right\}$$
$$= \left\{\sum_{i=1}^{N} x_i \frac{\partial P_n}{\partial x_i}\right\}\frac{\omega}{r} = \frac{n}{r} P_n\omega$$

by the homogeneity of P_n, so that the lemma is proved.

LEMMA 2. *If ∇ is the gradient operator in E^N, write $Q_n(\mathbf{x}) \equiv \{(\mathbf{x}\cdot\nabla)^n f(\mathbf{u})\}_{\mathbf{u}=\mathbf{0}}$. Then*

$$\int_{V(r)} Q_n(\mathbf{x})\omega = \begin{cases} 0 & (n \text{ odd}), \\ \sum_{\alpha} B_\alpha \int_{V(r)} \prod_{i=1}^{N} x_i^{2\alpha_i}\omega & (n \text{ even}). \end{cases}$$

Here the summation is taken over $1 \leqq \alpha_i \leqq m$: $\sum_{i=1}^N \alpha_i = m$, where $m = \frac{1}{2}n$ $(n = 0, 2, 4, \ldots)$ and B_α denotes the expression

$$(2m)!\left\{\prod_{i=1}^{N} (2\alpha_i)!\right\}^{-1}\left\{\frac{\partial^{2m} f(\boldsymbol{u})}{\partial u_1^{2\alpha_1} \ldots\ldots \partial u_N^{2\alpha_N}}\right\}_{\mathbf{u}=\mathbf{0}}.$$

Proof. Q_n is a homogeneous polynomial of degree n in the coordinates of $\mathbf{x}$ in which a typical term is

$$n!\left\{\prod_{i=1}^{N} (h_i)!\right\}^{-1}\prod_{i=1}^{N} x_i^{h_i}\left\{\frac{\partial^n f(\boldsymbol{u})}{\partial u_1^{h_1} \ldots\ldots \partial u_N^{h_N}}\right\}_{\mathbf{u}=\mathbf{0}},$$

where $\sum_{i=1}^N h_i = n$. The integral of $\prod_{i=1}^N x_i^{h_i}$ taken over $V(r)$ is zero unless each h_i is even. Thus, if n is odd, the result follows. Otherwise, write $h_i = 2\alpha_i$, $n = 2m$, and the other result follows.

LEMMA 3.

$$\int_{V(r)} \prod_{i=1}^{N} x_i^{2\alpha_i}\omega = r^{N+2m}\frac{\prod_{i=1}^{N}\Gamma(\alpha_i + \frac{1}{2})}{\Gamma(m + \frac{1}{2}N + 1)}.$$

Proof. The proof is quite straightforward and therefore we do not give it here. We merely remark that by symmetry,

$$\int_{V(r)} = 2^N \int_{V^+(r)},$$

where $V^+(r)$ is that part of $V(r)$ with $x_i \geqq 0$ $(i = 1, 2, \ldots, N)$. The substitution $r^2 y_k = x_k^2$ $(k = 1, 2, \ldots, N)$ then reduces the integral to one over the simplex whose vertices are the origin and the end points of the base vectors.

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Proof of Theorem 1. Since f possesses a Taylor series in $\|\mathbf{x}\| < R$, we have $f(\mathbf{x}) = \sum_{n=0}^{\infty} (1/n!)Q_n(\mathbf{x})$ there. If $0 < r < R$, the convergence is uniform on $S(r)$ and so

$$\int_{S(r)} f(\mathbf{x})\sigma_r = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{S(r)} Q_n(\mathbf{x})\sigma_r = A \quad \text{(say)} \qquad (r < R).$$

Q_n is a homogeneous polynomial in the coordinates of $\mathbf{x}$ and so by Lemma 1,

$$A = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{N+n}{r} \int_{V(r)} Q_n \omega.$$

Then by Lemmas 2 and 3,

$$A = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \frac{N+2m}{r} P(N, m, \alpha), \tag{3.1}$$

where

$$P(N, m, \alpha) = \frac{(2m)!}{\Gamma(m + \frac{1}{2}N + 1)} r^{N+2m} \sum_{\alpha} \left\{ \prod_{i=1}^{N} \frac{\Gamma(\alpha_i + \frac{1}{2})}{(2\alpha_i)!} \right\} \times \left\{ \frac{\partial^{2m} f(\mathbf{u})}{\partial u_1^{2\alpha_1} \ldots \partial u_N^{2\alpha_N}} \right\}_{\mathbf{u}=\mathbf{0}}$$

Using the reduplication formula for the Γ-function, this becomes

$$P(N, m, \alpha) = \frac{(2m)! r^{N+2m} \pi^{\frac{1}{2}N}}{m! 2^{2m} \Gamma(m + \frac{1}{2}N + 1)} \left\{ \left(\frac{\partial^2}{\partial u_1^2} + \ldots + \frac{\partial^2}{\partial u_N^2} \right)^m f(\mathbf{u}) \right\}_{\mathbf{u}=\mathbf{0}},$$

and so (3.1) reads

$$A = \sum_{m=0}^{\infty} \frac{2r^{N-1}\pi^{\frac{1}{2}N}}{\Gamma(m+1)\Gamma(m + \frac{1}{2}N)} \left\{ \left(\frac{r^2 \nabla^2}{2^2} \right)^m f(\mathbf{u}) \right\}_{\mathbf{u}=\mathbf{0}}.$$

With $f \equiv 1$ we obtain the familiar result $S(r) = 2r^{N-1}\pi^{\frac{1}{2}N}/\Gamma(\frac{1}{2}N)$ and on dividing by this we obtain the result stated.

4. For $r < b_1$ the function g of Theorem 2 satisfies the hypotheses of Theorem 1 and we obtain the result of Theorem 2 for $r < b_1$ since, by (1.3), the right side of (2.1) now reads

$$\{G_\nu(ri\lambda)g(\mathbf{u})\}_{\mathbf{u}=\mathbf{0}} = \begin{cases} \Gamma(\nu + 1)\left(\dfrac{2}{\lambda r}\right)^{\nu} J_\nu(\lambda r) g(\mathbf{0}) & (\lambda \neq 0), \\ g(\mathbf{0}) & (\lambda = 0). \end{cases}$$

To complete the proof of Theorem 2, the lemmas of the present section are needed and we shall employ the notation

$$V(r) = \frac{1}{S(r)} \int_{S(r)} g(\mathbf{x})\sigma_r, \qquad W(r) = r^{1-N} S(r) V(r),$$

so that $2\pi^{\frac{1}{2}N} V(r) \equiv \Gamma(\frac{1}{2}N) W(r)$.

LEMMA 4. *If* $\sum_{k=1}^{N} x_k^2 = 1$, *then*

$$x_k \sum_{i=1}^{N} (-1)^{i-1} x_i dx_1 dx_2 \ldots \widehat{dx_i} \ldots dx_N = (-1)^{k-1} dx_1 dx_2 \ldots \widehat{dx_k} \ldots dx_N \qquad (k = 1, 2, \ldots, N). \tag{4.1}$$

Proof. Since $\sum_{k=1}^{N} x_k^2 = 1$, we have

$$x_k dx_k = -\sum_{i \neq k} x_i dx_i. \tag{4.2}$$

The left-hand side of (4.1) can be written as

$$x_k dx_k \sum_{i<k} (-1)^{i+k-1} x_i dx_1 \ldots \widehat{dx_i} \ldots \widehat{dx_k} \ldots dx_N + x_k^2 (-1)^{k-1} dx_1 dx_2 \ldots \widehat{dx_k} \ldots dx_N + x_k dx_k \sum_{i>k} (-1)^{i+k} x_i dx_1 \ldots \widehat{dx_k} \ldots \widehat{dx_i} \ldots dx_N.$$

Substitute for $x_k dx_k$ from (4.2) and this reads

$$\sum_{i<k} (-1)^{k-1} x_i^2 dx_1 \ldots \widehat{dx_k} \ldots dx_N + (-1)^{k-1} x_k^2 dx_1 \ldots \widehat{dx_k} \ldots dx_N + \sum_{i>k} (-1)^{k-1} x_i^2 dx_1 \ldots \widehat{dx_k} \ldots dx_N = (-1)^{k-1} dx_1 \ldots \widehat{dx_k} \ldots dx_N$$

since $\sum_{i=1}^{N} x_i^2 = 1$. This completes the proof of the lemma.

LEMMA 5. *In the open interval* (b_m, b_{m+1}) *we have*

$$\frac{d}{dr} [r^{N-1} V'(r)] + \lambda^2 r^{N-1} V(r) = 0. \tag{4.3}$$

Proof. It suffices to prove that $W(r)$ satisfies (4.3). Now

$$W(r) = r^{1-N} \int_{S(r)} g(\mathbf{x}) \sigma_r = \int_{S(1)} g(r\mathbf{x}) \sigma_1.$$

Hence

$$W'(r) = \int_{S(1)} \left\{ \sum_{k=1}^{N} g_k x_k \right\} \sigma_1, \quad \text{where } g_k \equiv \left\{ \frac{\partial g(\mathbf{u})}{\partial u_k} \right\}_{\mathbf{u}=r\mathbf{x}}.$$

Now on $S(1)$ we have $\sum_{k=1}^{N} x_k^2 = 1$ and so writing σ_1 in full and using Lemma 4 we obtain

$$W'(r) = \int_{S(1)} \sum_{k=1}^{N} g_k (-1)^{k-1} dx_1 dx_2 \ldots \widehat{dx_k} \ldots dx_N$$

so that

$$W'(r) = r^{1-N} \int_{S(r)} (*dg).$$

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Now let c be chosen so that $b_m < c < r < b_{m+1}$ and we have

$$r^{N-1}W'(r) = \int_{S(r)} - \int_{S(c)} + \int_{S(c)} (*dg)$$
$$= \int_{\partial\{c \leqq ||\mathbf{x}|| \leqq r\}} (*dg) + \text{const.}$$

Hence by Stokes' theorem and the fact that $d(*dg) = (\nabla^2 g)\omega$ we have

$$r^{N-1}W'(r) = \int_{\{c \leqq ||\mathbf{x}|| \leqq r\}} (\nabla^2 g)\omega + \text{const.}$$

Since g satisfies (1.3) we have

$$r^{N-1}W'(r) = -\lambda^2 \int_{\{c \leqq ||\mathbf{x}|| \leqq r\}} g\omega + \text{const.} \tag{4.4}$$

Now writing $\rho^2 = \sum_{k=1}^{N} x_k^2$ we find that

$$\omega = d\rho\ (*d\rho) = d\rho\ \sigma_\rho.$$

The integral in (4.4) can be written as a repeated integral, its value being independent of the order in which the integration is carried out. Thus we write

$$\int_{\{c \leqq ||\mathbf{x}|| \leqq r\}} g\omega = \int_c^r \int_{S(\rho)} g\sigma_\rho\, d\rho$$

so that differentiating (4.4) we obtain

$$\frac{d}{dr}\left[r^{N-1}\frac{dW}{dr}\right] = -\lambda^2 \int_{S(r)} g\sigma_r = -\lambda^2 r^{N-1} W,$$

and the lemma is proved.

LEMMA 6. *Let* $\mathbf{b}$ *be any one of the singularities of g and let its weight be K. Then writing* $||\mathbf{b}|| = b$ *we have:*

(i) $V(r)$ *is continuous at* $r = b$,

(ii) $V'(b + 0) - V'(b - 0)$ *is equal to* $(2 - N)b^{1-N} K$ *for* $N \geqq 3$ *and equal to* $b^{-1}K$ *for* $N = 2$.

Proof. (i) By our definition of "harmonic singularity" in § 2, it will suffice, in case $N \geqq 3$, to show that

$$\int_{S(r)} \frac{1}{||\mathbf{x} - \mathbf{b}||^{N-2}} \sigma_r$$

is continuous at $r = b$. A similar proof deals with the case $N = 2$. Write T_ϵ for the manifold $||\mathbf{x} - \mathbf{b}|| = \epsilon$. Let $U(r)$ be the set of points $\mathbf{x}$ satisfying both $||\mathbf{x}|| \leqq r$ and $||\mathbf{x} - \mathbf{b}|| > \epsilon$. Then, if $r < b - \epsilon$, $U(r)$ is the union of the interior of $S(r)$ and $S(r)$, whilst if $r > b - \epsilon$, it is the set common to the closure

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of the interior of $S(r)$ and the exterior of T_ϵ. By Stokes' theorem,

$$\int_{S(r)} \frac{1}{||\mathbf{x}-\mathbf{b}||^{N-2}}\sigma_r = \int_{U(r)} d\left\{\frac{1}{||\mathbf{x}-\mathbf{b}||^{N-2}}\sigma_r\right\} + k_\epsilon(r), \tag{4.5}$$

where

$$k_\epsilon(r) = \begin{cases} 0 & \text{if } r < b - \epsilon, \\ \int_{T_\epsilon} ||\mathbf{x}-\mathbf{b}||^{2-N}\mu_\epsilon & \text{if } r > b + \epsilon, \end{cases}$$

in which μ_ϵ is the volume element in the manifold T_ϵ. In the case

$$b - \epsilon \leqq r \leqq b + \epsilon$$

the description of $k_\epsilon(r)$ is somewhat complicated to write out, but this is unnecessary for the proof. The integral on the right of (4.5) is continuous at $r = b$ for any fixed ϵ and so it is enough to show that as $\epsilon \to 0$ we have

$$\int_{T_\epsilon} ||\mathbf{x}-\mathbf{b}||^{2-N}\mu_\epsilon \to 0.$$

However, the value of this integral is $2\pi^{\frac{1}{2}N}\epsilon/\Gamma(\frac{1}{2}N)$ and so the proof of (i) is complete.

(ii) From the proof of Lemma 5 we have

$$r^{N-1}W'(r) = \int_{S(r)} (*dg) = G(r) \quad \text{(say)}$$

and it will suffice to show that

$$\lim_{\epsilon\to 0+} \{G(b+\epsilon) - G(b-\epsilon)\} = \begin{cases} (4-2N)K\pi^{\frac{1}{2}N}/\Gamma(\frac{1}{2}N) & (N \geqq 3), \\ 2\pi K & (N = 2). \end{cases}$$

Again we will confine ourselves to the case $N \geqq 3$. Let T_ϵ have the same meaning as before and let P_ϵ be the set of $\mathbf{x}$ satisfying both

$$b - \epsilon \leqq ||\mathbf{x}|| \leqq b + \epsilon \quad \text{and} \quad ||\mathbf{x}-\mathbf{b}|| > \epsilon.$$

Then by our definition of "harmonic singularity" we can write

$$G(r) = \int_{S(r)} (*dh) + \int_{S(r)} *d\left(\frac{K}{||\mathbf{x}-\mathbf{b}||^{N-2}}\right)$$

when r is sufficiently close to b and where h has continuous first partial derivatives in $b - \epsilon < r < b + \epsilon$. Clearly the first integral here is continuous at $r = b$ and so

$$G(b+\epsilon) - G(b-\epsilon) = \int_{S(b+\epsilon)} - \int_{S(b-\epsilon)} *d\left(\frac{K}{||\mathbf{x}-\mathbf{b}||^{N-2}}\right).$$

Applying Stokes' theorem we have

$$G(b+\epsilon) - G(b-\epsilon) = \int_{P_\epsilon} d\left[*d\left(\frac{K}{||\mathbf{x}-\mathbf{b}||^{N-2}}\right)\right] + \int_{T_\epsilon} *d\left(\frac{K}{||\mathbf{x}-\mathbf{b}||^{N-2}}\right).$$

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Here the former integral is zero since $||\mathbf{x} - \mathbf{b}||^{N-2}$ is harmonic in P_ϵ, and it remains to evaluate the latter. Its value is clearly the same as that of

$$K \int_{||\mathbf{x}||=\epsilon} {}^*d(||\mathbf{x}||^{2-N}) = K \int_{\rho=\epsilon} {}^*d(\rho^{2-N})$$

and this equals

$$K \int_{\rho=\epsilon} \frac{(2-N)}{\rho^{N-1}} \sigma_\rho$$

since $\rho^2 = \sum_{k=1}^N x_k^2$. Hence $G(b+\epsilon) - G(b-\epsilon) = K(2-N)2\pi^{\frac{1}{2}N}/\Gamma(\frac{1}{2}N)$, and the proof of the lemma is complete.

Proof of Theorem 2. This is simply a matter of collecting the results of the lemmas above. As already stated, Theorem 1 provides the result for $b_0 \leqq r < b_1$. Then in $b_m < r < b_{m+1}$ the function $V(r)$ satisfies (4.3). Hence we have

$$V(r) = \begin{cases} A_m r^{-\nu} J_\nu(\lambda r) + B_m r^{-\nu} Y_\nu(\lambda r) & (N \geqq 2,\ \lambda \neq 0), \\ A_m + B_m r^{-2\nu} & (N \geqq 3,\ \lambda = 0), \\ A_m + B_m \log r & (N = 2,\ \lambda = 0). \end{cases}$$

In any of these cases, A_0 and B_0 are found by using Theorem 1, and then A_m, B_m $(m = 1, 2, 3, \ldots)$ can be calculated successively using the two results of Lemma 6.

5. We conclude by mentioning a few special cases of the above results. Theorem 2 is probably the more interesting in this respect but from Theorem 1 we obtain, for example, the results

(a) $$f(\mathbf{a}) = \frac{1}{S(r)} \int_{S(r)} f(\mathbf{a} + \mathbf{x})\sigma_r \quad \text{if } f \text{ is harmonic;}$$

this is Gauss' Mean Value Theorem.

(b) $$\frac{1}{S(r)} \int_{S(r)} f(\mathbf{a} + \mathbf{x})\sigma_r = f(\mathbf{a}) + \frac{r^2}{2N} \quad \text{if } \nabla^2 f = 1,$$

(c) $$\nabla^2 f(\mathbf{a}) = \lim_{r\to 0} \frac{2N}{r^2 S(r)} \int_{S(r)} \{f(\mathbf{a} + \mathbf{x}) - f(\mathbf{a})\}\sigma_r.$$

Turning to Theorem 2 we have:

(d) Take

$$g(\mathbf{x}) = \prod_{k=1}^N \cos(\lambda_k x_k)$$

so that $\nabla^2 g + \lambda^2 g = 0$, where $\lambda^2 = \sum_{k=1}^N \lambda_k^2$. It follows that

$$\frac{1}{S(r)} \int_{S(r)} \left\{\prod_{k=1}^N \cos(\lambda_k x_k)\right\} \sigma_r = \Gamma(\tfrac{1}{2}N)\left(\frac{2}{\lambda r}\right)^\nu J_\nu(\lambda r) \qquad (\nu = \tfrac{1}{2}N - 1).$$

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When $N = 2$, the above equation becomes

$$\frac{1}{2\pi}\int_0^{2\pi} \cos ax \cos by \, d\theta = J_0((a^2 + b^2)^{\frac{1}{2}} r).$$

Multiplying this by r and integrating from 0 to r we obtain a formula used in the two-dimensional lattice-point problem [**2**].

(e) Suppose that in the case $N = 2$, $G(\mathbf{x})$ is a function of $||\mathbf{x}||$ only. Then $g(\mathbf{x}) = G(\mathbf{a} + \mathbf{x}) = G\{a^2 + r^2 - 2ar\cos(\theta - \alpha)\}^{\frac{1}{2}})$, where (r, θ) and (a, α) are the polar coordinates of $\mathbf{x}$ and $\mathbf{a}$. Suppose that $G(\mathbf{x})$ satisfies (1.3). Then so does $g(\mathbf{x})$. $G(\mathbf{x})$ is a linear combination of $J_0(\lambda r)$ and $Y_0(\lambda r)$, and taking each of these separately we obtain

$$\text{(i)}\quad \frac{1}{2\pi}\int_0^{2\pi} J_0(\lambda\{r^2 + a^2 - 2ar\cos\phi\}^{\frac{1}{2}})\, d\phi = J_0(\lambda r) J_0(\lambda a) \quad (r \geqq 0),$$

$$\text{(ii)}\quad \frac{1}{2\pi}\int_0^{2\pi} Y_0(\lambda\{r^2 + a^2 - 2ar\cos\phi\}^{\frac{1}{2}})\, d\phi$$

$$= \begin{cases} J_0(\lambda r) Y_0(\lambda a) & (0 \leqq r < a), \\ Y_0(\lambda r) J_0(\lambda a) & (r > a). \end{cases}$$

The two alternatives in case (ii) arise since $g(\mathbf{x}) = Y_0(\lambda||\mathbf{a} + \mathbf{x}||)$ has a harmonic singularity at $\mathbf{x} = -\mathbf{a}$ of strength $2/\pi$.

(f) Finally, consider the case of Theorem 2 in which $N = 2$ and $\lambda = 0$. Let $F(z)$ be an integral function with zeros at z_k $(k = 1, 2, 3, \ldots)$ where $0 < |z_1| < |z_2| < \ldots$ of multiplicities n_k. Take $g(\mathbf{x}) = \log|F(z)|$, where $z = x + iy$. g is then harmonic with harmonic singularities at (x_k, y_k) of strengths n_k. Thus we obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \log|F(z)|\, d\theta = \log|F(0)| + \sum_{k=1}^{m} \log\left(\frac{r}{r_k}\right)^{nk} \qquad (r_m < r < r_{m+1})$$

which is Jensen's theorem.

Finally, we refer to (c) above to mention that the operator on the right there is a higher-dimensional form of Riemann's generalized second derivative. Suppose that $g(\mathbf{x})$, and hence $g(\mathbf{a} + \mathbf{x})$, is a solution of (1.3) and that no singularities are present. Then Theorem 2 yields.

$$(5.1)\quad \lim_{r \to 0+} \frac{2N}{r^2 S(r)} \int_{S(r)} \{g(\mathbf{a} + \mathbf{x}) - g(\mathbf{a})\}\sigma_r$$

$$= \lim_{r \to 0+} \frac{2N}{r^2}\left\{\Gamma(\nu + 1)\left(\frac{2}{\lambda r}\right)^{\nu} J_\nu(\lambda r) - 1\right\} g(\mathbf{a}) = -\lambda^2 g(\mathbf{a}).$$

This is a generalization of the result that when $g'' + \lambda^2 g = 0$, we have

$$\lim_{h \to 0+} \frac{1}{h^2}\{g(a + h) - 2g(a) + g(a - h)\} = \lim_{h \to 0+} \frac{2}{h^2}\{\cos\lambda h - 1\} g(a) = -\lambda^2 g(a),$$

which is fundamental in Riemann's investigation of the uniqueness of trigonometrical series. The case $N = 2$ of (5.1) has been used in a similar way in [**3**] and one would expect (5.1), generally, to be of use in studying the uniqueness of eigenfunction expansions arising from the equation

$$\nabla^2 y + \lambda^2 y = 0.$$

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