Canad. J. Math. Vol. **51** (6), 1999 pp. 1194–1225

Subregular Nilpotent Elements and Bases in K-Theory

Dedicated to Professor H. S. M. Coxeter

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Abstract. In this paper we describe a canonical basis for the equivariant K-theory (with respect to a $\mathbf{C}^*$-action) of the variety of Borel subalgebras containing a subregular nilpotent element of a simple complex Lie algebra of type D or E.

Introduction

Let e be a nilpotent element in a semisimple Lie algebra $\mathfrak{g}$ over $\mathbf{C}$. Let $\mathcal{B}_e$ be the variety of all Borel subalgebras of $\mathfrak{g}$ that contain e. This variety has a very complicated geometry which is of great interest for representation theory. For example, the ordinary cohomology of $\mathcal{B}_e$ carries representations of the Weyl groups (Springer) which enter in the character theory of reductive groups over a finite field; on the other hand, the equivariant K-theory $K_H(\mathcal{B}_e)$ of $\mathcal{B}_e$ (with respect to a torus H acting on $\mathcal{B}_e$ and maximal in a suitable sense) carries a representation of an affine Hecke algebra which enters in the representation theory of reductive groups over a p-adic field.

It is known [S] that $\mathcal{B}_e$ lies naturally inside a smooth variety Λ_e of twice its dimension, with the same homotopy type as $\mathcal{B}_e$.

In [L4], [L5] I gave a conjectural definition of a canonical (signed) basis $\mathbf{B}^{\pm}_{\mathcal{B}_e}$ of $K_H(\mathcal{B}_e)$ and one, $\mathbf{B}^{\pm}_{\Lambda_e}$, of $K_H(\Lambda_e)$, as modules over the representation ring $R_{\mathbf{C}^*}$. This conjectural definition is trivially correct in the case where e is regular; as shown in [L4], it is also correct in the case where $e = 0$ and in the case where e is subregular in type D_4.

In this paper we show that the conjectural definition of $\mathbf{B}^{\pm}_{\mathcal{B}_e}$, $\mathbf{B}^{\pm}_{\Lambda_e}$ is correct in the case where e is subregular in type $D_n (n \geq 5)$ or E_6, E_7, E_8. (Here we have $H = \mathbf{C}^*$.) In these cases it turns out that $\mathbf{B}^{\pm}_{\mathcal{B}_e}$ is just $\pm$ the canonical basis of the reflection representation of the affine Hecke algebra considered in [L1]. On the other hand, it turns out that $\mathbf{B}^{\pm}_{\Lambda_e}$, which in some definite sense, is dual to $\mathbf{B}^{\pm}_{\mathcal{B}_e}$, consists of certain natural vector bundles on Λ_e. These vector bundles can be considered as examples of the "tautological vector bundles" on quiver varieties (Nakajima [N1]), via Kronheimer's realization [Kr] of Λ_e, and seem to be also related to the vector bundles considered by Gonzales-Sprinberg and Verdier [GV].

This leads us to the following question (for not necessarily subregular e): can one represent any element in the conjectural signed basis $\mathbf{B}^{\pm}_{\Lambda_e}$ as $\pm$ a vector bundle on Λ_e?

Received by the editors October 22, 1998; revised June 25, 1998.
Supported by the Ambrose Monnel Foundation and the National Science Foundation.
AMS subject classification: 20G99.

1 Preliminaries on Hilbert Schemes

1.1

Let Γ be a finite group. Let $\mathcal{C}_\Gamma$ be the category whose objects are **C**-vector spaces with a given linear Γ-action and such that the space of morphisms from M to M' is the set $\mathrm{Hom}_\Gamma(M, M')$ of linear maps from M to M' compatible with the Γ-action. Let $\mathcal{C}^0_\Gamma$ be the full subcategory of $\mathcal{C}_\Gamma$ whose objects are finite dimensional over **C**. For $M, M' \in \mathcal{C}^0_\Gamma$ we set $(M, M')_\Gamma = \dim \mathrm{Hom}_\Gamma(M, M')$.

1.2

Let T be a two-dimensional **C**-vector space with a given non-singular symplectic form $\langle , \rangle \colon T \times T \to \mathbf{C}$. For $r \in \mathbf{N}$ let $T^r = T \otimes T \otimes \cdots \otimes T$ (r factors) and let S^r be the r-th symmetric power of T regarded as a quotient of T^r. Let $S^\dagger = \bigoplus_{r \in \mathbf{N}} S^r$ be the symmetric algebra of T (a quotient of the tensor algebra $T^\dagger = \bigoplus_{r \in \mathbf{N}} T^r$). Let T' be the dual space of T.

1.3

Assume now that Γ is a finite subgroup $\neq \{1\}$ of the symplectic group $\mathrm{Sp}(T)$. Then Γ acts naturally on $T^\dagger, S^\dagger$ preserving each subspace T^r, S^r.

Let $\tilde{I}$ be the set of isomorphism classes of irreducible Γ-modules over **C**. For each $i \in \tilde{I}$ we assume given a simple Γ-module ρ_i in the class i. Following McKay [M], we regard $\tilde{I}$ as the set of vertices of a graph in which $i \neq i' \in \tilde{I}$ are joined by

$$(\rho_i \otimes T, \rho_{i'})_\Gamma = (\rho_{i'} \otimes T, \rho_i)_\Gamma$$

edges. (The number above will be denoted by $-i \cdot i'$; we also set $i \cdot i = 2$.) This graph is an *affine Coxeter graph*.

Let $\heartsuit \in \tilde{I}$ be the class containing the unit representation **C** of Γ. Let $I = \tilde{I} - \{\heartsuit\}$. We regard I as the set of vertices of a full subgraph of the affine Coxeter graph; this is called the *Coxeter graph*.

The quiver varieties attached by Nakajima [N1] to the affine Coxeter graph can be also described directly in terms of objects of $\mathcal{C}^0_\Gamma$ as follows.

Let M, M' be objects of $\mathcal{C}^0_\Gamma$. Let $\Lambda^s_{M,M'}$ be the set of all triples (x, p, q) where x is a $T^\dagger$-algebra structure on M' compatible with the natural Γ-action, $p \in \mathrm{Hom}_\Gamma(M, M'), q \in \mathrm{Hom}_\Gamma(M', M)$ and the following hold:

(a) if e, e' is any basis of T such that $\langle e, e' \rangle = 1$, then $e \otimes e' - e' \otimes e \in T^2$ acts on M' as the map pq;

(b) $p(M)$ generates M' as a $T^\dagger$-module.

Let $\Lambda^{sn}_{M,M'}$ be the set of all triples $(x, p, q) \in \Lambda^s_{M,M'}$ such that $q = 0$ and such that, for the $T^\dagger$-module structure defined by x, there exists $r_0 \geq 1$ such that T^r acts on M' as zero for all $r \geq r_0$.

1.4

Let $G_{M'}$ be the group of automorphisms of the Γ-module M'. Then $G_{M'}$ acts naturally on $\Lambda^s_{M,M'}$ leaving stable the subset $\Lambda^{sn}_{M,M'}$, and these actions are free. Nakajima [N1] shows that

(a) $G_{M'} \setminus \Lambda^s_{M,M'}$ *is naturally a smooth variety of pure dimension*

$$(M', M' \otimes T)_\Gamma - 2(M', M')_\Gamma + 2(M, M')_\Gamma$$

and with trivial canonical bundle.

On the other hand, as a consequence of [L2, 12.3]:

(b) $G_{M'} \setminus \Lambda^{sn}_{M,M'}$ *is naturally a projective variety of pure dimension*

$$\frac{1}{2}(M', M' \otimes T)_\Gamma - (M', M')_\Gamma + (M, M')_\Gamma.$$

1.5

For an integer $r \geq 1$, let $T'^{[r]}$ be the set of all ideals J of $S^\dagger$ of codimension r. This is naturally an algebraic variety, the Hilbert scheme of r points on T'. Let $\mathrm{Sym}^r(T')$ be the r-th symmetric product of T', that is, the quotient of the r-fold product $T' \times T' \times \cdots \times T'$ by the natural action of the symmetric group $\mathfrak{S}_r$. Let $\pi \colon T'^{[r]} \to \mathrm{Sym}^r(T')$ be the canonical (Hilbert-Chow) morphism. The fibre $T'^{[r]}_0 = \pi(0, 0, \ldots, 0)$ is the subvariety of $T'^{[r]}$ consisting of the ideals $J \in T'^{[r]}$ such that $S^{r'} \subset J$ for large enough r'.

For $M' \in \mathcal{C}^0_\Gamma$, we denote by $\mathbf{H}^{M'}$ the set of all ideals J in $S^\dagger$ which are Γ-submodules such that $S^\dagger/J \cong M'$ in $\mathcal{C}_\Gamma$. Note that $\mathbf{H}^{M'}$ is a closed subvariety of the Hilbert scheme $T'^{[\dim M']}$. Let $\mathbf{H}^{M'}_0 = \mathbf{H}^{M'} \cap T'^{[\dim M']}_0$, that is, the set of all ideals J in $S^\dagger$ which are Γ-submodules such that $S^\dagger/J \cong M'$ in $\mathcal{C}_\Gamma$ and such that J contains S^r for large enough r. (A closed subvariety of $\mathbf{H}^{M'}$.)

1.6

Assume now that $M = \mathbf{C}$ (the unit representation of Γ). If $(x, p, q) \in \Lambda^s_{\mathbf{C},M'}$, then we have automatically $q = 0$. Indeed, applying [N2, Proposition 2.7] to (x, p, q) (with the Γ-module structures ignored), we see that $q = 0$ on the $T^\dagger$-submodule M'_1 of M' generated by $p(\mathbf{C})$. But $M'_1 = M'$ by 1.3(b). Hence $q = 0$.

We now apply [L3, 6.14] (which simplifies due to the previous paragraph) and we see that there is a natural isomorphism

$$G_{M'} \setminus \Lambda^s_{\mathbf{C},M'} \xrightarrow{\sim} \mathbf{H}^{M'}.$$

Similarly, applying [L3, 6.15] we see that there is a natural isomorphism

$$G_{M'} \setminus \Lambda^{sn}_{\mathbf{C},M'} \xrightarrow{\sim} \mathbf{H}^{M'}_0.$$

From 1.4(a), (b) we deduce:

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(a) $\mathbf{H}^{M'}$ *is naturally a smooth variety of pure dimension*

$$(M', M' \otimes T)_\Gamma - 2(M', M')_\Gamma + 2(\mathbf{C}, M')_\Gamma$$

and with trivial canonical bundle;

(b) $\mathbf{H}_0^{M'}$ *is naturally a projective variety of pure dimension*

$$\frac{1}{2}(M', M' \otimes T)_\Gamma - (M', M')_\Gamma + (\mathbf{C}, M')_\Gamma.$$

In the remainder of this section, let $M' = [\Gamma]$ be the regular representation of Γ. We have $[\Gamma] \otimes T \cong [\Gamma] \oplus [\Gamma]$ in $\mathcal{C}_\Gamma$ and $(\mathbf{C}, [\Gamma])_\Gamma = 1$. Hence

(c) $\mathbf{H}^{[\Gamma]}$ *is a smooth variety of pure dimension* 2 *and with trivial canonical bundle;* $\mathbf{H}_0^{[\Gamma]}$ *is a projective subvariety of* $\mathbf{H}^{[\Gamma]}$ *of pure dimension* 1.

1.7

Let $r = |\Gamma|$. Let $\left(\mathrm{Sym}^r(T')\right)^\Gamma$ be the fixed point set of the natural Γ-action on $\mathrm{Sym}^r(T')$. Note that the obvious map

$$\Gamma \setminus T' \longrightarrow \left(\mathrm{Sym}^r(T')\right)^\Gamma$$

is an isomorphism. (We use the fact that Γ acts freely on $T' - \{0\}$.)

Ito and Nakamura [IN] have proved that

(a) *The map* $\mathbf{H}^{[\Gamma]} \to \left(\mathrm{Sym}^r(T')\right)^\Gamma = \Gamma \setminus T'$ *(restriction of* π*) is a minimal resolution of singularities of* $\Gamma \setminus T'$.

We sketch a proof. It is easy to see that our map restricts to an isomorphism $\mathbf{H}^{[\Gamma]} - \mathbf{H}_0^{[\Gamma]} \to \Gamma \setminus (T' - \{0\})$. Since $\mathbf{H}^{[\Gamma]}$ is smooth of pure dimension 2 and the fibre at 0, that is $\mathbf{H}_0^{[\Gamma]}$, is of pure dimension 1 (see 1.6), it follows that $\mathbf{H}^{[\Gamma]} - \mathbf{H}_0^{[\Gamma]}$ is dense in $\mathbf{H}^{[\Gamma]}$. Hence our map is a resolution of singularities of $\Gamma \setminus T'$. This resolution is minimal since $\mathbf{H}^{[\Gamma]}$ has trivial canonical bundle. (a) follows.

1.8

From now on we assume that Γ is not cyclic. Let $(S^r)^\Gamma$ be the space of Γ-invariants in S^r and let $(S^\dagger)^\Gamma$ be the algebra of Γ-invariants in $S^\dagger$. Then $(S^\dagger)^\Gamma = \bigoplus_r (S^r)^\Gamma$ is generated as an algebra by three elements P_1, P_2, P_3 with $P_j \in S^{r_u}$ for $u = 1, 2, 3$ where $0 < r_1 \leq r_2 < r_3$. Moreover, the vector spaces $\mathbf{C}P_1 + \mathbf{C}P_2$ and $\mathbf{C}P_3$ are independent of the choice of P_1, P_2, P_3, that is, they are canonically attached to Γ. Also, r_1, r_2, r_3 are canonically attached to Γ; we have $r_1 r_2 = 2|\Gamma|$, $r_1 + r_2 = r_3 - 2$ and $h' = r_3/2$ is an integer equal to half of the *Coxeter number* of the Coxeter graph. (We have $h' = n - 1$ in type D_n and $h' = 6, 9, 15$ in type E_6, E_7, E_8 respectively.)

Let $\tilde{\Gamma}$ be the set of all $g \in GL(T)$ such that g acts trivially on $\mathbf{C}P_1 + \mathbf{C}P_2$ and acts by multiplication by ± 1 on $\mathbf{C}P_3$. It is known that $\tilde{\Gamma}$ is a subgroup of $GL(T)$ containing Γ with index 2 and that $\tilde{\Gamma}$ is generated by the (complex) reflections of order 2 in T that it contains. Now $\tilde{\Gamma}$ acts naturally on $S^\dagger$ by algebra automorphisms. Let $(S^r)^{\tilde{\Gamma}}$ be the space of

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$\tilde{\Gamma}$-invariants on S^r. Let $\mathcal{I}$ be the ideal in $S^\dagger$ generated by $\bigoplus_{r>0}(S^r)^{\tilde{\Gamma}}$. We have an induced action of $\tilde{\Gamma}$ on the algebra $\tilde{S} = S^\dagger/\mathcal{I}$ which, by a theorem of Chevalley, is isomorphic in $\mathcal{C}_{\tilde{\Gamma}}$ to the regular representation of $\tilde{\Gamma}$. By restricting to Γ, we see that $\tilde{S} \cong [\Gamma] \oplus [\Gamma]$ in $\mathcal{C}_\Gamma$.

Let $\tilde{\mathbf{H}}_0$ be the set of all ideals $\tilde{J}$ of $\tilde{S}$ such that $\tilde{J}$ is a Γ-submodule and $\tilde{S}/\tilde{J} \cong [\Gamma]$ in $\mathcal{C}_\Gamma$.

(a) *We have an isomorphism* $\tilde{\mathbf{H}}_0 \xrightarrow{\sim} \mathbf{H}_0^{[\Gamma]}$.

(It attaches to $\tilde{J}$ the inverse image of $\tilde{J}$ under the canonical map $S^\dagger \to \tilde{S}$.)

We shall only verify that the map (a) is an isomorphism at the level of sets. It suffices to show that

(b) *any ideal* J in $\mathbf{H}_0^{[\Gamma]}$ *must contain* $\mathcal{I}$.

Let $J \in \mathbf{H}_0^{[\Gamma]}$. Let $P \in S^r$ be a Γ-invariant element with $r > 0$. Assume that $P \notin J$. We show that

(c) *the* Γ*-linear map* $\mathbf{C} \oplus \mathbf{C} \to S^\dagger/J$ *given by* $(a, b) \mapsto a1 + bP \mod J$ *is injective.*

Indeed, assume that $a1 + bP \in J$ with $(a, b) \neq (0, 0)$. From our assumption on P we see that $a \neq 0$. Hence $1 - cP \in J$, where $c = -b/a$.

Since $S^{r'} \subset J$ for large enough r', we have $(1 - cP)(1 + cP + c^2P^2 + \cdots + c^sP^s) = 1 \mod J$ if s is large enough. (We use $r > 0$.) Hence $1 \in J$, so that $J = S^\dagger$, a contradiction. This proves (c).

From (c) we see that $[\Gamma] \cong S^\dagger/J$ contains the trivial representation of Γ with multiplicity at least 2. This is absurd. Thus, our assumption that $P \notin J$ leads to a contradiction.

We see therefore that J contains any Γ-invariant element in S^r where $r > 0$. In particular, J contains any Γ'-invariant element in S^r where $r > 0$. Since these elements generate the ideal $\mathcal{I}$, we see that J contains $\mathcal{I}$. This proves (b), hence (a).

We have clearly $\mathcal{I} = \bigoplus_r(\mathcal{I} \cap S^r)$. Hence $\tilde{S} = \bigoplus_r \tilde{S}^r$ where $\tilde{S}^r = S^r/(\mathcal{I} \cap S^r)$ is the image of S^r in $\tilde{S}$.

1.9

We have

$$I = \{i_0^1, i_1^1, \ldots, i_{a_1}^1\} \cup \{i_0^2, i_1^2, \ldots, i_{a_2}^2\} \cup \{i_0^3, i_1^3, \ldots, i_{a_3}^3\}$$

(a disjoint union except for $i_0^1 = i_0^2 = i_0^3$) where a_1, a_2, a_3 are ≥ 1, $i, i' \in I$ satisfy $i \cdot i' = -1$ precisely when $\{i, i'\} = \{i_t^u, i_{t+1}^u\}$ with $u \in \{1, 2, 3\}, 0 \leq t < a_u$.

We denote $i_0^1 = i_0^2 = i_0^3$ by i_0.

1.10 The Polynomials B_i

The requirements

$$B_\heartsuit = 1,$$

$$(v + v^{-1})B_i - \sum_{j \in \tilde{I}; i \cdot j = -1} B_j = 0, \quad \text{if } i \in I - \{i_0\},$$

$$(v + v^{-1})B_i - \sum_{j \in \tilde{I}; i \cdot j = -1} B_j = v^{h'}(v - v^{-1}), \quad \text{if } i = i_0,$$

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define uniquely elements $B_i \in \mathbf{Q}(v)$ for all $i \in \tilde{I}$. Here v is an indeterminate. One can easily compute the elements B_i in each case. In the following tables the elements B_i are attached to the elements of $\tilde{I}$ in an obvious way (two vertices are joined in $\tilde{I}$ if they are consecutive in the same horizontal line or the same vertical line). The vertex $\heartsuit$ is marked with the polynomial 1.

Type D_n.

$$\begin{array}{ccc} v^{n-2} & & \\ v^{n-3}+v^{n-1} & & v^{n-2} \\ v^{n-4}+v^{n-2} & & \\ \dots & & \\ v^2+v^4 & & \\ v+v^3 & & 1 \\ v^2 & & \end{array}$$

Type E_6.

$$\begin{array}{ccccc} v^4 & & & & \\ v^3+v^5 & & & & \\ v^2+v^4+v^6 & & v+v^5 & & 1 \\ v^3+v^5 & & & & \\ v^4 & & & & \end{array}$$

Type E_7.

$$\begin{array}{ccc} 1 & & \\ v+v^7 & & \\ v^2+v^6+v^8 & & \\ v^3+v^5+v^7+v^9 & & v^4+v^8 \\ v^4+v^6+v^8 & & \\ v^5+v^7 & & \\ v^6 & & \end{array}$$

Type E_8.

$$\begin{array}{ccc} v^7+v^{13} & & \\ v^6+v^8+v^{12}+v^{14} & & \\ v^5+v^7+v^9+v^{11}+v^{13}+v^{15} & & v^6+v^{10}+v^{14} \\ v^4+v^8+v^{10}+v^{12}+v^{14} & & \\ v^3+v^9+v^{11}+v^{13} & & \\ v^2+v^{10}+v^{12} & & \\ v+v^{11} & & \\ 1 & & \end{array}$$

In particular, we have $B_i \in \mathbf{Z}[v]$ for all $i \in \tilde{I}$.

The polynomials B_i were introduced in [L1, p. 647].

1.11

From [GV, 5.3] one can extract that

(a) $$\sum_{r \geq 0} (\tilde{S}^r, \rho_i)_\Gamma v^r = B_i + v^{2h'} \bar{B}_i$$

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for any $i \in \tilde{I}$. Here $\bar{}: \mathbf{Q}(v) \to \mathbf{Q}(v)$ is the field involution such that $\bar{v} = v^{-1}$.

1.12

Let $\tilde{S}_i^r$ be the ρ_i-isotypic component of $\tilde{S}^r$. Using 1.11(a) and the tables in 1.10, we see that the following hold.

(a) $\tilde{S}_i^r \neq 0$ implies $0 \le r \le 2h'$.
(b) $\tilde{S}_i^r \cong \tilde{S}_i^{2h'-r}$ for $0 \le r \le 2h'$.
(c) $\tilde{S}_{i_0}^{h'} \cong \rho_{i_0} \oplus \rho_{i_0}$.
(d) If $i \neq \heartsuit$ and $i = i_t^u$ with $t > 0$ then $\tilde{S}_i^{h'-t} \cong \tilde{S}_i^{h'+t} \cong \rho_i$ and $\tilde{S}_i^{h'-t+1} = \tilde{S}_i^{h'-t+2} = \cdots = \tilde{S}_i^{h'+t-1} = 0$.
(e) If $i = \heartsuit$ then $\tilde{S}_i^0 \cong \tilde{S}_i^{2h'} \cong \rho_i$ and $\tilde{S}_i^r = 0$ for $0 < r < 2h'$.

Lemma 1.13 *Let V be a Γ-submodule of $\tilde{S}_{i_0}^{h'}$ such that $V \cong \rho_{i_0}$. For any $k \in \tilde{I}$, define a subspace $\tilde{J}_k$ of $\bigoplus_{r\ge 0} \tilde{S}_k^r$ by*

$$\tilde{J}_k = \begin{cases} \bigoplus_{r>h'} \tilde{S}_k^r \oplus V, & \text{if } k = i_0, \\ \bigoplus_{r>h'} \tilde{S}_k^r, & \text{if } k \neq i_0. \end{cases}$$

Then $\tilde{J}^V = \bigoplus_{k\in\tilde{I}} \tilde{J}_k \subset \tilde{S}$ belongs to $\tilde{\mathbf{H}}_0$.

Lemma 1.14 *Assume that $i \in I$ is of the form i_t^u where $t > 0$. Let $j = i_1^u$. Let V be a Γ-submodule of $\tilde{S}_{i_0}^{h'}$ such that*

$$V \cong \rho_{i_0}, \quad \tilde{S}^1 \tilde{S}_j^{h'-1} \subset V, \quad \tilde{S}^1 V \cap \tilde{S}_j^{h'+1} = 0.$$

Let V' be a Γ-submodule of $\tilde{S}_i^{h'-t} \oplus \tilde{S}_i^{h'+t}$ such that $V' \cong \rho_i$. For any $k \in \tilde{I}$, define a subspace $\tilde{J}_k$ of $\bigoplus_{r\ge 0} \tilde{S}_k^r$ by

$$\tilde{J}_k = \begin{cases} \bigoplus_{r>h'+t} \tilde{S}_k^r \oplus V', & \text{if } k = i_t^u, \\ \bigoplus_{r>h'+t'} \tilde{S}_k^r \oplus \tilde{S}_k^{h'-t'}, & \text{if } k = i_{t'}^u, 0 < t' < t, \\ \bigoplus_{r>h'} \tilde{S}_k^r \oplus V, & \text{if } k = i_0, \\ \bigoplus_{r>h'} \tilde{S}_k^r, & \text{for all other } k \in \tilde{I}. \end{cases}$$

Then $\tilde{J}^{V,V'} = \bigoplus_{k\in\tilde{I}} \tilde{J}_k \subset \tilde{S}$ belongs to $\tilde{\mathbf{H}}_0$.

Let $\tilde{J}$ be $\tilde{J}^{V,V'}$ or $\tilde{J}^V$ in 1.13. It is clear that $\tilde{J} \cong [\Gamma]$ in $\mathcal{C}_\Gamma$. Since $\tilde{S} \cong [\Gamma] \oplus [\Gamma]$ in $\mathcal{C}_\Gamma$, it follows that $\tilde{S}/\tilde{J} \cong [\Gamma]$ in $\mathcal{C}_\Gamma$. To prove that $\tilde{J}$ is an ideal of $\tilde{S}$, it is enough to check that multiplication by $\tilde{S}^1$ maps $\tilde{J}$ into itself. This follows immediately from the assumptions and the properties 1.12(a)–(e), using the inclusion

$$\tilde{S}^1 \tilde{S}_k^r \subset \sum_{k' \in \tilde{I}; k\cdot k' = -1} \tilde{S}_{k'}^{r+1}.$$

Lemma 1.15 *Assume that M is both an $S^\dagger$-module and a Γ-module, so that the module structure $S^\dagger \otimes M \to M$ is Γ-linear. Assume also that the Γ-module M has at most two non-zero isotypic components. Then $S^2 M = 0$.*

As explained in [L3, Section 6], giving M is the same as giving a module $\underline{M}$ over the preprojective algebra of the corresponding affine Coxeter graph. Our assumption on M implies that

(a) $\underline{M}$ has a zero component at all but two vertices.

We must show that any path of length 2 acts as 0 on $\underline{M}$. But this clearly follows, using (a), from the relations of the preprojective algebra. The lemma is proved.

Lemma 1.16 *Let $u \in \{1,2,3\}$. Let $j = i_1^u$. There exists a unique Γ-submodule $V(u)$ of $\tilde{S}_{i_0}^{h'}$ such that*

$$V(u) \cong \rho_{i_0}, \quad \tilde{S}^1 \tilde{S}_j^{h'-1} \subset V(u), \quad \tilde{S}^1 V(u) \cap \tilde{S}_j^{h'+1} = 0.$$

To prove this, we define subspaces $\tilde{S}' = \bigoplus_{k \in \tilde{I}} \tilde{S}'_k, \tilde{S}'' = \bigoplus_{k \in \tilde{I}} \tilde{S}''_k$ of $\tilde{S}$ by

$$\tilde{S}'_k = \bigoplus_{r > h'+1} \tilde{S}_k^r \quad \text{for } k = j,$$

$$\tilde{S}'_k = \bigoplus_{r > h'} \tilde{S}_k^r \quad \text{for } k \neq j,$$

$$\tilde{S}''_k = \bigoplus_{r \geq h'-1} \tilde{S}_k^r \quad \text{for } k = j,$$

$$\tilde{S}''_k = \bigoplus_{r \geq h'} \tilde{S}_k^r \quad \text{for } k \neq j.$$

Then $\tilde{S}' \subset \tilde{S}''$ are ideals of $\tilde{S}$. Hence $M = \tilde{S}''/\tilde{S}'$ is naturally an $\tilde{S}$-module (hence an $S^\dagger$-module) and it is also a Γ-module with only two isotypic components M_{i_0}, M_j (corresponding to i_0 and j). Moreover, M_{i_0}, M_j inherit $\mathbf{Z}$-gradings from $\tilde{S}$. We have $M_{i_0} = M_{i_0}^{h'} \cong \rho_{i_0} \oplus \rho_{i_0}$ and $M_j = M_j^{h'-1} \oplus M_j^{h'+1}$ with $M_j^{h'-1} \cong M_j^{h'+1} \cong \rho_j$. Let $X = \tilde{S}^1 M_j^{h'-1}$. Equivalently, X is the image of the Γ-linear map $\tilde{S}^1 \otimes M_j^{h'-1} \to M_{i_0}^{h'}$ given by the $\tilde{S}$-module structure. Since $M_j^{h'-1} \cong \rho_j$ and $T \otimes \rho_j$ contains ρ_{i_0} with multiplicity one, it follows that either $X = 0$ or $X \cong \rho_{i_0}$ in $\mathcal{C}_\Gamma$.

Let X' be the set of all $m \in M_{i_0}^{h'}$ such that $fm = 0$ for any $f \in \tilde{S}^1$. Equivalently, X' is the kernel of the Γ-linear map $M_{i_0}^{h'} \to \tilde{S}^1 \otimes M_j^{h'+1}$ given by $m \mapsto e \otimes (e'm) - e' \otimes (em)$, where e, e' form a symplectic basis of T. Since $M_j^{h'-1} \cong \rho_j$ and $T \otimes \rho_j$ contains ρ_{i_0} with multiplicity one, it follows that either $X' = M_{i_0}^{h'}$ or $X' \cong \rho_{i_0}$ in $\mathcal{C}_\Gamma$. Applying Lemma 1.15 to M we see that $\tilde{S}^2 M = 0$. In particular, we have $X \subset X'$. Hence there are four possibilities:

(a) $X = 0, X' \cong \rho_{i_0}$;
(b) $X = X' \cong \rho_{i_0}$;
(c) $X \cong \rho_{i_0}, X' = M_{i_0}^{h'}$;
(d) $X = 0, X' = M_{i_0}^{h'}$.

To prove the lemma, it is enough to show that there is a unique Γ-submodule X_0 of $M_{i_0}^{h'}$ such that $X_0 \cong \rho_{i_0}$ and $X \subset X_0 \subset X'$. This is clear in cases (a), (b), (c): we take X_0 to be $X', X = X', X$ respectively.

It remains to show that the case (d) cannot occur. Assume that we are in case (d). Then any Γ-submodule V of $\tilde{S}_{i_0}^{h'}$ such that $V \cong \rho_{i_0}$ automatically satisfies $\tilde{S}^1\tilde{S}_j^{h'-1} \subset V$, $\tilde{S}^1 V \cap \tilde{S}_j^{h'+1} = 0$. Applying Lemma 1.14 with $i = i_1^u = j$ for any V as above and any Γ-submodule V' of $\tilde{S}_i^{h'-1} \oplus \tilde{S}_i^{h'+1}$ such that $V' \cong \rho_i$, we obtain a two-parameter family of distinct points of $\tilde{\mathbf{H}}_0$. (Both V and V' run through a P^1.) This contradicts the fact that $\tilde{\mathbf{H}}_0 = \mathbf{H}_0^{[\Gamma]}$ has pure dimension 1. The lemma is proved.

1.17

Let Π_{i_0} be the set of points $\tilde{J}^V \in \tilde{\mathbf{H}}_0 = \mathbf{H}_0^{[\Gamma]}$ attached in Lemma 1.13 to the various Γ-submodules V of $\tilde{S}_{i_0}^{h'}$ such that $V \cong \rho_{i_0}$. This is a projective line contained in $\tilde{\mathbf{H}}_0 = \mathbf{H}_0^{[\Gamma]}$.

For any $i \in I$ of the form $i = i_t^u$ with $t > 0$, let Π_i be the set of points $\tilde{J}^{V,V'} \in \tilde{\mathbf{H}}_0 = \mathbf{H}_0^{[\Gamma]}$ attached in Lemma 1.14 to $V = V(u)$ (as in 1.16) and to the various Γ-submodules V' of $\tilde{S}_i^{h'-t} \oplus \tilde{S}_i^{h'+t}$ such that $V' \cong \rho_i$. This is a projective line contained in $\tilde{\mathbf{H}}_0 = \mathbf{H}_0^{[\Gamma]}$.

The projective lines $\Pi_i (i \in I)$ are clearly distinct. From 1.7(a) it follows that $\mathbf{H}_0^{[\Gamma]}$ has exactly $|I|$ irreducible components, each of dimension 1. It follows that $\Pi_i (i \in I)$ are exactly the irreducible components of $\mathbf{H}_0^{[\Gamma]}$ so that $\mathbf{H}_0^{[\Gamma]} = \bigcup_{i \in I} \Pi_i$.

1.18

Let $k \in \tilde{I}$. We consider the vector bundle E^k over $\mathbf{H}^{[\Gamma]}$ whose fibre E_J^k at $J \in \mathbf{H}^{[\Gamma]}$ is $\mathrm{Hom}_\Gamma(\rho_k, S^\dagger/J)$. This is a vector bundle with fibres of dimension $\dim \rho_k$.

The action of $\mathbf{C}^*$ on T given by $\lambda \colon x \mapsto \lambda x$ extends to an action of $\mathbf{C}^*$ on $S^\dagger$ by algebra automorphisms; an element $\lambda \in \mathbf{C}^*$ acts on S^r as multiplication by λ^r. We denote this automorphism of $S^\dagger$ by τ_λ. Note that, if J is an ideal of $S^\dagger$, then $\tau_\lambda(J)$ is an ideal of $S^\dagger$. If furthermore, $J \in \mathbf{H}^{[\Gamma]}$, then $\tau_\lambda(J) \in \mathbf{H}^{[\Gamma]}$. (This is because the $\mathbf{C}^*$-action on $S^\dagger$ commutes with the Γ-action on $S^\dagger$.) Note also that, if $J \in \mathbf{H}^{[\Gamma]}$, then τ_λ induces an isomorphism $S^\dagger/J \xrightarrow{\sim} S^\dagger/\tau_\lambda(J)$ in $\mathcal{C}_\Gamma$ and this, in turn, induces an isomorphism $E_J^k \xrightarrow{\sim} E_{\tau_\lambda(J)}^k$ of vector spaces. We see that $\mathbf{H}^{[\Gamma]}$ has a natural $\mathbf{C}^*$-action and that the vector bundle E^k is naturally $\mathbf{C}^*$-equivariant. Now $\tilde{\mathbf{H}}_0 = \mathbf{H}_0^{[\Gamma]}$ is a $\mathbf{C}^*$-stable subvariety of $\mathbf{H}^{[\Gamma]}$; hence each of its irreducible components Π_i, $(i \in I)$ is $\mathbf{C}^*$-stable.

The $\mathbf{C}^*$-action $\lambda \colon x \mapsto \lambda^{-1}x$ on T' induces a $\mathbf{C}^*$-action on $\Gamma \setminus T'$ and one on $\mathrm{Sym}^r(T')$; the last action is $\lambda \colon (x_1, x_2, \dots, x_r) \mapsto (\lambda^{-1}x_1, \lambda^{-1}x_2, \dots, \lambda^{-1}x_r)$. This, in turn, restricts to a $\mathbf{C}^*$-action on $\bigl(\mathrm{Sym}^r(T')\bigr)^\Gamma$ when $r = \dim([\Gamma])$ which is compatible with the $\mathbf{C}^*$-action on $\Gamma \setminus T'$ under the identification in 1.7. Note that the map $\mathbf{H}^{[\Gamma]} \to \bigl(\mathrm{Sym}^r(T')\bigr)^\Gamma = \Gamma \setminus T'$ in 1.7(a) is $\mathbf{C}^*$-equivariant. Indeed it is enough to show that $p \colon T'^{[r]} \to \mathrm{Sym}^r(T')$ in 1.5 is $\mathbf{C}^*$-equivariant. This follows immediately from the definitions.

Lemma 1.19 *Let V be a Γ-submodule of $\tilde{S}_{i_0}^{h'}$ such that $V \cong \rho_{i_0}$. The fibre of E^k at $\tilde{J}^V \in \Pi_{i_0}$*

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is canonically

$$\bigoplus_{r<h'} \mathrm{Hom}_\Gamma(\rho_k, \tilde{S}_k^r) \oplus \mathrm{Hom}_\Gamma(\rho_k, \tilde{S}_k^{h'}/V), \quad \text{if } k = i_0,$$

$$\bigoplus_{r<h'} \mathrm{Hom}_\Gamma(\rho_k, \tilde{S}_k^r), \quad \text{if } k \neq i_0.$$

Lemma 1.20 *Assume that $i \in I$ is of the form i_t^u where $t > 0$. Let $V(u) \subset \tilde{S}_{i_0}^{h'}$ be as in 1.16. Let V' be a Γ-submodule of $\tilde{S}_i^{h'-t} \oplus \tilde{S}_i^{h'+t}$ such that $V' \cong \rho_i$. The fibre of E^k at $\tilde{J}^{V(u),V'} \in \Pi_i$ is canonically*

$$\bigoplus_{r<h'-t} \mathrm{Hom}_\Gamma(\rho_k, \tilde{S}_k^r) \oplus \mathrm{Hom}_\Gamma\big(\rho_k, (\tilde{S}_i^{h'-t} \oplus \tilde{S}_i^{h'+t})/V'\big), \quad \text{if } k = i_t^u,$$

$$\bigoplus_{r<h'-t'} \mathrm{Hom}_\Gamma(\rho_k, \tilde{S}_k^r) \oplus \mathrm{Hom}_\Gamma(\rho_k, \tilde{S}_k^{h'+t'}), \quad \text{if } k = i_{t'}^u,\ 0 < t' < t,$$

$$\bigoplus_{r<h'} \mathrm{Hom}_\Gamma(\rho_k, \tilde{S}_k^r) \oplus \mathrm{Hom}_\Gamma\big(\rho_k, \tilde{S}_k^{h'}/V(u)\big), \quad \text{if } k = i_0,$$

$$\bigoplus_{r<h'} \mathrm{Hom}_\Gamma(\rho_k, \tilde{S}_k^r), \quad \text{for all other } k \in \tilde{I}.$$

This and the previous lemma follow directly from definitions, since the fibre of E^k at a point $\tilde{J} \in \tilde{\mathbf{H}}_0$ is $\mathrm{Hom}_\Gamma(\rho_k, \tilde{S}/\tilde{J})$.

1.21

Let $i \in I$. We define a line bundle O_i on Π_i as follows. If $i = i_0$, the fibre of O_i at $\tilde{J}^V \in \Pi_{i_0}$ is the line

$$\mathrm{Hom}(\rho_i, \tilde{S}_{i_0}^{h'}/V).$$

If $i = i_t^u$ with $t > 0$, the fibre of O_i at $\tilde{J}^{V(u),V'} \in \Pi_i$ is the line

$$\mathrm{Hom}\big(\rho_i, (\tilde{S}_i^{h'-t} \oplus \tilde{S}_i^{h'+t})/V'\big).$$

O_i has a unique $\mathbf{C}^*$-equivariant structure such that the following holds:

If $i = i_0$ (so that $\mathbf{C}^*$ acts trivially on Π_i), then $\mathbf{C}^*$ acts trivially on each fibre of O_i. If $i = i_t^u$ with $t > 0$ (so that $\mathbf{C}^*$ acts on Π_i with exactly two fixed points, $\tilde{J}^{V(u),\tilde{S}_i^{h'-t}}$ and $\tilde{J}^{V(u),\tilde{S}_i^{h'+t}}$), then $\lambda \in \mathbf{C}^*$ acts on the fibre of O_i at $\tilde{J}^{V(u),\tilde{S}_i^{h'-t}}$ as multiplication by λ^t and on the fibre of O_i at $\tilde{J}^{V(u),\tilde{S}_i^{h'+t}}$ as multiplication by λ^{-t}.

For any $m \in \mathbf{Z}$ we define the line bundle O_i^m on Π_i to be $O_i^{\otimes m}$, if $m \geq 0$, or the dual of $O_i^{\otimes(-m)}$ if $m < 0$. This line bundle inherits a $\mathbf{C}^*$-equivariant structure from O_i.

We shall generally use the following notation. If $\mathcal{E}$ is a $\mathbf{C}^*$-equivariant vector bundle on a variety with $\mathbf{C}^*$-action and $r \in \mathbf{Z}$, we denote by $v^r\mathcal{E}$ the $\mathbf{C}^*$-equivariant vector bundle given

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by the tensor product of $\mathcal{E}$ with the trivial line bundle $\mathbf{C}$ with $\mathbf{C}^*$-equivariant structure in which $\lambda \in \mathbf{C}^*$ acts as multiplication by λ^r. We denote by $\mathbf{C}$ the trivial vector bundle with the obvious $\mathbf{C}^*$-equivariant structure.

Proposition 1.22 *(a) If $k = \heartsuit$, then $E^k = \mathbf{C}$.*

(b) If $k \in \tilde{I}$ and $i \in I$ are such that $k \neq i$, then $E^k|_{\Pi_i}$ is a trivial vector bundle (if we forget the $\mathbf{C}^*$-equivariant structure).

(c) For any $\tilde{J} \in \Pi_{i_0}$ (necessarily a fixed point of the $\mathbf{C}^*$-action) we have $E^k|_{\tilde{J}} \cong v^{c_1} \oplus v^{c_2} \oplus \cdots \oplus v^{c_s}$ as a $\mathbf{C}^*$-equivariant vector bundle over a point. (Here $B_k = v^{c_1} + v^{c_2} + \cdots + v^{c_s}$ is as in 1.10.)

(d) If $k \in I$, then $E^k|_{\Pi_k} \cong v^{h'} O^1_k \oplus U$, where U is a $\mathbf{C}^*$-equivariant vector bundle over Π_k which is trivial if we forget the $\mathbf{C}^*$-action.

This follows immediately from Lemmas 1.19, 1.20 and from 1.11(a).

Corollary 1.23 *For $k \in \tilde{I}$, let E'^k be the vector bundle on $\mathbf{H}^{[\Gamma]}$ dual to E^k with the $\mathbf{C}^*$-equivariant structure inherited from E^k.*

(a) If $k = \heartsuit$, then $E'^k = \mathbf{C}$.

(b) If $k \in \tilde{I}$ and $i \in I$ are such that $k \neq i$, then $E'^k|_{\Pi_i}$ is a trivial vector bundle (if we forget the $\mathbf{C}^*$-equivariant structure).

(c) For any $\tilde{J} \in \Pi_{i_0}$ we have $E'^k|_{\tilde{J}} \cong v^{-c_1} \oplus v^{-c_2} \oplus \cdots \oplus v^{-c_s}$ as a $\mathbf{C}^*$-equivariant vector bundle over a point. (Here $B_k = v^{c_1} + v^{c_2} + \cdots + v^{c_s}$ is as in 1.10.)

(d) If $k \in I$, then $E'^k|_{\Pi_k} \cong v^{-h'} O^{-1}_k \oplus U'$, where U' is a $\mathbf{C}^*$-equivariant vector bundle over Π_k which is trivial if we forget the $\mathbf{C}^*$-action.

1.24

For $u \in \{1,2,3\}$, $0 \le t < a_u$, we denote by $p^u_{t,t+1}$ the unique point in the intersection $\Pi_{i^u_t} \cap \Pi_{i^u_{t+1}}$, that is,

$$p^u_{t,t+1} = \tilde{J}^{V(u),\tilde{S}_i^{h'-t}} = \tilde{J}^{V(u),\tilde{S}_{i'}^{h'+t+1}}, \quad \text{if } t > 0,\ i = i^u_t,\ i' = i^u_{t+1},$$

$$p^u_{0,1} = \tilde{J}^{V(u)} = \tilde{J}^{V(u),\tilde{S}_{i'}^{h'+1}}, \quad \text{if } t = 0,\ i' = i^u_1.$$

Note that $p^1_{0,1}, p^2_{0,1}, p^3_{0,1}$ are distinct points of Π_{i_0} (a consequence of 1.7(a)) and that all intersections $\Pi_i \cap \Pi_j$ other than those just considered are empty.

For $u \in \{1,2,3\}$, let $i = i^u_{a_u}$ and let $q^u = \tilde{J}^{V(u),\tilde{S}_i^{h'-a_u}} \in \Pi_i$.

The $\mathbf{C}^*$-actions on $\mathbf{H}^{[\Gamma]}, \mathbf{H}_0^{[\Gamma]}$ have the same fixed point set:

$$(\mathbf{H}^{[\Gamma]})^{\mathbf{C}^*} = (\mathbf{H}_0^{[\Gamma]})^{\mathbf{C}^*} = \bigsqcup_{i \in I} \mu_i$$

where μ_i is the connected component of $(\mathbf{H}^{[\Gamma]})^{\mathbf{C}^*} = (\mathbf{H}_0^{[\Gamma]})^{\mathbf{C}^*}$ defined as

$$\Pi_{i_0} \quad \text{if } i = i_0,$$

$$\{p^u_{t,t+1}\}, \quad \text{if } i = i^u_t \text{ with } u \in \{1,2,3\} \text{ and } 0 < t < a_u,$$

$$\{q^u\}, \quad \text{if } i = i^u_t \text{ with } u \in \{1,2,3\} \text{ and } t = a_u.$$

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1.25

The equivariant K-groups $K_{\mathbf{C}^*}()$ are as in [L4, 6.1]; $R_{\mathbf{C}^*}$ is the representation ring of $\mathbf{C}^*$, that is, $K_{\mathbf{C}^*}$ of a point.

Consider the homomorphism

$$\bigoplus_{u,t;0\le t<a_u} K_{\mathbf{C}^*}(p^u_{t,t+1}) \xrightarrow{a} \bigoplus_i K_{\mathbf{C}^*}(\Pi_i)$$

with components $K_{\mathbf{C}^*}(p^u_{t,t+1}) \to K_{\mathbf{C}^*}(\Pi_{i^u_t})$ (direct image map) and $K_{\mathbf{C}^*}(p^u_{t,t+1}) \to K_{\mathbf{C}^*}(\Pi_{i^u_{t+1}})$ (minus the direct image map); the other components are 0. The homomorphism $\bigoplus_{i\in I} K_{\mathbf{C}^*}(\Pi_i) \to K_{\mathbf{C}^*}(\mathbf{H}_0^{[\Gamma]})$ with components given by the direct image maps is zero on the image of a hence it induces a homomorphism $\operatorname{coker}(a) \to K_{\mathbf{C}^*}(\mathbf{H}_0^{[\Gamma]})$.

Lemma 1.26 *a is injective and $K_{\mathbf{C}^*}(\mathbf{H}_0^{[\Gamma]}) = \operatorname{coker}(a)$.*

The same statement can be formulated in the case where $\mathbf{H}_0^{[\Gamma]}$ is replaced by a variety X of pure dimension 1 with $\mathbf{C}^*$-action such that each irreducible component is a P^1, any two components are either disjoint or intersect at exactly one point, no point belongs to three components and the pattern of intersection of the components is given by a tree. We prove this more general statement by induction on the number of irreducible components of X. If X has exactly one component, the result is clear. Assume now that X has $N \ge 2$ components. Then we have $X = X' \cup X''$ where X' is a closed subset of X of the same type as X but with only $N-1$ components and X'' is a component of X which intersects X' in exactly one point p. The desired result holds for X' by the induction hypothesis; it gives an exact sequence of the form

$$0 \longrightarrow A' \longrightarrow A \longrightarrow K_{\mathbf{C}^*}(X') \longrightarrow 0.$$

We would like to show that we have an analogous exact sequence

$$0 \longrightarrow A' \oplus K_{\mathbf{C}^*}(p) \longrightarrow A \oplus K_{\mathbf{C}^*}(X'') \longrightarrow K_{\mathbf{C}^*}(X) \longrightarrow 0.$$

We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & A' \oplus K_{\mathbf{C}^*}(p) & \longrightarrow & K_{\mathbf{C}^*}(p) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A \oplus K_{\mathbf{C}^*}(X'') & \longrightarrow & K_{\mathbf{C}^*}(X'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{\mathbf{C}^*}(X') & \longrightarrow & K_{\mathbf{C}^*}(X) & \longrightarrow & R_{\mathbf{C}^*} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

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with exact horizontal lines. The vertical lines (except possibly for the middle one) are exact. But then the middle vertical line is automatically exact. The desired statement for X follows. The lemma is proved.

2 Preliminaries on $\mathcal{B}_e, \Lambda_e$

2.1

Let G be a connected, semisimple, almost simple, simply connected algebraic group of simply laced type. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{g}_n$ be the variety of nilpotent elements in $\mathfrak{g}$. Let $\mathcal{B}$ be the variety of all Borel subalgebras of $\mathfrak{g}$. A parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is said to be *almost minimal* if there exists $\mathfrak{b} \in \mathcal{B}$ such that $\mathfrak{b} \subset \mathfrak{p}$, $\dim(\mathfrak{p}/\mathfrak{b}) = 1$.

Let I' be a finite set indexing the set of G-orbits on the set of almost minimal parabolic subalgebras (for the adjoint action). A parabolic subalgebra in the G-orbit indexed by i is said to have type i. Let $\mathcal{P}_i$ be the variety of all parabolic subalgebras of type i. Let $\pi_i\colon \mathcal{B} \to \mathcal{P}_i$ be the morphism defined by $\pi_i(\mathfrak{b}) = \mathfrak{p}$ where $\mathfrak{b} \in \mathcal{B}$, $\mathfrak{p} \in \mathcal{P}_i$, $\mathfrak{b} \subset \mathfrak{p}$.

Let $\mathbf{X}$ be the set of isomorphism classes of algebraic G-equivariant line bundles on $\mathcal{B}$ where G acts on $\mathcal{B}$ by the adjoint action. Then $\mathbf{X}$ is a finitely generated free abelian group under the operation given by tensor product of line bundles. For each $i \in I'$, let $L_i \in \mathbf{X}$ be the tangent bundle along the fibres of $\pi_i\colon \mathcal{B} \to \mathcal{P}_i$.

Let $\mathcal{X}$ be a free abelian group (in additive notation) with a given isomorphism $\mathcal{X} \xrightarrow{\sim} \mathbf{X}$ denoted by $x \mapsto L_x$. Let $\alpha_i \in \mathcal{X}$ be defined by $L_{\alpha_i} = L_i$. If $x \in \mathcal{X}$, the Euler characteristic of any fibre of π_i (a projective line) with coefficients in the restriction of L_x is equal to $\check{\alpha}_i(x) + 1$ where $\check{\alpha}_i(x) \in \mathbf{Z}$. Then $\check{\alpha}_i\colon \mathcal{X} \to \mathbf{Z}$ is a homomorphism. For $i \in I'$, let $x \mapsto {}^{\sigma_i}x$ be the (involutive) map $\mathcal{X} \to \mathcal{X}$ given by ${}^{\sigma_i}x = x - \check{\alpha}_i(x)\alpha_i$. The involutions $x \mapsto {}^{\sigma_i}x$ are the standard generators of the Weyl group W, a finite *Coxeter group* with length function $l\colon W \to \mathbf{N}$. Let w_0 be the longest element of W.

2.2

Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. Let $\mathcal{A}\mathcal{X}$ be the group algebra of $\mathcal{X}$ with coefficients in $\mathcal{A}$. The basis element of $\mathcal{A}\mathcal{X}$ corresponding to $x \in \mathcal{X}$ is denoted by $[x]$. The affine Hecke algebra $\mathcal{H}$ is the $\mathcal{A}$-algebra with generators $\tilde{T}_w (w \in W)$ and $\theta_x (x \in \mathcal{X})$ subject to the relations

(a) $(\tilde{T}_{\sigma_i} + v^{-1})(\tilde{T}_{\sigma_i} - v) = 0, \quad (i \in I')$;
(b) $\tilde{T}_w \tilde{T}_{w'} = \tilde{T}_{ww'}$ if $l(ww') = l(w) + l(w')$;
(c) $\theta_x \tilde{T}_{\sigma_i} - \tilde{T}_{\sigma_i} \theta_{{}^{\sigma_i}x} = (v - v^{-1})\theta_{\frac{[x]-[{}^{\sigma_i}x]}{1-[-\alpha_i]}}$;
(d) $\theta_x \theta_{x'} = \theta_{x+x'}$;
(e) $\theta_0 = 1$.

Here we use the following convention: for $p = \sum_{x\in\mathcal{X}} c_x[x] \in \mathcal{A}\mathcal{X}$ (finite sum with $c_x \in \mathcal{A}$) we set $\theta_p = \sum_{x\in\mathcal{X}} c_x\theta_x \in \mathcal{H}$.

Let $\mathcal{H}_0$ be the subalgebra of $\mathcal{H}$ generated by the elements $\tilde{T}_{\sigma_i} (i \in I')$ or equivalently, the $\mathcal{A}$-submodule of $\mathcal{H}$ generated by the elements $\tilde{T}_w (w \in W)$.

Let $\chi \mapsto \chi^{\blacktriangle}$ be the involutive antiautomorphism of the $\mathcal{A}$-algebra $\mathcal{H}$ defined by $\tilde{T}_w \mapsto \tilde{T}_{w^{-1}}$ for all $w \in W$ and $\tilde{T}_{w_0}^{-1}\theta_{{}^{w_0}x}\tilde{T}_{w_0} \mapsto \theta_{-x}$ for all $x \in \mathcal{X}$. (See [L4, 1.22, 1.24, 1.25]).

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2.3

We fix an $\mathfrak{sl}_2$-triple (e, f, h) in $\mathfrak{g}$ that is, three elements e, f, h of $\mathfrak{g}$ such that $[h, e] = 2e, [h, f] = -2f, [e, f] = h$.

Let $\zeta\colon \mathrm{SL}_2 \to G$ be the homomorphism of algebraic groups whose tangent map at 1 carries

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ to } e, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ to } f, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ to } h.$$

2.4

Let $\Lambda = \{(y, \mathfrak{b}) \in \mathfrak{g}_n \times \mathcal{B} \mid y \in \mathfrak{b}\}$. Let $\mathfrak{z}(f)$ be the centralizer of f in $\mathfrak{g}$ and let

$$\Sigma = \{y \in \mathfrak{g}_n \mid y - e \in \mathfrak{z}(f)\},$$
$$\Lambda_e = (\Sigma \times \mathcal{B}) \cap \Lambda,$$
$$\mathcal{B}_e = \{\mathfrak{b} \in \mathcal{B} \mid e \in \mathfrak{b}\}.$$

We identify $\mathcal{B}_e$ with a closed subvariety of Λ_e by $\mathfrak{b} \mapsto (e, \mathfrak{b})$, that is, $\mathcal{B}_e$ is the fibre at 0 of $pr_1\colon \Lambda_e \to \Sigma$.

Now $\mathbf{C}^*$ acts on Λ_e by

$$\lambda\colon (y, \mathfrak{b}) \mapsto \left(\lambda^{-2} \operatorname{Ad} \zeta \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} y, \quad \operatorname{Ad} \zeta \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mathfrak{b}\right).$$

This restricts to a $\mathbf{C}^*$- action on $\mathcal{B}_e$.

Throughout this paper we assume that e is *subregular*. Then, for each $i \in I'$ there is a unique irreducible component V_i of $\mathcal{B}_e$ which is a single fibre of $\pi_i\colon \mathcal{B} \to \mathcal{P}_i$ (hence a P^1) and any irreducible component of $\mathcal{B}_e$ is equal to V_i for a unique $i \in I'$ (a result of Tits).

According to Brieskorn [B], we can find $\Gamma \subset \mathrm{Sp}(T)$ as in 1.3 and an isomorphism

(a) $\Gamma \setminus T' \xrightarrow{\sim} \Sigma$;

moreover, according to Slodowy [S], the isomorphism (a) can be chosen so that the $\mathbf{C}^*$-action

$$\lambda\colon y \mapsto \lambda^{-2} \operatorname{Ad} \zeta \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} y$$

on Σ corresponds to the $\mathbf{C}^*$-action on $\Gamma \setminus T'$ induced by the $\mathbf{C}^*$-action on $\lambda, x \mapsto \lambda^{-1}x$ on T'. We shall assume that (a) has been chosen with this additional property.

Brieskorn also shows that $pr_1\colon \Lambda_e \to \Sigma$ is a minimal resolution of singularities of Σ; using 1.7(a), we see that there exists a unique isomorphism

(b) $\mathbf{H}^{[\Gamma]} \xrightarrow{\sim} \Lambda_e$

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such that the diagram

$$\begin{array}{ccc} \mathbf{H}^{[\Gamma]} & \xrightarrow{\sim} & \Lambda_e \\ \big\downarrow & & \big\downarrow{\scriptstyle pr_1} \\ \Gamma \setminus T' & \xrightarrow{\sim} & \Sigma \end{array}$$

is commutative. (Here $[\Gamma]$ is the regular representation of Γ, the lower horizontal map is as above, and the left vertical map is as in 1.7(a).) In particular, Λ_e is irreducible, smooth, of dimension 2.

In the remainder of this paper we shall assume that G is of type D_n ($n \geq 4$) or E_n ($n \in \{6, 7, 8\}$).

This is equivalent to the assumption in 1.8 that Γ is not cyclic. It is also equivalent to the equality

$$\{y \in \mathfrak{g} \mid [y, e] = [y, f] = [y, h] = 0\} = 0.$$

The isomorphism (b) automatically carries the subvariety $\mathbf{H}_0^{[\Gamma]}$ of $\mathbf{H}^{[\Gamma]}$ onto the subvariety $\mathcal{B}_e$ of Λ_e (these are fibres of the vertical maps over corresponding points). Hence it carries an irreducible component Π_i of $\mathbf{H}_0^{[\Gamma]}$ (where $i \in I$) onto an irreducible component $V_{i'}$ of $\mathcal{B}_e$ (where $i' \in I'$). The map $i \mapsto i'$ is a bijection $I \xrightarrow{\sim} I'$. We use this bijection to identify $I = I'$. We identify $\mathbf{H}^{[\Gamma]} = \Lambda_e$, $\mathbf{H}_0^{[\Gamma]} = \mathcal{B}_e$ using the isomorphisms above. This identification is compatible with the $\mathbf{C}^*$-actions. Indeed, we know already that in the commutative diagram above, all maps except possibly for the upper horizontal one are compatible with the $\mathbf{C}^*$-actions. But then the upper horizontal isomorphism is compatible with the $\mathbf{C}^*$-actions at least when restricted to the complement of the exceptional divisors; then it must be compatible everywhere.

We also identify $\Pi_i = V_i$ for $i \in I = I'$.

2.5

The equivariant K-groups $K_{\mathbf{C}^*}(\mathcal{B}_e)$, $K_{\mathbf{C}^*}(\Lambda_e)$ will be regarded as $\mathcal{H}$-modules as in [L4, 12.5]. Note that $K_{\mathbf{C}^*}(\mathcal{B}_e), K_{\mathbf{C}^*}(\Lambda_e)$ are naturally $R_{\mathbf{C}^*}$-modules. We will identify $R_{\mathbf{C}^*} = \mathcal{A}$ in such a way that v^m corresponds to the one dimensional representation of $\mathbf{C}^*$ in which λ acts by multiplication by λ^m.

3 Matrix Entries of the Action of the Generators $\tilde{T}_{\sigma_i}$ on $K_{\mathbf{C}^*}(\mathcal{B}_e)$

3.1

There is a unique homomorphism $n_0 \colon \mathcal{X} \to \mathbf{Z}$ such that

$$n_0(\alpha_j) = -2 \quad \text{if } j \neq i_0,\ n_0(\alpha_{i_0}) = 0.$$

For $i \in I = I'$ we define a homomorphism $n_i \colon \mathcal{X} \to \mathbf{Z}$ by $n_{i_0} = n_0$ and

$$n_i(x) = n_0({}^{\sigma_{i_1^u} \sigma_{i_2^u} \cdots \sigma_{i_t^u}} x)$$

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if $i = i_t^u$, $u \in \{1, 2, 3\}$, $0 < t \le a_u$.

If $x \in \mathcal{X}$, then the G-equivariant line bundle L_x on $\mathcal{B}$ will be regarded as a $\mathbf{C}^*$-equivariant line bundle by restriction, via the homomorphism $\mathbf{C}^* \to G$ given by $\lambda \mapsto \mathrm{Ad}\,\zeta \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. In particular, we obtain a $\mathbf{C}^*$-action on the fibre of L_x at a $\mathbf{C}^*$-fixed point on $\mathcal{B}_e$.

Lemma 3.2 *Let $i \in I$, $x \in \mathcal{X}$ and let $\mathfrak{b} \in \mu_i \subset \mathcal{B}_e^{\mathbf{C}^*}$. Then $\mathbf{C}^*$ acts on the fibre of L_x at $\mathfrak{b}$ through the character $v^{n_i(x)}$.*

We prove the result for $i = i_t^u$ with fixed u by induction on $t \ge 0$. The case $t = 0$ is left to the reader. Assume now that $t \ge 1$ and that the result is known for $t - 1$. Let $i' = i_{t-1}^u$. We have $\mathfrak{b} \in V_i$. We can find $\mathfrak{b}' \in V_i$ such that $\mathfrak{b}' \in \mu_{i'}$. Since $\mathfrak{b}, \mathfrak{b}'$ are distinct points in the same fibre of π_i, we can use [L4, 7.4] and we see that the fibre of L_x at $\mathfrak{b}$ is canonically isomorphic to the fibre of $L_{\sigma_i x}$ at $\mathfrak{b}'$. Using the induction hypothesis, we deduce that $\mathbf{C}^*$ acts on the fibre of L_x at $\mathfrak{b}$ through the character $v^{n_{i'}(^{\sigma_i}x)} = v^{n_i(x)}$. This yields the induction step. The lemma is proved.

3.3

For $i \in I$ and $m \in \mathbf{Z}$ we shall regard O_i^m as a $\mathbf{C}^*$-equivariant line bundle on V_i. (Recall that $\Pi_i = V_i$.) If $i = i_0$, we have

$$j_*(\mathbf{C}) = O_i^0 - O_i^{-1} \in K_{\mathbf{C}^*}(V_i)$$

where $j \colon \{p_{0,1}^u\} \to V_i$ is the inclusion. Moreover, $O_i^1 + O_i^{-1} = 2$ in $K_{\mathbf{C}^*}(V_i)$.

If $i \ne i_0$ (so that $i = i_t^u, 0 < t \le a_u$), we note that the $\mathbf{C}^*$-equivariant structure of O_i^m is such that the action of $\mathbf{C}^*$ on the fibre of O_i^m at μ_i is tm; we have

$$j_*(\mathbf{C}) = O_i^0 - v^{-t} O_i^{-1} \in K_{\mathbf{C}^*}(V_i), \quad j'_*(\mathbf{C}) = O_i^0 - v^t O_i^{-1} \in K_{\mathbf{C}^*}(V_i)$$

where j is the inclusion of μ_i into V_i and j' is the inclusion of the other $\mathbf{C}^*$-fixed point into V_i. (See [L4, 13.5].) Moreover, $O_i^1 + O_i^{-1} = v^t + v^{-t}$ in $K_{\mathbf{C}^*}(V_i)$.

3.4

Let o_i^m be the $\mathbf{C}^*$-equivariant coherent sheaf on $\mathcal{B}_e$ given by the direct image of O_i^m under the inclusion $V_i \subset \mathcal{B}_e$. From Lemma 1.26 we see that $K_{\mathbf{C}^*}(\mathcal{B}_e)$ is the $\mathcal{A}$-module with generators $o_i^m (i \in I, m \in \mathbf{Z})$ and relations:

$$o_{i_t^u}^0 - v^{-t} o_{i_t^u}^{-1} = o_{i_{t+1}^u}^0 - v^{t+1} o_{i_{t+1}^u}^{-1}$$

for $u \in \{1, 2, 3\}$, $0 \le t < a_u$, $o_i^{m+1} + o_i^{m-1} = (v^t + v^{-t}) o_i^m$ for $i = i_t^u$, $u \in \{1, 2, 3\}$, $0 \le t \le a_u$, $m \in \mathbf{Z}$.

It follows that

(a) *an $\mathcal{A}$-basis of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ is given by o_i^{-1} $(i \in I)$ and $p = o_{i_0}^0 - o_{i_0}^{-1}$.*

Note that

(b) $p = j_*(\mathbf{C})$

where j is the imbedding of $p_{0,1}^u$ into $\mathcal{B}_e$. (This holds for any $u \in \{1, 2, 3\}$.)

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3.5

For $x \in \mathfrak{X}$, the restriction of L_x to V_i is $v^s O_i^{\check{\alpha}_i(x)}$ where $s = n_i(x) - t\check{\alpha}_i(x)$ (with $i = i_t^u$). Indeed, the fibre of L_x at a point of μ_i is $v^{n_i(x)} = v^s v^{t\check{\alpha}_i(x)}$.

Lemma 3.6 *(a) $\theta_x p = v^{n_0(x)} p$.*
(b) If $i = i_t^u$ and $\check{\alpha}_i(x) = 1$, then $\theta_x o_i^m = v^{n_i(x)-t} o_i^{m+1}$ and $\theta_{x-\alpha_i} o_i^m = v^{n_i(x)-t} o_i^{m-1}$.

(a) follows from 3.4(b) and 3.2. In the case (b), we have by 3.5:

$$\theta_x o_i^m = v^{n_i(x)-t\check{\alpha}_i(x)} o_i^{m+\check{\alpha}_i(x)} = v^{n_i(x)-t} o_i^{m+1},$$

$$\theta_{x-\alpha_i} o_i^m = v^{n_i(x-\alpha_i)-t\check{\alpha}_i(x-\alpha_i)} o_i^{m+\check{\alpha}_i(x-\alpha_i)} = v^{n_i(x)-t} o_i^{m-1}.$$

The lemma is proved.

Lemma 3.7 *For any $i \in I - \{i_0\}$ we have $\tilde{T}_{\sigma_i} p = -v^{-1} p$.*

One can argue as in the proof of [L4, 13.11]. A slightly simpler proof goes as follows. We can find $i' = i_1^u \in I, i' \neq i$. We have $p = j_*(\mathbf{C})$ where j is the imbedding of $\{p_{0,1}^u\}$ into $\mathcal{B}_e$. Clearly, $\{p_{0,1}^u\}$ is an i-saturated subvariety of $\mathcal{B}_e$, in the sense of [L4, 10.22]. Since $p = j_*(\mathbf{C})$ (j as in 3.4(b)), it follows (see [L4, 10.22(a)]) that the $\mathcal{A}$-submodule of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ generated by p is stable under $\tilde{T}_{\sigma_i}$. Hence $\tilde{T}_{\sigma_i} p = cp$ where $c \in \mathcal{A}$. Let $x \in \mathfrak{X}$ be such that $\check{\alpha}_i(x) = 1$. We have

$$\theta_{x-\alpha_i} \tilde{T}_{\sigma_i} p = (\tilde{T}_{\sigma_i} + v^{-1} - v)\theta_x p.$$

Hence

$$c\theta_{x-\alpha_i} p = (\tilde{T}_{\sigma_i} + v^{-1} - v) v^{n_0(x)} p,$$

$$c v^{n_0(x-\alpha_i)} p = v^{n_0(x)} (c + v^{-1} - v) p,$$

$$c v^2 = c + v^{-1} - v,$$

$$c = -v^{-1}.$$

The lemma is proved.

Lemma 3.8 *For any $i \in I$ we have $\tilde{T}_{\sigma_i}(o_i^{-1}) = v o_i^{-1}$.*

In the following proof we shall consider the $\mathbf{C}^*$-action on Λ given by the same formula as for Λ_e.

For each $z \in \mathbf{C}$ we consider the $\mathbf{C}^*$-stable subvariety $V_{i,z} = \{(ze, \mathfrak{b}) \in \Lambda \mid \mathfrak{b} \in V_i\}$ of Λ. Then $pr_2 \colon V_{i,z} \to V_i$ is a $\mathbf{C}^*$-equivariant isomorphism. The line bundle O_i^{-1} on V_i can be regarded via this isomorphism as a line bundle on $V_{i,z}$. Since $V_{i,z}$ is an i-saturated subvariety of Λ, one can define as in [L4, 8.1] an $R_{\mathbf{C}^*}$-linear map $\tilde{T}_{s_i} \colon K_{\mathbf{C}^*}(V_{i,z}) \to K_{\mathbf{C}^*}(V_{i,z})$ which has the following properties:

(a) if we regard $\mathbf{C}[v, v^{-1}] \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(V_{i,z})$ as the fibres of a vector bundle over $\mathbf{C} \times \mathbf{C}^*$ (z varies in $\mathbf{C}$) then $\tilde{T}_{s_i}$ is a (semisimple) vector bundle map;

(b) for $z = 1$, $\tilde{T}_{s_i} \colon K_{\mathbf{C}^*}(V_{i,1}) \to K_{\mathbf{C}^*}(V_{i,1})$, $\tilde{T}_{\sigma_i} \colon K_{\mathbf{C}^*}(\mathcal{B}_e) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ are compatible under direct image map $K_{\mathbf{C}^*}(V_{i,1}) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ induced by $V_{i,1} = V_i \subset \mathcal{B}_e$;

(c) for $z = 0$, $\tilde{T}_{s_i}: K_{\mathbf{C}^*}(V_{i,0}) \to K_{\mathbf{C}^*}(V_{i,0})$, $\tilde{T}_{\sigma_i}: K_{\mathbf{C}^*}(\mathcal{B}_0) \to K_{\mathbf{C}^*}(\mathcal{B}_0)$ are compatible under the direct image map $K_{\mathbf{C}^*}(V_{i,0}) \to K_{\mathbf{C}^*}(\mathcal{B}_0)$ induced by $V_{i,0} \subset \mathcal{B}_0$.

Now to prove the lemma, it is enough (by (b)) to show that $\tilde{T}_{\sigma_i}(O_i^{-1}) = vO_i^{-1}$ in $K_{\mathbf{C}^*}(V_{i,1})$. Using (a), we see that it is enough to show that $\tilde{T}_{\sigma_i}(O_i^{-1}) = vO_i^{-1}$ in $K_{\mathbf{C}^*}(V_{i,0})$. Let $\mathcal{F}$ be the direct image of O_i^{-1} under the imbedding $V_{i,0} \subset \mathcal{B}_0$ (a $\mathbf{C}^*$-equivariant coherent sheaf on $\mathcal{B}_0$). Since $K_{\mathbf{C}^*}(V_{i,0}) \to K_{\mathbf{C}^*}(\mathcal{B}_0)$ (direct image) is injective, we see from (c) that it is enough to show that $\tilde{T}_{\sigma_i}(\mathcal{F}) = v\mathcal{F}$ in $K_{\mathbf{C}^*}(\mathcal{B}_0)$. It is easy to see that $\mathcal{F}$ is an $R_{\mathbf{C}^*}$-linear combination of elements of $K_{\mathbf{C}^*}(\mathcal{B}_0)$ represented by line bundles L_x on $\mathcal{B}$ such that $\check{\alpha}_i(x) = -1$. Hence it is enough to show that for any such L_x we have $\tilde{T}_{\sigma_i}(L_x) = vL_x$ in $K_{\mathbf{C}^*}(\mathcal{B}_0)$. It is also enough to show that the analogous equality holds in $K_{G\times\mathbf{C}^*}(\mathcal{B}_0)$ (equivariant structure as in [L4, 7.5]). But this follows from [L4, 7.23]. The lemma is proved.

Lemma 3.9 *Assume that $i = i_t^u, i' = i_{t-1}^u$ with $u \in \{1,2,3\}$ and $0 < t \le a_u$. Let $\tilde{p} = j_*(\mathbf{C})$ where $j: \{p_{t-1,t}^u\} \to \mathcal{B}_e$ is the inclusion. We have*

(a) $\tilde{T}_{\sigma_i}\tilde{p} = -v^{-1}\tilde{p} + (v^{t-1} - v^{-t+1})o_i^{-1}$,
(b) $\tilde{T}_{\sigma_{i'}}\tilde{p} = -v^{-1}\tilde{p} + (v^t - v^{-t})o_{i'}^{-1}$.

We prove (a). Since V_i is an i-saturated subvariety of $\mathcal{B}_e$ and the image of $K_{\mathbf{C}^*}(V_i) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ has $\mathcal{A}$-basis $\{\tilde{p}, o_i^{-1}\}$, we have $\tilde{T}_{\sigma_i}\tilde{p} = a\tilde{p} + bo_i^{-1}$ for some $a, b \in \mathcal{A}$. By 3.8 we have $\tilde{T}_{\sigma_i}o_i^{-1} = vo_i^{-1}$. The eigenvalues of the 2×2 matrix describing $\tilde{T}_{\sigma_i}$ in the basis $\{\tilde{p}, o_i^{-1}\}$ belong to $\{v, -v^{-1}\}$. Hence either $a = -v^{-1}$ or $a = v$. Moreover, if $a = v$ and $b \ne 0$, then the 2×2 matrix above is not semisimple, a contradiction. Hence there are two possibilities: either $a = v, b = 0$ or $a = -v^{-1}$.

Let $x \in \mathcal{X}$ be such that $\check{\alpha}_i(x) = 1$. We have

$$\theta_{x-\alpha_i}\tilde{T}_{\sigma_i}\tilde{p} = (\tilde{T}_{\sigma_i} + v^{-1} - v)\theta_x\tilde{p},$$

$$\theta_{x-\alpha_i}(a\tilde{p} + bo_i^{-1}) = (\tilde{T}_{\sigma_i} + v^{-1} - v)v^{n_{i'}(x)}\tilde{p},$$

$$av^{n_{i'}(x-\alpha_i)}\tilde{p} + bv^{n_i(x)-t}o_i^{-2} = v^{n_{i'}(x)}(a\tilde{p} + bo_i^{-1}) + (v^{-1} - v)v^{n_{i'}(x)}\tilde{p}.$$

Note that $n_{i'}(x - \alpha_i) = n_i(x)$ and $n_{i'}(x) = n_i(x) - 2t$. Hence

$$a\tilde{p} + bv^{-t}o_i^{-2} = v^{-2t}(a\tilde{p} + bo_i^{-1}) + (v^{-1} - v)v^{-2t}\tilde{p}.$$

Recall that $\tilde{p} = o_i^0 - v^t o_i^{-1}$. Hence

$$o_i^{-2} = -o_i^0 + (v^t + v^{-t})o_i^{-1} = -\tilde{p} + v^{-t}o_i^{-1}.$$

We deduce that

$$a\tilde{p} + bv^{-t}(-\tilde{p} + v^{-t}o_i^{-1}) = v^{-2t}(a\tilde{p} + bo_i^{-1}) + (v^{-1} - v)v^{-2t}\tilde{p}.$$

Taking the coefficient of $\tilde{p}$ we deduce

(c) $a - bv^{-t} = v^{-2t}a + (v^{-1} - v)v^{-2t}$.

Assume that $a = v, b = 0$. Then from (c) we see that $v^{2t+2} = 1$. This is impossible since $t \ge 1$. Hence we must have $a = -v^{-1}$ and then (c) yields $b = v^{t-1} - v^{-t+1}$. This completes the proof of (a).

We prove (b). Since $V_{i'}$ is an i'-saturated subvariety of $\mathcal{B}_e$ and the image of $K_{\mathbf{C}^*}(V_{i'}) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ has $\mathcal{A}$-basis $\{\tilde{p}, o_{i'}^{-1}\}$, we have $\tilde{T}_{\sigma_{i'}}\tilde{p} = a'\tilde{p} + b'o_{i'}^{-1}$ for some $a', b' \in \mathcal{A}$. By 3.8, we have $\tilde{T}_{\sigma_{i'}}o_{i'}^{-1} = vo_{i'}^{-1}$. Just as in the proof of (a), we see that there are two possibilities: either $a' = v, b' = 0$ or $a' = -v^{-1}$.

Let $x \in \mathcal{X}$ be such that $\check{\alpha}_{i'}(x) = 1$. We have

$$\theta_{x-\alpha_{i'}}\tilde{T}_{\sigma_{i'}}\tilde{p} = (\tilde{T}_{\sigma_{i'}} + v^{-1} - v)\theta_x\tilde{p},$$

$$\theta_{x-\alpha_{i'}}(a'\tilde{p} + b'o_{i'}^{-1}) = (\tilde{T}_{\sigma_{i'}} + v^{-1} - v)v^{n_{i'}(x)}\tilde{p},$$

$$a'v^{n_{i'}(x-\alpha_{i'})}\tilde{p} + b'v^{n_{i'}(x)-t+1}o_{i'}^{-2} = v^{n_{i'}(x)}(a'\tilde{p} + b'o_{i'}^{-1}) + (v^{-1} - v)v^{n_{i'}(x)}\tilde{p}.$$

Note that $n_{i'}(\alpha_{i'}) = 2(t-1)$. Hence

$$a'v^{-2t+2}\tilde{p} + b'v^{-t+1}o_{i'}^{-2} = a'\tilde{p} + b'o_{i'}^{-1} + (v^{-1} - v)\tilde{p}.$$

Recall that $\tilde{p} = o_{i'}^{0} - v^{-t+1}o_{i'}^{-1}$ hence

$$o_{i'}^{-2} = -o_{i'}^{0} + (v^{t-1} + v^{-t+1})o_{i'}^{-1} = -\tilde{p} + v^{t-1}o_{i'}^{-1}.$$

We deduce that

$$a'v^{-2t+2}\tilde{p} + b'v^{-t+1}(-\tilde{p} + v^{t-1}o_{i'}^{-1}) = a'\tilde{p} + b'o_{i'}^{-1} + (v^{-1} - v)\tilde{p}.$$

Taking the coefficient of $\tilde{p}$ we deduce

(d) $a'v^{-2t+2} + b'v^{-t+1}(-1) = a' + (v^{-1} - v)$.

Assume that $a' = v, b = 0$. Then from (c) we see that $v^{-2t+4} = 1$. Hence $t = 2$. From (a) applied to i_1^u, i_0^u (instead of i_2^u, i_1^u), we see that $\tilde{T}_{\sigma_{i'}} \colon K_{\mathbf{C}^*}(V_{i'}) \to K_{\mathbf{C}^*}(V_{i'})$ is not equal to multiplication by v. We have a contradiction. Thus we must have $a' = -v^{-1}$ and then (d) yields $b' = v^t - v^{-t}$. The lemma is proved.

The following lemma is a special case of the previous lemma (take $t = 1$).

Lemma 3.10 *We have $\tilde{T}_{\sigma_{i_0}}p = -v^{-1}p + (v - v^{-1})o_{i_0}^{-1}$.*

Lemma 3.11 *Assume that $i = i_t^u, i' = i_{t-1}^u$ with $u \in \{1,2,3\}$ and $0 < t \le a_u$. Let $\tilde{p}$ be as in 3.9. Then*

(a) $\tilde{T}_{\sigma_{i'}}o_i^{-1} = -v^{-1}o_i^{-1} - o_{i'}^{-1}$,
(b) $\tilde{T}_{\sigma_i}o_{i'}^{-1} = -v^{-1}o_{i'}^{-1} - o_i^{-1}$.

Clearly, $V_i \cup V_{i'}$ is an i-saturated and i'-saturated subvariety of $\mathcal{B}_e$. Hence the $\mathcal{A}$-submodule $\mathcal{V}$ of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ with basis $\{o_i^{-1}, \tilde{p}, o_{i'}^{-1}\}$ is stable under the operators $\tilde{T}_{\sigma_{i'}}, \tilde{T}_{\sigma_i}$.

We prove (a). This proof is a generalization of that of [L4, 13.13]. We have $\tilde{T}_{\sigma_{i'}}o_i^{-1} = ao_i^{-1} + b\tilde{p} + co_{i'}^{-1}$ for some $a, b, c \in \mathcal{A}$.

Let $x \in \mathcal{X}$ be such that $\check{\alpha}_i(x) = \check{\alpha}_{i'}(x) = 1$. We have $\theta_{x-\alpha_{i'}}\tilde{T}_{\sigma_{i'}}o_i^{-1} = (\tilde{T}_{\sigma_{i'}} + v^{-1} - v)\theta_x o_i^{-1}$,

$$\begin{aligned}
\theta_{x-\alpha_{i'}}(ao_i^{-1} + b\tilde{p} + co_{i'}^{-1}) &= v^{n_i(x)-t}(\tilde{T}_{\sigma_{i'}} + v^{-1} - v)o_i^{0} \\
&= v^{n_i(x)-t}(\tilde{T}_{\sigma_{i'}} + v^{-1} - v)(\tilde{p} + v^t o_i^{-1}) \\
&= v^{n_i(x)-t}\big(-v^{-1}\tilde{p} + (v^t - v^{-t})o_{i'}^{-1} + v^t(ao_i^{-1} + b\tilde{p} + co_{i'}^{-1}) \\
&\qquad + (v^{-1} - v)\tilde{p} + (v^{-1} - v)v^t o_i^{-1}\big).
\end{aligned}$$

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Now

$$\theta_{x-\alpha_{i'}} o_i^{-1} = \theta_x \theta_{-\alpha_{i'}} o_i^{-1} = v^{n_i(-\alpha_{i'})-t} \theta_x o_i^0 = v^{n_i(-\alpha_{i'})-t} v^{n_i(x)-t} o_i^1$$
$$= v^{2-t} v^{n_i(x)-t} o_i^1,$$

$$\theta_{x-\alpha_{i'}} \tilde{p} = v^{n_{i'}(x-\alpha_{i'})} \tilde{p} = v^{n_i(x)-2t-2(t-1)} \tilde{p},$$
$$\theta_{x-\alpha_{i'}} o_{i'}^{-1} = v^{n_{i'}(x)-t+1} o_{i'}^{-2} = v^{n_i(x)-2t-t+1} o_{i'}^{-2},$$

hence

$$av^{2-t} o_i^1 + bv^{-3t+2} \tilde{p} + cv^{-2t+1} o_{i'}^{-2}$$
$$= -v^{-1}\tilde{p} + (v^t - v^{-t}) o_{i'}^{-1} + v^t (a o_i^{-1} + b\tilde{p} + c o_{i'}^{-1}) + (v^{-1} - v)\tilde{p} + (v^{-1} - v) v^t o_i^{-1}.$$

We have

$$o_{i'}^{-2} = -\tilde{p} + v^{t-1} o_{i'}^{-1},$$
$$o_i^1 = -o_i^{-1} + (v^t + v^{-t}) o_i^0 = (v^t + v^{-t})\tilde{p} + v^{2t} o_i^{-1},$$

hence

$$av^{2-t}\big((v^t + v^{-t})\tilde{p} + v^{2t} o_i^{-1}\big) + bv^{-3t+2}\tilde{p} + cv^{-2t+1}(-\tilde{p} + v^{t-1} o_{i'}^{-1})$$
$$= -v^{-1}\tilde{p} + (v^t - v^{-t}) o_{i'}^{-1} + v^t (a o_i^{-1} + b\tilde{p} + c o_{i'}^{-1}) + (v^{-1} - v)\tilde{p} + (v^{-1} - v) v^t o_i^{-1},$$

which yields $a = -v^{-1}$, $c = -1$, $b = 0$. This proves (a).

We prove (b). From

$$\tilde{T}_{\sigma_{i'}} o_{i'}^{-1} = v o_{i'}^{-1},$$
$$\tilde{T}_{\sigma_{i'}} o_i^{-1} = -v^{-1} o_i^{-1} - o_{i'}^{-1},$$
$$\tilde{T}_{\sigma_{i'}} \tilde{p} = -v^{-1}\tilde{p} + (v^t - v^{-t}) o_{i'}^{-1},$$

we see that $\{\xi \in \mathcal{V} \mid \tilde{T}_{\sigma_{i'}} \xi = v\xi\} = \mathcal{A} o_{i'}^{-1}$. Since

(c) $\tilde{T}_{\sigma_i} = \tilde{T}_{\sigma_{i'}}^{-1} \tilde{T}_{\sigma_i}^{-1} \tilde{T}_{\sigma_{i'}} \tilde{T}_{\sigma_i} \tilde{T}_{\sigma_{i'}}$,

it follows that

(d) $\{\xi \in \mathcal{V} \mid \tilde{T}_{\sigma_i} \xi = v\xi\}$

is the $\mathcal{A}$-submodule generated by a single element of $\mathcal{V}$. Since this submodule contains o_i^{-1} it must be equal to $\mathcal{A} o_i^{-1}$. Now $\tilde{T}_{\sigma_i} o_{i'}^{-1} + v^{-1} o_{i'}^{-1}$ clearly belongs to (d), hence

(e) $\tilde{T}_{\sigma_i} o_{i'}^{-1} = -v^{-1} o_{i'}^{-1} + y o_i^{-1}$

for some $y \in \mathcal{A}$. Using (e) and (a) we compute

$$\tilde{T}_{\sigma_i} \tilde{T}_{\sigma_{i'}} \tilde{T}_{\sigma_i} o_{i'}^{-1} = (-1 - y)(-v^{-1} o_{i'}^{-1} + y o_i^{-1}) - y o_i^{-1},$$
$$\tilde{T}_{\sigma_{i'}} \tilde{T}_{\sigma_i} \tilde{T}_{\sigma_{i'}} o_{i'}^{-1} = -v o_{i'}^{-1} + yv(-v^{-1} o_i^{-1} - o_{i'}^{-1}).$$

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Since $\tilde T_{\sigma_i}\tilde T_{\sigma_{i'}}\tilde T_{\sigma_i}o_{i'}^{-1}=\tilde T_{\sigma_{i'}}\tilde T_{\sigma_i}\tilde T_{\sigma_{i'}}o_{i'}^{-1}$, we have

$$(-1-y)(-v^{-1}o_{i'}^{-1}+yo_i^{-1})-yo_i^{-1}=-vo_{i'}^{-1}+yv(-v^{-1}o_i^{-1}-o_{i'}^{-1}).$$

We pick the coefficient of $o_{i'}^{-1}$ in both sides. We get $y=-1$. Hence (e) reduces to (b). The lemma is proved.

Lemma 3.12 *Assume that $i,i'\in I$ satisfy $i\cdot i'=0$. Then $\tilde T_{\sigma_i}(o_{i'}^{-1})=-v^{-1}o_{i'}^{-1}$.*

Note that $V_{i'}$ is an i-saturated and i'-saturated subvariety of $\mathcal{B}_e$. Hence the image of $K_{\mathbf{C}^*}(V_{i'})\to K_{\mathbf{C}^*}(\mathcal{B}_e)$ is stable under $\tilde T_{\sigma_i}$ and under $\tilde T_{\sigma_{i'}}$. The set of vectors in this image that are annihilated by $\tilde T_{\sigma_{i'}}-v$ consists of all $\mathcal{A}$-multiples of $o_{i'}^{-1}$. (This follows from 3.8, 3.9.) This set is stable under the action of $\tilde T_{\sigma_i}$ since $\tilde T_{\sigma_i},\tilde T_{\sigma_{i'}}$ commute. It follows that

(a) $\tilde T_{\sigma_i}o_{i'}^{-1}=a_{i,i'}o_{i'}^{-1}$ for some $a_{i,i'}\in\mathcal{A}$.

We show that,

(b) *if $i'=i_t^u$ where $t>0$, then $\tilde T_{\sigma_i}\colon K_{\mathbf{C}^*}(V_{i'})\to K_{\mathbf{C}^*}(V_{i'})$ is scalar multiplication by $a_{i,i'}$.*

Let p',p'' be the two $\mathbf{C}^*$-fixed points on $V_{i'}$. Note that $\{p'\}$ and $\{p''\}$ are i-saturated subvarieties of $\mathcal{B}_e$. It follows that $\tilde T_{\sigma_i}p'=a'p'$, $\tilde T_{\sigma_i}p''=a''p''$ in $K_{\mathbf{C}^*}(\mathcal{B}_e)$ where $a',a''\in\mathcal{A}$. (We denote the direct image of $\mathbf{C}$ under the direct image map $K_{\mathbf{C}^*}(p')\to K_{\mathbf{C}^*}(\mathcal{B}_e)$ again by p'; we use a similar notation for p''.) We may arrange notation so that $p'=o_{i'}^0-v^{-t}o_{i'}^{-1}$, $p''=o_{i'}^0-v^to_{i'}^{-1}$. Hence $p'-p''=(v^t-v^{-t})o_{i'}^{-1}$. Applying $\tilde T_{\sigma_i}$ yields $a'p'-a''p''=a_{i,i'}(v^t-v^{-t})o_{i'}^{-1}$. Hence $a_{i,i'}(p'-p'')=a'p'-a''p''$. Now p,p' are linearly independent in $K_{\mathbf{C}^*}(\mathcal{B}_e)$ over the field of quotients of R_H (since $t\neq 0$). It follows that $a'=a''=a_{i,i'}$. This proves (b). In particular, in the setup of (b) we have

(c) $\tilde T_{\sigma_i}p'=a_{i,i'}p'$, $\tilde T_{\sigma_i}p''=a_{i,i'}p''$.

Let π be the $\mathbf{C}^*$-fixed point on V_j where $j=i_1^u$ with $\pi\notin V_{i_0}$. (We denote the direct image of $\mathbf{C}$ under the direct image map $K_{\mathbf{C}^*}(\pi)\to K_{\mathbf{C}^*}(\mathcal{B}_e)$ again by π. We have

$$p=o_j^0-vo_j^{-1},\pi=o_j^0-v^{-1}o_j^{-1}=p+(v-v^{-1})o_j^{-1}.$$

Recall that

$$\tilde T_{\sigma_{i_0}}p=-v^{-1}p+(v-v^{-1})o_{i_0}^{-1},\quad \tilde T_{\sigma_{i_0}}o_j^{-1}=-v^{-1}o_j^{-1}-o_{i_0}^{-1}$$

(see Lemmas 3.9, 3.11) so that

$$\begin{aligned}\tilde T_{\sigma_{i_0}}\pi=\tilde T_{\sigma_{i_0}}(p+(v-v^{-1})o_j^{-1})&=-v^{-1}p+(v-v^{-1})o_{i_0}^{-1}+(v-v^{-1})(-v^{-1}o_j^{-1}-o_{i_0}^{-1})\\&=-v^{-1}(p+(v-v^{-1})o_j^{-1})=-v^{-1}\pi.\end{aligned}$$

Thus,

(d) $\tilde T_{\sigma_{i_0}}\pi=-v^{-1}\pi$.

We now show that

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(e) $a_{i_0,i^u_t} = -v^{-1}$ for any $t \geq 2$.

We argue by induction on t. Assume first that $t = 2$. Then the intersection $V_{i^u_t} \cap V_{i^u_{t-1}}$ is on the one hand the point π above and on the other hand it is one of the points p', p'' in (c) (with $i = i_0, i' = i^u_2$). Hence from (c), (d) we deduce that $a_{i_0,i^u_2} = -v^{-1}$. Assume now that $t \geq 3$. Consider the point $\tilde{p} = V_{i^u_t} \cap V_{i^u_{t-1}}$. Then $\tilde{p}$ is one of the points p', p'' in (c) (with $i = i_0, i' = i^u_t$) and also one of the points p', p'' in (c) (with $i = i_0, i' = i^u_{t-1}$). Hence from (c) we deduce that $a_{i_0,i^u_t} = a_{i_0,i^u_{t-1}}$. By the induction hypothesis we have $a_{i_0,i^u_{t-1}} = -v^{-1}$. It follows that $a_{i_0,i^u_t} = -v^{-1}$. This proves (e).

From the identities

$$\tilde{T}_{\sigma_{i_0}} o_{i'}^{-1} = -v^{-1} o_{i'}^{-1} \quad \text{for } i' = i^u_t,\ t \geq 2,$$
$$\tilde{T}_{\sigma_{i_0}} o_{i'}^{-1} = -v^{-1} o_{i'}^{-1} - o_{i_0}^{-1} \quad \text{for } i' = i^u_1,$$
$$\tilde{T}_{\sigma_{i_0}} p = -v^{-1} p + (v - v^{-1}) o_{i_0}^{-1},$$
$$\tilde{T}_{\sigma_{i_0}} o_{i_0}^{-1} = v o_{i_0}^{-1},$$

we see that the trace of $\tilde{T}_{\sigma_{i_0}} \colon K_{\mathbf{C}^*}(\mathcal{B}_e) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ is $v - |I|v^{-1}$. If $i \in I$, then the automorphisms $\tilde{T}_{\sigma_i}, \tilde{T}_{\sigma_{i_0}}$ of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ are conjugate under an automorphism of $K_{\mathbf{C}^*}(\mathcal{B}_e)$. (This follows by using several times 3.11(c) and the fact that the Coxeter graph is connected.) It follows that

(f) *for $i \in I$, the trace of $\tilde{T}_{\sigma_i} \colon K_{\mathbf{C}^*}(\mathcal{B}_e) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ is $v - |I|v^{-1}$.*

Assume now that $i \neq i_0$. From the identities

$$\tilde{T}_{\sigma_i} o_j^{-1} = a_{i,j} o_j^{-1} \quad \text{if } i \cdot j = 0,$$
$$\tilde{T}_{\sigma_i} o_j^{-1} = -v^{-1} o_j^{-1} - o_i^{-1} \quad \text{if } i \cdot j = -1,$$
$$\tilde{T}_{\sigma_i} o_i^{-1} = v o_i^{-1},$$
$$\tilde{T}_{\sigma_i} p = -v^{-1} p,$$

we see that the trace of $\tilde{T}_{\sigma_i} \colon K_{\mathbf{C}^*}(\mathcal{B}_e) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ is equal to

$$\sum_{j; i \cdot j = 0} a_{i,j} - v^{-1} n' - v^{-1} + v$$

where n' is the number of elements $j \in I$ such that $i \cdot j = -1$. Comparing with (f) we see that $\sum_{j; i \cdot j = 0} (a_{i,j} + v^{-1}) = 0$. Since $a_{i,j} \in \{v, -v^{-1}\}$, we deduce that $a_{i,j} = -v^{-1}$ for all j such that $i \cdot j = 0$. The lemma is proved.

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4 Action of $\tilde{T}_{w_0}^{\pm 1}$ on $K_{\mathbf{C}^*}(\mathcal{B}_e)$

4.1

For $i \in I$ we set $A_i = \frac{\bar{B}_i - B_i}{v^{h'} + v^{-h'}} \in \mathbf{Q}(v)$.

Lemma 4.2 *We have*

$$(v + v^{-1})A_i = \sum_{j \in I; i \cdot j = -1} A_j, \quad \text{if } i \in I - \{i_0\},$$

$$(v + v^{-1})A_i = \sum_{j \in I; i \cdot j = -1} A_j - (v - v^{-1}), \quad \text{if } i = i_0.$$

This follows immediately from the identities defining B_i, using $\bar{B}_\heartsuit = B_\heartsuit$.

4.3

Let $\nu = l(w_0)$. Let w_1 be a *Coxeter element* in W (see [C]) and let $\Delta \in \mathcal{A}$ be the determinant of $v - v^{-1}w_1$ in the reflection representation of W. For any integer $m \geq 0$ we set $[m] = \frac{v^m - v^{-m}}{v - v^{-1}} \in \mathcal{A}$.

Lemma 4.4 *For any $i \in I$ we have*

$$A_i = -(v - v^{-1})\Delta^{-1} \frac{[a_u + 1 - t]}{[a_u + 1]} \prod_{u' \in \{1,2,3\}} [a_{u'} + 1] \in \mathbf{Q}(v)$$

where $i = i_t^u$.

One can check that the elements above form a solution of the equations in 4.2. We then use the uniqueness of such a solution.

Lemma 4.5 *Let $i \mapsto i^*$ be the involution of I defined by $w_0 \sigma_i w_0^{-1} = \sigma_{i^*}$. The action of $\tilde{T}_{w_0}^{-1}$ on $K_{\mathbf{C}^*}(\mathcal{B}_e)$ is as follows.*
(a) $\tilde{T}_{w_0}^{-1}(o_i^{-1}) = -(-v)^{\nu - 2h'} o_{i^*}^{-1}$ for all $i \in I$,
(b) $\tilde{T}_{w_0}^{-1}(p) = (-v)^\nu p + (-v)^\nu (1 + v^{-2h'}) \sum_{j \in I} A_j o_j^{-1}$.

Let $\mathcal{M}$ be the $\mathcal{A}$-submodule of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ with basis $\{o_i^{-1} \mid i \in I\}$. Note that $\mathcal{M}$ is an $\mathcal{H}_0$-submodule of $K_{\mathbf{C}^*}(\mathcal{B}_e)$. Since the set of vectors $m \in \mathcal{M}$ satisfying $\tilde{T}_{\sigma_i} m = vm$ is equal to $\mathcal{A} o_i^{-1}$ and $\tilde{T}_{w_0} \tilde{T}_{\sigma_i} \tilde{T}_{w_0}^{-1} = \tilde{T}_{\sigma_{i^*}}$, it follows that $\tilde{T}_{w_0}(\mathcal{A} o_i^{-1}) = \mathcal{A} o_{i^*}^{-1}$. Hence $\tilde{T}_{w_0} o_i^{-1} = b_i o_{i^*}^{-1}$ where $b_i \in \mathcal{A}$. Note that b_i is invertible in $\mathcal{A}$ since $\tilde{T}_{w_0} \colon \mathcal{M} \to \mathcal{M}$ is an isomorphism.

We show that b_i is independent of i. Assume that $j \in I$, $i \cdot j = -1$. We have $\tilde{T}_{\sigma_i} o_j^{-1} = -v^{-1} o_j^{-1} - o_i^{-1}$, hence

$$\tilde{T}_{w_0} \tilde{T}_{\sigma_i} o_j^{-1} = -v^{-1} \tilde{T}_{w_0} o_j^{-1} - \tilde{T}_{w_0} o_i^{-1},$$

$$\tilde{T}_{\sigma_{i^*}} \tilde{T}_{w_0} o_j^{-1} = \tilde{T}_{\sigma_{i^*}} b_j o_{j^*}^{-1} = -v^{-1} b_j o_{j^*}^{-1} - b_i o_{i^*}^{-1},$$

$$\tilde{T}_{\sigma_{i^*}} o_{j^*}^{-1} = -v^{-1} o_{j^*}^{-1} - b_i b_j^{-1} o_{i^*}^{-1}.$$

Since $b_i b_j^{-1} \neq 0$, it follows that $b_i b_j^{-1} = 1$. Since the Coxeter graph is connected, it follows that b_i is indeed independent of i. Thus there exists an invertible element $\epsilon v^c \in \mathcal{A}$ (with $\epsilon \in \{1, -1\}, c \in \mathbf{Z}$) such that $\tilde{T}_{w_0} o_i^{-1} = \epsilon v^c o_{i^*}^{-1}$ for all $i \in I$. The determinant of $\tilde{T}_{w_0} \colon \mathcal{M} \to \mathcal{M}$ is on the one hand equal to $\pm(v^c)^{|I|}$ (the determinant of a monomial matrix), and on the other hand is equal to the ν-th power of the determinant of $\tilde{T}_{\sigma_i} \colon \mathcal{M} \to \mathcal{M}$ where $i \in I$, that is, to $((-1)^{|I|-1} v^{-|I|+2})^{\nu}$. Thus, $\pm v^{c|I|} = ((-1)^{|I|-1} v^{-|I|+2})^{\nu}$. It follows that $c = (-|I| + 2)\nu/|I| = -\nu + 2h'$. To determine the sign ϵ, we specialize $v = 1$. Under this specialization, $\mathcal{M}$ becomes the reflection representation of W tensor the sign representation. The trace of w_0 on this representation is well known to be $-(-1)^{\nu} \sharp\{i \in I \mid i = i^*\}$. On the other hand, we have $w_0 o_i^{-1} = \epsilon o_{i^*}^{-1}$ for all $i \in I$. Hence the trace of w_0 is $\epsilon \sharp\{i \in I \mid i = i^*\}$. Since $\sharp\{i \in I \mid i = i^*\} \neq 0$, it follows that $\epsilon = -(-1)^{\nu}$. This proves (a).

We prove (b). Let

$$\xi = p + \sum_{j \in I} A_j o_j^{-1} \in \mathbf{Q}(v) \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\mathcal{B}_e).$$

The equations in 4.2 show that

$$\tilde{T}_{\sigma_i} \xi = -v^{-1} \xi \quad \text{for all } i \in I.$$

It follows that $\tilde{T}_{w_0}^{-1}(\xi) = (-v)^{\nu} \xi$ or equivalently

$$\tilde{T}_{w_0}^{-1} \Big(p + \sum_{j \in I} A_j o_j^{-1} \Big) = (-v)^{\nu} \Big(p + \sum_{j \in I} A_j o_j^{-1} \Big).$$

Note that $A_{j^*} = A_j$. Using (a), we deduce that

$$\tilde{T}_{w_0}^{-1} p - (-v)^{\nu - 2h'} \sum_{j \in I} A_j o_j^{-1} = (-v)^{\nu} \Big(p + \sum_{j \in I} A_j o_j^{-1} \Big),$$

and (b) follows. The lemma is proved.

Lemma 4.6 *The action of $\tilde{T}_{w_0}$ on $K_{\mathbf{C}^*}(\mathcal{B}_e)$ is as follows.*

(a) $\tilde{T}_{w_0}(o_i^{-1}) = -(-v)^{-\nu+2h'} o_{i^*}^{-1}$ for all $i \in I$,

(b) $\tilde{T}_{w_0}(p) = (-v)^{-\nu} p + (-v)^{-\nu}(1 + v^{2h'}) \sum_{j \in I} A_j o_j^{-1}$.

(a) follows immediately from 4.5(a). We prove (b). If ξ is as in 4.5, we have $\tilde{T}_{w_0}(\xi) = (-v)^{-\nu} \xi$, or equivalently

$$\tilde{T}_{w_0} p - (-v)^{-\nu + 2h'} \sum_{j \in I} A_j o_j^{-1} = (-v)^{-\nu} \Big(p + \sum_{j \in I} A_j o_j^{-1} \Big).$$

(b) follows. The lemma is proved.

Lemma 4.7 *Let $\mathbf{p} = p - \sum_{j \in I} B_j v^{-h'} o_j^{-1}$. We have*

$$\tilde{T}_{w_0} \mathbf{p} = (-v)^{-\nu} \Big(p + \sum_{j \in I} v^{h'} \bar{B}_j o_j^{-1} \Big).$$

Using Lemma 4.6, we have

$$\begin{aligned}\tilde{T}_{w_0}\mathbf{p} &= \tilde{T}_{w_0}\Big(p - \sum_j B_j v^{-h'} o_j^{-1}\Big) \\ &= (-v)^{-\nu} p + \sum_j (-v)^{-\nu} v^{h'}(\bar{B}_j - B_j) o_j^{-1} + \sum_j B_j v^{-h'} (-v)^{-\nu+2h'} o_j^{-1},\end{aligned}$$

as desired.

5 Inner Product on $K_{\mathbf{C}^*}(\mathcal{B}_e)$

Lemma 5.1 *Consider an $R_{\mathbf{C}^*}$-bilinear inner product $(,)$ on $K_{\mathbf{C}^*}(\mathcal{B}_e)$ with values in $R_{\mathbf{C}^*} = \mathcal{A}$ such that $(\chi\xi, \xi') = (\xi, \chi^{\blacktriangle}\xi')$ and $(\xi, \xi') = (\xi', \xi)$ for $\xi, \xi' \in K_{\mathbf{C}^*}(\mathcal{B}_e), \chi \in \mathcal{H}$. There exists $c \in \mathcal{A}$ such that*

(a) $(o_i^{-1}, o_j^{-1}) = c$ for $i, j \in I$ such that $i \cdot j = -1$,
(b) $(o_i^{-1}, o_i^{-1}) = -[2]c$ for all $i \in I$,
(c) $(o_i^{-1}, o_j^{-1}) = 0$ for $i, j \in I$ such that $i \cdot j = 0$,
(d) $(p, o_i^{-1}) = 0$ for $i \in I - \{i_0\}$,
(e) $(p, o_{i_0}^{-1}) = -c(v - v^{-1})$,
(f) $(p, p) = cv^{-2h'}(1 + v^{2h'})A_{i_0}(v - v^{-1})$.

Assume that $i \cdot j = -1$. We have $(\tilde{T}_{\sigma_i} o_j^{-1}, o_i^{-1}) = (o_j^{-1}, \tilde{T}_{\sigma_i} o_i^{-1})$, hence

$$(-v^{-1}o_j^{-1} - o_i^{-1}, o_i^{-1}) = (o_j^{-1}, v o_i^{-1}), \quad (o_i^{-1}, o_i^{-1}) = -(v + v^{-1})(o_j^{-1}, o_i^{-1}).$$

Similarly, $(o_j^{-1}, o_j^{-1}) = -(v+v^{-1})(o_j^{-1}, o_i^{-1})$; hence there exists $c \in \mathcal{A}$ so that (a),(b) hold.

Assume that $i \cdot j = 0$. We have

$$(\tilde{T}_{\sigma_i} o_j^{-1}, o_i^{-1}) = (o_j^{-1}, \tilde{T}_{\sigma_i} o_i^{-1}), \quad (-v^{-1}o_j^{-1}, o_i^{-1}) = (o_j^{-1}, v o_i^{-1}).$$

Hence $(v + v^{-1})(o_j^{-1}, o_i^{-1}) = 0$ and (c) follows. For $i \neq i_0$, we have

$$(\tilde{T}_{\sigma_i} p, o_i^{-1}) = (p, \tilde{T}_{\sigma_i} o_i^{-1}), \quad (-v^{-1}p, o_i^{-1}) = (p, v o_i^{-1})$$

and (d) follows. We have

$$(\tilde{T}_{\sigma_{i_0}} p, o_{i_0}^{-1}) = (p, \tilde{T}_{\sigma_{i_0}} o_{i_0}^{-1}), \quad (-v^{-1}p + (v - v^{-1})o_{i_0}^{-1}, o_{i_0}^{-1}) = (p, v o_{i_0}^{-1}).$$

Hence

$$(v + v^{-1})(p, o_{i_0}^{-1}) = (v - v^{-1})(o_{i_0}^{-1}, o_{i_0}^{-1}) = -c(v - v^{-1})(v + v^{-1})$$

and (e) follows.

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Let $x \in \mathcal{X}$ be such that $\check{\alpha}_0(x) = 1$. We have

$$\theta_x o_{i_0}^{-1} = v^{n_0(x)} o_{i_0}^{0} = v^{n_0(x)}(o_{i_0}^{-1} + p).$$

Using Lemma 4.6 we have

$$(\theta_x p, o_{i_0}^{-1}) = (p, \tilde{T}_{w_0}^{-1}\theta_{-^{w_0}x}\tilde{T}_{w_0} o_{i_0}^{-1}) = (\tilde{T}_{w_0}^{-1} p, -(-v)^{-\nu+2h'}\theta_{-^{w_0}x} o_{i_0}^{-1}),$$

hence

$$\begin{aligned} &v^{n_0(x)}(p, o_{i_0}^{-1}) \\ &\quad = \Big((-v)^{\nu} p + (-v)^{\nu}(1 + v^{-2h'}) \sum_{j \in I} A_j o_j^{-1}, -(-v)^{-\nu+2h'} v^{n_0(x)}(o_{i_0}^{-1} + p)\Big), \end{aligned}$$

$(p, o_{i_0}^{-1}) = (v^{2h'} p + (1 + v^{2h'}) \sum_{j \in I} A_j o_j^{-1}, -o_{i_0}^{-1} - p)$. Using now (a)–(e), we deduce

$$\begin{aligned} &- c(v - v^{-1}) \\ &\quad = v^{2h'} c(v - v^{-1}) - v^{2h'}(p, p) \\ &\qquad - (1 + v^{2h'}) \sum_{j; j \cdot i_0 = -1} A_j c + (1 + v^{2h'}) A_{i_0} c(v + v^{-1}) + (1 + v^{2h'}) A_{i_0} c(v - v^{-1}). \end{aligned}$$

Here we substitute $\sum_{j; j\cdot i_0 = -1} A_j = (v + v^{-1}) A_{i_0} + (v - v^{-1})$ and we obtain (f). The lemma is proved.

5.2

Let $(|)_{\mathcal{B}_e} \colon K_{\mathbf{C}^*}(\mathcal{B}_e) \times K_{\mathbf{C}^*}(\mathcal{B}_e) \to R_{\mathbf{C}^*} = \mathcal{A}$ be the $R_{\mathbf{C}^*}$-bilinear inner product defined in [L4, 12.16]. According to [L4, 12.17], we have

$$(\xi \mid \xi')_{\mathcal{B}_e} = (\xi' \mid \xi)_{\mathcal{B}_e},$$
$$(\chi\xi \mid \xi')_{\mathcal{B}_e} = (\xi \mid \chi^{\blacktriangle}\xi')_{\mathcal{B}_e},$$

for all $\xi, \xi' \in K_{\mathbf{C}^*}(\mathcal{B}_e), \chi \in \mathcal{H}$. Hence Lemma 5.1 is applicable to $(,) = (|)_{\mathcal{B}_e}$. We show that in this case, c from Lemma 5.1 is given by

(a) $c = -v^{2h'-1}$.

It is enough to show that $(o_i^{-1} \mid o_{i_0}^{-1})_{\mathcal{B}_e} = -v^{2h'-1}$ for $i = i_1^u$. By definition, we have

$$(\xi \mid \xi')_{\mathcal{B}_e} = \big(\xi \parallel k_*(\xi')\big)$$

where $k \colon \mathcal{B}_e \to \Lambda_e$ is the inclusion and $(\|) \colon K_{\mathbf{C}^*}(\mathcal{B}_e) \times K_{\mathbf{C}^*}(\Lambda_e) \to R_{\mathbf{C}^*}$ is given by

$$(\xi \parallel \tilde{\xi}) = (-v)^{\nu-2}\big(\xi \tilde{T}_{w_0}\varpi^*(\tilde{\xi})\big) = (-v)^{\nu-2}\big(\tilde{T}_{w_0}\varpi^*(\xi) : \tilde{\xi}\big);$$

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$\varpi\colon \mathcal{B}_e \to \mathcal{B}_e$ and $\varpi\colon \Lambda_e \to \Lambda_e$ are the involutions defined in [L4, 12.6] and $(:)\colon K_{\mathbf{C}^*}(\mathcal{B}_e) \times K_{\mathbf{C}^*}(\Lambda_e) \to R_{\mathbf{C}^*}$ is the "intersection product" in Λ_e (see [L4, 12.11]).

Since V_{i_0}, V_i intersect transversally in Λ_e (at $p^u_{0,1}$), we have $\big(o_i^{-1} : k_*(o_{i_0}^{-1})\big) = v^N$ where N is the weight of the $\mathbf{C}^*$-action on the tensor product of the fibres of $O_i^{-1}, O_{i_0}^{-1}$ at $p^u_{0,1}$, that is, $N = 0 + 1 = 1$. We have $\varpi^*(o_{i_0}^{-1}) = o_{i_0}^{-1}$ and $\tilde{T}_{w_0} o_{i_0}^{-1} = -(-v)^{-\nu+2h'} o_{i_0}^{-1}$, hence

$$(o_i^{-1} \mid o_{i_0}^{-1})_{\mathcal{B}_e} = (-v)^{\nu-2}(o_i^{-1} : -(-v)^{-\nu+2h'} o_{i_0}^{-1}) = -(-v)^{2h'-2} v^N = -v^{2h'-1}.$$

Thus, (a) is proved.

5.3

Using 3.4(a), we see that

(a) *an $\mathcal{A}$-basis of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ is given by $v^{-h'} o_i^{-1} (i \in I)$ and* $\mathbf{p}$ (see 4.7).

Lemma 5.4 *We have*

(a) $(v^{-h'} o_i^{-1} \mid v^{-h'} o_j^{-1})_{\mathcal{B}_e} = -v^{-1}$ *for $i, j \in I$ such that $i \cdot j = -1$,*
(b) $(v^{-h'} o_i^{-1} \mid v^{-h'} o_i^{-1})_{\mathcal{B}_e} = 1 + v^{-2}$ *for all $i \in I$,*
(c) $(v^{-h'} o_i^{-1} \mid v^{-h'} o_j^{-1})_{\mathcal{B}_e} = 0$ *for $i, j \in I$ such that $i \cdot j = 0$,*
(d) $(\mathbf{p} \mid v^{-h'} o_i^{-1})_{\mathcal{B}_e} = -v^{-1}$ *for $i \in I$ such that $i \cdot \heartsuit = -1$,*
(e) $(\mathbf{p} \mid v^{-h'} o_i^{-1})_{\mathcal{B}_e} = 0$ *for $i \in I$ such that $i \cdot \heartsuit = 0$,*
(f) $(\mathbf{p} \mid \mathbf{p})_{\mathcal{B}_e} = 1 + v^{-2}$.

The proof is based on Lemma 5.1 and 5.2(a). Thus, (a), (b), (c) follow from 5.1(a), (b), (c). Now (d), (e) follow from 5.1(a)–(e), using the equations defining B_i. Finally, (f) is proved using 5.1(a)–(f) by a brute force computation using the explicit values of B_i given in the tables in 1.10.

6 The Canonical Signed Basis of $K_{\mathbf{C}^*}(\mathcal{B}_e)$

6.1

Let $\bar{\ }\colon K_{\mathbf{C}^*}(\mathcal{B}_e) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ be the involution defined in [L4, 12.9]. This is antilinear with respect to the involution of $\mathcal{A}$ given by restricting $\bar{\ }\colon \mathbf{Q}(v) \to \mathbf{Q}(v)$. (See 1.11.) Recall that

$$\bar{\xi} = (-v)^{-\nu} \tilde{T}_{w_0}^{-1} \varpi^* D_{\mathcal{B}_e}(\xi)$$

where $D_{\mathcal{B}_e}\colon K_{\mathbf{C}^*}(\mathcal{B}_e) \to K_{\mathbf{C}^*}(\mathcal{B}_e)$ is the Serre-Grothendieck duality (see [L4, 6.10]).

Lemma 6.2 *We have*

(a) $\overline{v^{-h'} o_i^{-1}} = v^{-h'} o_i^{-1}$ *for all $i \in I$,*
(b) $\bar{\mathbf{p}} = \mathbf{p}$.

Using [L4, 6.11, 6.12], we see that $D_{\mathcal{B}_e}(o_i^{-1}) = -o_i^{-1}$. Note also that $\varpi^* o_i^{-1} = o_{i^*}^{-1}$. Hence

$$\overline{v^{-h'} o_i^{-1}} = -v^{h'} (-v)^{-\nu} \tilde{T}_{w_0}^{-1} o_{i^*}^{-1} = v^{h'} (-v)^{-\nu} (-v)^{\nu-2h'} o_i^{-1}$$

and (a) follows.

We have $D_{\mathcal{B}_e}(p) = p$ and $\varpi^*(p) = p$ hence

$$\bar{p} = (-v)^{-\nu}\tilde{T}_{w_0}^{-1}(p) = p + (1 + v^{-2h'})\sum_{j\in I} A_j o_j^{-1} = p + \sum_{j\in I}(\bar{B}_j - B_j)v^{-h'}o_j^{-1}$$

$$= p - \sum_{j\in I} B_j v^{-h'} o_j^{-1} + \overline{\sum_{j\in I} B_j v^{-h'} o_j^{-1}}.$$

Thus,

$$\overline{p - \sum_{j\in I} B_j v^{-h'} o_j^{-1}} = p - \sum_{j\in I} B_j v^{-h'} o_j^{-1}.$$

The lemma is proved.

6.3

As in [L4, 12.18] we set

$$\mathbf{B}^{\pm}_{\mathcal{B}_e} = \{\xi \in K_{\mathbf{C}^*}(\mathcal{B}_e) \mid \bar{\xi} = \xi, (\xi \mid \xi)_{\mathcal{B}_e} \in 1 + v^{-1}\mathbf{Z}[v^{-1}]\}.$$

Theorem 6.4 *$\mathbf{B}^{\pm}_{\mathcal{B}_e}$ is the signed basis of the $\mathcal{A}$-module $K_{\mathbf{C}^*}(\mathcal{B}_e)$ consisting of $\pm$ the elements $v^{-h'}o_i^{-1} (i \in I)$ and $\mathbf{p}$.*

The fact that the elements above are contained in $\mathbf{B}^{\pm}_{\mathcal{B}_e}$ follows from Lemmas 5.4, 6.2. The fact that $\pm$ these elements (which form a signed basis) exhaust $\mathbf{B}^{\pm}_{\mathcal{B}_e}$ follows from [L4, 12.21], using Lemma 5.4. The theorem is proved.

7 The Canonical Signed Basis of $K_{\mathbf{C}^*}(\Lambda_e)$

7.1

For $i \in I$, let V_i' be the set of all $(y, \mathfrak{b}) \in \Lambda_e$ with the following property: under the $\mathbf{C}^*$-action on Λ_e,

$$\lim_{\lambda\mapsto\infty} \lambda \cdot (y, b)$$

is defined and belongs to μ_i. The limit above is denoted by $\pi'_{\mu_i}(y, \mathfrak{b})$. By [KL, 4.6], the V_i' form a partition of Λ_e into locally closed subsets and for each i, $\pi'_{\mu_i}\colon V_i' \to \mu_i$ is naturally a vector bundle of dimension, say, δ_i. Since the analogue of Λ_e over the finite field with $\mathbf{q}$ elements is well known to have $\mathbf{q}^2 + |I|\mathbf{q}$ rational points, it follows that

$$(\mathbf{q} + 1)\mathbf{q}^{\delta_{i_0}} + \sum_{i\neq i_0} \mathbf{q}^{\delta_i} = \mathbf{q}^2 + |I|\mathbf{q}.$$

Since this holds for all prime powers $\mathbf{q}$, it follows that

(a) $\delta_i = 1$ for all i.

Lemma 7.2 *(a) V'_{i_0} is an open set in Λ_e.*

(b) $V'_{i^u_t} = V_{i^u_{t+1}} - \mu_{i^u_{t+1}}$ if $0 < t < a_u$.

(c) $V'_{i^u_{a_u}}$ is a line in Λ_e such that $V'_{i^u_{a_u}} \cap \mathcal{B}_e = \{q^u\}$.

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(a) follows from 7.1(a) since μ_{i_0} is a P^1. Using 7.1(a), we see that for $i \neq i_0$, V'_i is a line. Using the definitions we see that (c) holds and that, for $0 < t < a_u$,

$$V'_{i^u_t} \cap \mathcal{B}_e = V_{i^u_{t+1}} - \mu_{i^u_{t+1}}.$$

Since $V'_{i^u_t}$ is a line containing $V_{i^u_{t+1}} - \mu_{i^u_{t+1}}$, we must have $V'_{i^u_t} = V_{i^u_{t+1}} - \mu_{i^u_{t+1}}$. The lemma is proved.

Lemma 7.3 *The $R_{\mathbf{C}^*}$-module $K_{\mathbf{C}^*}(\Lambda_e)$ is projective of rank $|I| + 1$.*

We consider the partition into the locally closed $\mathbf{C}^*$-stable pieces $V'_i (i \in I)$ which are either an affine line or a line bundle over P^1. Each of these pieces has a $K_{\mathbf{C}^*}$ which is free and a $K^1_{\mathbf{C}^*} = 0$. It follows that $K_{\mathbf{C}^*}(\Lambda_e)$ is projective of rank equal to the sum of ranks of the $K_{\mathbf{C}^*}$ of the pieces, that is, $|I| + 1$.

Lemma 7.4 *Let $i \in I$. Let $(\|)$ be as in 5.2. We have*

(a) $(v^{-h'} o_i^{-1} \parallel E'^i) = v^{-2}$,
(b) $(v^{-h'} o_j^{-1} \parallel E'^i) = 0$ *for* $j \in I - \{i\}$,
(c) $(\mathbf{p} \parallel E'^i) = 0$.

Using 4.6, we have for $j \in I$:

$$(v^{-h'} o_j^{-1} \parallel E'^i) = (-v)^{\nu-2} v^{-h'} (\varpi^* \tilde{T}_{w_0} o_j^{-1} : E'^i)$$
$$= -(-v)^{\nu-2} v^{-h'} (-v)^{-\nu+2h'} (o_j^{-1} : E'^i) = -v^{h'-2} (o_j^{-1} : E'^i).$$

Now $(o_j^{-1} : E'^i)$ is the alternating sum of cohomologies of V_j with coefficients in $O_j^{-1} \otimes E'^i|_{V_j}$. If $i \neq j$ then, by 1.23, the last vector bundle on V_j is isomorphic to a direct sum of copies of O_j^{-1} (except for the $\mathbf{C}^*$-action) hence the corresponding cohomologies of V_j are 0. We see that

(d) $(o_j^{-1} : E'^i) = 0$ for $i \neq j$

and (b) follows. If $i = j$ then, by 1.23, the vector bundle $O_i^{-1} \otimes E'^i|_{V_i}$ is isomorphic to $v^{-h'} O_i^{-2} \oplus U''$, where U'' is a $\mathbf{C}^*$-equivariant vector bundle on V_i, isomorphic to a direct sum of copies of O_i^{-1} (except for the $\mathbf{C}^*$-action). Note that U'' has 0 contribution to the cohomology of V_i. On the other hand, the alternating sum of cohomologies of V_i with coefficients in O_i^{-2} is $-1 \in R_{\mathbf{C}^*}$. We see that

(e) $(o_i^{-1} : E'^i) = -v^{-h'}$

and (a) follows.

Using 4.7 and (d),(e), we have

$$(\mathbf{p} \parallel E'^i) = (-v)^{\nu-2} (\varpi^* \tilde{T}_{w_0} \mathbf{p} : E'^i) = v^{-2} (p + \sum_{j \in I} v^{h'} \bar{B}_j o_j^{-1} : E'^i)$$
$$= v^{-2} (p : E^i) - v^{-2} \bar{B}_i = 0.$$

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We have used that $(p : E'^i)$ is equal to $E'^i|_{P^u_{0,1}} = \bar{B}_i \in R_{\mathbf{C}^*}$ (see 1.23(c)). The lemma is proved.

Lemma 7.5 *Let* $\mathbf{C}$ *be the trivial one dimensional vector bundle on* Λ_e *with the trivial* $\mathbf{C}^*$*-equivariant structure. We have*

(a) $(v^{-h'} o_j^{-1} \parallel \mathbf{C}) = 0$ *for any* $j \in I$,
(b) $(\mathbf{p} \parallel \mathbf{C}) = v^{-2}$.

As in the proof of 7.4, we have

$$(v^{-h'} o_j^{-1} \parallel \mathbf{C}) = -(-v)^{\nu-2} v^{-h'} (-v)^{-\nu+2h'} (o_j^{-1} : \mathbf{C})$$

and this is zero since the cohomologies of V_j with coefficients in o_j^{-1} are 0. Similarly, using (a), we have

$$(\mathbf{p} \parallel \mathbf{C}) = v^{-2}\Big(p + \sum_{j \in I} v^{h'} \bar{B}_j o_j^{-1} : \mathbf{C}\Big) = v^{-2}(p : \mathbf{C}) = v^{-2}.$$

The lemma is proved.

7.6

Consider the commutative diagram

$$\begin{array}{ccc} K_{\mathbf{C}^*}(\mathcal{B}_e) & \xrightarrow{k_*} & K_{\mathbf{C}^*}(\Lambda_e) \\ \downarrow & & \downarrow \\ \mathbf{Q}(v) \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\mathcal{B}_e) & \xrightarrow{1 \otimes k_*} & \mathbf{Q}(v) \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\Lambda_e) \end{array}$$

where $k\colon \mathcal{B}_e \to \Lambda_e$ is the inclusion and the vertical maps are the obvious ones. Note that the vertical maps are injective since $K_{\mathbf{C}^*}(\mathcal{B}_e), K_{\mathbf{C}^*}(\Lambda_e)$ are projective of finite rank over $\mathcal{A} = R_{\mathbf{C}^*}$. (See 3.4(a), 7.3.) The lower horizontal map is an isomorphism (see [L4, 11.8]). It follows that k_* is also injective. Hence we may identify $K_{\mathbf{C}^*}(\Lambda_e)$ with an $\mathcal{A}$-submodule of $\mathcal{E} = \mathbf{Q}(v) \otimes_{\mathcal{A}} K_{\mathbf{C}^*}(\Lambda_e)$ and $K_{\mathbf{C}^*}(\mathcal{B}_e)$ with a $\mathcal{A}$-submodule of $K_{\mathbf{C}^*}(\Lambda_e)$ (via k_*). There is a well defined symmetric $\mathbf{Q}(v)$-linear form $(,)$ on $\mathcal{E}$ with values in $\mathbf{Q}(v)$ whose restriction to $K_{\mathbf{C}^*}(\mathcal{B}_e)$ is $(|)_{\mathcal{B}_e}$, whose restriction to $K_{\mathbf{C}^*}(\Lambda_e)$ is $(|)_{\Lambda_e}$ (see [L4, 12.16]) and such that $(b, a) = (b \parallel a)$ for $b \in K_{\mathbf{C}^*}(\mathcal{B}_e), a \in K_{\mathbf{C}^*}(\Lambda_e)$.

Proposition 7.7 *The elements*

(a) $v^2 E'^i (i \in I), v^2 \mathbf{C}$

form an $\mathcal{A}$*-basis of* $K_{\mathbf{C}^*}(\Lambda_e)$ *dual to the basis*

(b) $v^{-h'} o_i^{-1} (i \in I)$, $\mathbf{p}$

of $K_{\mathbf{C}^*}(\mathcal{B}_e)$ *with respect to the pairing* $(\parallel)\colon K_{\mathbf{C}^*}(\mathcal{B}_e) \times K_{\mathbf{C}^*}(\Lambda_e) \to R_{\mathbf{C}^*}$.

The fact that the matrix of inner products under $(\|)$ (or $(,)$, see 7.6) of an element in (b) with an element in (a) is the unit matrix is contained in Lemmas 7.4, 7.5. This shows in particular that the form $(,)$ on $\mathcal{E}$ (see 7.6) is non-singular. Now let ξ be an element of $K_{\mathbf{C}^*}(\Lambda_e)$. Then $c_i = (v^{-h'} o_i^{-1}, \xi) \in \mathcal{A}, c' = (\mathbf{p}, \xi) \in \mathcal{A}$. Let $\xi' = \sum_{i \in I} c_i v^2 E'^i + c' v^2 \mathbf{C}$. Then $(b, \xi') = (b, \xi)$ for any b in the set (b). Since this set is a $\mathbf{Q}(v)$-basis of $\mathcal{E}$ and $(,)$ is non-singular on $\mathcal{E}$, it follows that $\xi = \xi'$. Thus, the elements (a) generate the $\mathcal{A}$-module $K_{\mathbf{C}^*}(\Lambda_e)$. They are linearly independent over $\mathbf{Q}(v)$, hence they form an $\mathcal{A}$-basis of $K_{\mathbf{C}^*}(\Lambda_e)$. The proposition is proved.

7.8

Let $\bar{}: K_{\mathbf{C}^*}(\Lambda_e) \to K_{\mathbf{C}^*}(\Lambda_e)$ be the involution defined in [L4, 12.9] or, alternatively by the requirement

$$(\bar{b}, a) = \overline{(b, \bar{a})} \in \mathcal{A}$$

for all $b \in K_{\mathbf{C}^*}(\mathcal{B}_e), a \in K_{\mathbf{C}^*}(\Lambda_e)$ (see [L4, 12.15]). Following [L4, 12.18] we define

$$\mathbf{B}^{\pm}_{\Lambda_e} = \{\xi \in K_{\mathbf{C}^*}(\Lambda_e) | \bar{\xi} = \xi, (\xi|\xi)_{\Lambda_e} \in \mathbf{Q}(v) \cap (1 + v^{-1}\mathbf{Z}[[v^{-1}]])\}.$$

Theorem 7.9 *$\mathbf{B}^{\pm}_{\Lambda_e}$ is the signed basis of the $\mathcal{A}$-module $K_{\mathbf{C}^*}(\Lambda_e)$ consisting of $\pm$ the elements $v^2 E'^i (i \in I), v^2 \mathbf{C}$.*

Note that if a is in the set 7.7(a), then $\bar{a} = a$. Indeed, $\bar{a}$ and a have the same inner products $(,)$ with any element b of the set 7.7(b) (using 7.7, 7.8 and the fact that any such b satisfies $\bar{b} = b$). Also, by 7.7, the matrix A with entries (a, a') where a, a' run through the set 7.7(a) is the inverse of the matrix B with entries (b, b') where a, a' run through the set 7.7(b). Since B is congruent to the identity matrix modulo $v^{-1}\mathbf{Z}[v^{-1}]$ (by Lemma 5.4), it follows that A is congruent to the identity matrix modulo $v^{-1}\mathbf{Z}[[v^{-1}]]$. It follows that $\pm$ the elements in 7.7(a) are contained in $\mathbf{B}^{\pm}_{\Lambda_e}$. Since the elements 7.7(a) form an $\mathcal{A}$-basis of $K_{\mathbf{C}^*}(\Lambda_e)$ (see 7.7), it follows by an argument similar to that in [L4, 12.21] that any element in $\mathbf{B}^{\pm}_{\Lambda_e}$ is, up to sign, as in 7.7(a). The theorem is proved.

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https://doi.org/10.4153/CJM-1999-053-1 Published online by Cambridge University Press