BULL. AUSTRAL. MATH. SOC. 47N10, 26E15, 46T20
VOL. 63 (2001) [337–339]

A SIMPLE PROOF OF THE SUM FORMULA

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In this note we present a simple, short proof of the sum formula for subdifferentials of convex functions.

1. INTRODUCTION

Let X be a Banach space, X^* be its dual endowed with the dual norm. Let $f, g : X \to R \cup \{+\infty\}$ be proper, lower semicontinuous, convex functions with (effective) domains $\operatorname{dom} f$ and $\operatorname{dom} g$ respectively, and let $\partial f, \partial g : X \to 2^{X^*}$ be their subdifferential operators respectively. It is straightforward to verify that $\partial f(x) + \partial g(x) \subseteq \partial(f+g)(x)$ for any $x \in X$. However, the converse inclusion is not always true (see for example the remark in [**2**, Theorem, 3.16]). It is therefore important to find conditions which assure that $\partial f(x) + \partial g(x) = \partial(f+g)(x)$ for any $x \in X$. If for example the domain of f and the interior of the domain of g have nonempty intersection, then the above equality is true (see for example [**2**, Theorem 3.16]). Attouch and Brezis proved in [**1**] the following more general result (see also [**3**] for a different proof):

THE SUM FORMULA. Let f and g be as above and assume that

$$\bigcup_{\lambda>0} \lambda(\operatorname{dom} f - \operatorname{dom} g) = \overline{\operatorname{lin}(\operatorname{dom} f - \operatorname{dom} g)}$$

Then $\partial f + \partial g = \partial(f+g)$.

(In the above statement, "lin" stands for the "linear span of".)

The aim of this note is to present a short and simple proof of the sum formula.

2. A PARTICULAR CASE

We begin by recalling the definition of the *epi-sum* (or *inf-convolution*) of two functions $f, g : R \to R \cup \{+\infty\}$:

$$(f \overset{e}{+} g)(x) = \inf_{u+v=x} \{f(u) + g(v)\}$$

Received 25th July, 2000

From the definition it follows that $\operatorname{dom}(f \overset{e}{+} g) = \operatorname{dom} f + \operatorname{dom} g$. The epi-sum $f \overset{e}{+} g$ is called *exact* at (u, v) if $(f \overset{e}{+} g)(u+v) = f(u) + g(v)$. If f and g are convex, then $f \overset{e}{+} g$ is also convex. Finally, a direct computation shows that if f and g are convex and $f \overset{e}{+} g$ is exact at (u, v) then

$$\partial(f \overset{e}{+} g)(u+v) = \partial f(u) \cap \partial g(v). \tag{1}$$

Lemma 1. *Let $f : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let Y be a closed subspace of X such that $Y + \operatorname{dom} f$ is an absorbing subset of X and $(f \overset{e}{+} I_Y)(0) > -\infty$. Then $0 \in \operatorname{Int}\big(\operatorname{dom}(f \overset{e}{+} I_Y)\big)$ and $f \overset{e}{+} I_Y$ is locally Lipschitz at 0. In particular $f \overset{e}{+} I_Y$ is subdifferentiable at 0, that is, $\partial(f \overset{e}{+} I_Y)(0) \neq \emptyset$.*

Proof: By [**3**, Theorem 3], there exist $\varepsilon > 0$ and $\lambda > 0$ such that

$$\varepsilon B \subseteq \{x \in X\,;\, \|x\| \leqslant \lambda,\ f(x) \leqslant \lambda\} + Y.$$

It follows immediately that $(f \overset{e}{+} I_Y)(x) \leqslant \lambda$ for any $x \in \varepsilon B$. Since $(f \overset{e}{+} I_Y)(0) > -\infty$, it follows that $f \overset{e}{+} I_Y$ is nowhere $-\infty$. It is well known (see for example [**2**, Proposition 1.6] and the remark following it) that f is locally Lipschitz on εB and therefore is subdifferentiable at 0 (see for example [**2**, Proposition 1.11]). ∎

Lemma 2. *Let $f : X \to R \cup \{+\infty\}$ be a proper lower semicontinuous convex function and let Y be a closed subspace of X such that $Y + \operatorname{dom} f$ is an absorbing subset of X. Then $\partial(f + I_Y) = \partial f + \partial I_Y$.*

Proof: It is well known and easy to check that $\partial f(x) + \partial I_Y(x) \subseteq \partial(f + I_Y)(x)$ for any $x \in X$. To prove the other inclusion, let $x^* \in \partial(f + I_Y)(x)$. Define $g : X \to R \cup \{+\infty\}$ by $g(u) = f(x+u) - \langle x^*, x+u\rangle$. Then

(i) $\operatorname{dom} g = \operatorname{dom} f - x$ and, since $x \in Y$, $Y + \operatorname{dom} g = Y + \operatorname{dom} f$ is absorbing;
(ii) $u^* \in \partial f(x)$ if and only if $u^* - x^* \in \partial g(0)$.
(iii) $u^* \in \partial(f + I_Y)(x)$ if and only if $u^* - x^* \in \partial(g + I_Y)(0)$.

Since $x^* \in \partial(f + I_Y)(x)$, (iii) implies that $0 \in \partial(g + I_Y)(0)$ and thus

$$g(0) + I_Y(0) = \inf_{u \in X}\{g(u) + I_Y(u)\}.$$

It follows that

$$(g \overset{e}{+} I_Y)(0) = \inf_{u \in X}\{g(u) + I_Y(-u)\} = \inf_{u \in X}\{g(u) + I_Y(u)\} = g(0) + I_Y(0) > -\infty.$$

Thus $g \overset{e}{+} I_Y$ is exact at $(0,0)$. From (i) and Lemma 1 it follows that $g \overset{e}{+} I_Y$ is subdifferentiable at 0 and therefore, from (1),

$$\emptyset \neq \partial(g \overset{e}{+} I_Y)(0) = \partial g(0) \cap \partial I_Y(0).$$

Thus, there exists $u^* \in X^*$ such that $u^* \in \partial g(0) \cap \partial I_Y(0)$. Clearly $-u^* \in \partial I_Y(x)$ and, by (ii), $x^* + u^* \in \partial f(x)$. Since $x^* = x^* + u^* + (-u^*) \in \partial f(x) + \partial I_Y(0)$, the lemma is proved. ∎

3. Proof of the Sum Formula

The proof is now standard, but we shall sketch it for sake of completeness.

First, in view of [**3**, Lemma 25 (c)] we can assume without any loss of generality that $\overline{\text{lin}\,(\text{dom}\, f - \text{dom}\, g)} = X$ and thus $\text{dom}\, f - \text{dom}\, g$ is absorbing in X. Define $h : X \times X \to R \cup \{+\infty\}$ by $h(x,y) = f(x) + g(y)$ and let $D = \{(x,x)\,;\, x \in X\}$. Then h is a proper lower semicontinuous convex function on $X \times X$, D is a closed subspace of $X \times X$, and $\text{dom}\, h - D$ is an absorbing subset of $X \times X$. The Sum Formula follows from Lemma 2 and the following statements which, with the exception of (c), follow more or less directly from the definitions; (c) is a particular case of Lemma 2. As usual we shall identify $(X \times X)^*$ with $X^* \times X^*$.

(a) $\partial I_D(z,z) = \{(z^*, -z^*)\,;\, z^* \in X^*\}$;

(b) $z^* \in \partial(f+g)(z) \iff (z^*, 0) \in \partial(h + I_D)(z,z)$

(c) $(z^*, 0) \in \partial(h + I_D)(z,z) \iff (z^*, 0) \in \partial h(z,z) + \partial I_D(z,z)$;

(d) $(z^*, 0) \in \partial h(z,z) + \partial I_D(z,z) \iff z^* = u^* + v^*,\ (u^*, v^*) \in \partial h(z,z)$;

(e) $(u^*, v^*) \in \partial h(z,z) \iff u^* \in \partial f(u)$ and $v^* \in \partial f(v)$.

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