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THE $c\mu$ RULE REVISITED

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Abstract

The $c\mu$ rule is optimal for arbitrary arrival processes provided that the service times are geometric and the service discipline is pre-emptive.

1. The problem

We consider a discrete-time model of N queues competing for a single server. Let $A_i(t)$ be the random number of new customers that join queue i. No statistical restriction is imposed on the arrival process $\{A_i(t)\}$, $i = 1, \cdots, N$. The buffer capacity at each queue is unlimited. The service time requirement of a customer in queue i is geometric with mean μ_i^{-1} and the service time requirements of different customers are statistically independent. In each period t, the service must be assigned to one of the queued customers. At the end of the period the server may be reassigned to another customer. That is, the service discipline is pre-emptive. Let $0 < \beta < 1$ be a fixed discount factor. The objective is to find an assignment policy π to minimize the expected total discounted waiting cost

$$J(\pi, T) := E \sum_{t=1}^{T} \beta^t \sum_{i=1}^{N} c_i X_1^\pi(t). \tag{1}$$

Here $T < \infty$, $c_1 \geqq 0$ are constants, and $X_i^\pi(t)$ is the length of the queue i under π. The control policy π may depend upon previous queue lengths and control actions.

The problem described above generalizes that considered by Baras, Dorsey and Makowski in [1]. They use dynamic programming to find the optimal policy for the special case $N = 2$, $T = \infty$, and when the arrivals in different time periods are independent and identically distributed.

The $c\mu$ rule denotes the policy which at time t assigns the server to a customer in queue i if $c_i\mu_i = \max\{c_j\mu_j \mid X_j(t) > 0\}$. The following result is proved in the next section.

Theorem. The $c\mu$ rule is optimal.

2. Proof of the theorem

The theorem can be proved using the results obtained in [2]. However, the simple structure of the present problem permits a direct proof based on an interchange argument.

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For $T=1$ the assertion is true since $J(\pi, 1)=E\beta\sum_1^N c_iX_i(1)$ does not depend on π.

Suppose the assertion is true for some $T\geqq 1$ and consider the horizon $T+1$. Let π be an optimal policy. By the induction hypothesis we may suppose that π follows the $c\mu$ rule for $t=2,\cdots, T+1$. By way of contradiction suppose π does not follow the $c\mu$ rule for $t=1$. Then there are customers i, j at time 1, with $c_i\mu_i > c_j\mu_j$, and π serves j at $t=1$. Let σ be the first time that π serves i. Then $\sigma\geqq 2$ a.s. Since the decision at $T+1$ does not affect the cost we may also assume that $\sigma\leqq T+1$ a.s. Since π follows the $c\mu$ rule for $t\geqq 2$, therefore π does not serve j during $2,\cdots,\sigma$.

Consider now the policy π which is identical to π except that the time when j is served for the first time, namely 1, and the time when i is served for the first time, namely σ, are interchanged. Then the states under both policies will be the same from time $\sigma+1$ onwards, and so

$$J(\pi, T+1)-J(\pi, T+1)=c_i\mu_iE\sum_{t=2}^{\sigma}\beta^t-c_j\mu_jE\sum_{t=2}^{\sigma}\beta^t>0. \tag{2}$$

Since μ_i is the probability that customer i will complete service in one step, the first term on the right in (2) is the reduction in cost achieved by serving i at $t=1$ instead of at $t=\sigma$; similarly, the second term is the increase in cost due to postponing service to j. The inequality implies that π cannot be optimal and the proof is complete.

Remark. Since the $c\mu$ rule is optimal for all $T<\infty$ and $\beta<0$, it is also optimal for the infinite-horizon problem with either the total discounted cost or the average cost per unit time criterion. Secondly, by introducing a customer 0 with $c_0=\mu_0=0$ we can see that it cannot be optimal to keep a server idle when customers are present.

References

[1] Baras, J. S., Dorsey, A. J. and Makowski, A. M. (1985) Two competing queues with linear costs: The μc rule is often optimal. *Adv. Appl. Prob.* **17,** 186–209.

[2] Varaiya, P., Walrand, J. and Buyukkoc, C. Extensions of the multi-armed bandit problem. *IEEE Trans. Automatic Control.*