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ON A SPACE OF SOME THETA FUNCTIONS

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In the theory of modular forms there is an interesting problem whether every modular form can be expressed as a linear comination of theta functions. For this Eichler proved in [1] that for a sufficiently large prime q all modular forms of degree $-2m (m = 1, 2, \cdots)$ for $\Gamma_0(q)$ can be represented by linear combinations of theta functios of degree $-2m$ with level 1 and q. We prove this theorem for $q = 2, 3, 5$ and 11 by using a theorem of Siegel for $q = 2, 3, 5$ and a general result of Eichler for $q = 11$. The former method is shown in Schoeneberg [2].

Before our statement, it should be recalled: for an even positive $4m \times 4m$ matrix Q with level N and square discriminant, the theta function

$$\vartheta(\tau, Q) = \sum_{\xi \in Z^{4m}} e^{\pi i {}^t\xi Q \xi \tau}$$

is a modular form of degree $-2m$ for $\Gamma_0(N)$, i.e. of type $(-2m, N, 1)$ in the sense of Hecke.

THEOREM. *For $q = 2, 3, 5$ and 11 all modular forms of degree $-2m (m = 1, 2, \cdots)$ for $\Gamma_0(q)$ can be represented by linear combinations of theta functions of type $(-2m, q, 1)$ and $(-2m, 1, 1)$.*

Proof for $q = 2$. Let d_m (resp. e_m) be the dimension of the space $\mathfrak{M}(m)$ (resp. $\mathfrak{S}(m)$) of modular forms (resp. cusp forms) of degree $-2m$ for $\Gamma_0(2)$. Then it is well known that

$$(1) \qquad \begin{cases} d_m = \left[\dfrac{m}{2}\right] + 1 & \text{for} \quad m \geq 1, \\ e_m = 0 & \text{for} \quad m = 1, \\ e_m = \left[\dfrac{m}{2}\right] - 1 & \text{for} \quad m \geq 2. \end{cases}$$

Let A be an even positive 4×4 matrix with level 2 and determinant 4, for

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example

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

and M be an eveen positve 8×8 matrix with determinat 1, for example

$$M = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 4 & 3 & & & & \\ & & 3 & 4 & 5 & & & \\ & & & 5 & 20 & 3 & & \\ & & & & 3 & 12 & 1 & \\ & & & & & 1 & 4 & 1 \\ & & & & & & 1 & 2 \end{pmatrix}$$

by Minkowski. Then $\vartheta(\tau, A)$ is a modular form of degree -2 for $\Gamma_0(2)$ and $\vartheta(\tau, M)$ is a modular form of degree -4 for $\Gamma(1)$. There are two inequivalent cusps 0 and ∞ for $\Gamma_0(2)$, and

$$\vartheta(\tau, A) = 1 \quad \text{at} \quad \tau = \infty, \quad \vartheta(\tau, M) = 1 \quad \text{at} \quad \tau = \infty,$$

$$\vartheta(\tau, A) = -\frac{1}{2} \quad \text{at} \quad \tau = 0, \quad \vartheta(\tau, M) = 1 \quad \text{at} \quad \tau = 0.$$

Under these preparations we prove the theorem inductively. Firstly it is clear by the dimension formula (1) that $\mathfrak{M}(1) = \boldsymbol{C}\{\vartheta(\tau, A)\}$,

$$\mathfrak{M}(2) = \boldsymbol{C}\{\vartheta(\tau, A)^2,\ \vartheta(\tau, M)\} \text{ and } \mathfrak{M}(3) = \boldsymbol{C}\{\vartheta(\tau, A)^3,\ \vartheta(\tau, A)\vartheta(\tau, M)\}.$$

Secondly we prove the theorem for $\mathfrak{M}(4)$ by using Siegel's theorem. Put

$$B = \begin{pmatrix} A & & & \\ & A & & \\ & & A & \\ & & & A \end{pmatrix}$$

Then B is an even positive 16×16 matrix with level 2 and determinant 4^4 and owing to Siegel [3]

$$F(\tau, B) = \frac{1}{M(B)} \sum_{B_k} \frac{\vartheta(\tau, B_k)}{E_k}$$

can be represented by Eisenstein series with level 2, where B_k runs over all representatives fo the classes in the genus of B, E_k is the ordre of the unit group of B_k and $M(B) = \sum \frac{1}{E_k}$. Since $F(\tau, B) = 1$ at $\tau = \infty$ and $F(\tau, B) = \frac{1}{16}$ at $\tau = 0$,

$$(2) \qquad F(\tau, B) = \frac{480}{17}(G_8(\tau) - G_8(2\tau)) + 480G_8(2\tau)$$
$$= 1 + \frac{480}{17} e^{2\pi i\tau} + \cdots,$$

where $G_l(\tau)$ is an Eisenstein series with level 1 defined by

$$G_l(\tau) = \frac{(l-1)!\,(-1)^{\frac{l}{2}}}{2(2\pi)^l} \sum_{c,d\in Z}' \frac{1}{(c\tau + d)^l}$$
$$= \frac{(l-1)!\,(-1)^{\frac{l}{2}}}{(2\pi)^l} \zeta(l) + \sum_{n=1}^{\infty} (\sum_{d|n} d^{l-1}) e^{2\pi i n\tau}.$$

Now $\frac{480}{17}$ is not an integer. Hence among the above theta functinos $\vartheta(\tau, B_k)$ we can take some $\vartheta(\tau, B_{k_0})$, linealy independent to $\vartheta(\tau, B)$. Consequently $\mathfrak{M}(4)$ is $\boldsymbol{C}\{\vartheta(\tau, M)^2, \vartheta(\tau, B), \vartheta(\tau, B_{k_0})\}$, since $\vartheta(\tau, B) = \vartheta(\tau, B_{k_0}) = 1$ at $\tau = \infty$ and $\vartheta(\tau, B) = \vartheta(\tau, B_{k_0}) = \frac{1}{16}$ at $\tau = 0$, and so $\vartheta(\tau, M)^2 \notin \boldsymbol{C}\{\vartheta(\tau, B), \vartheta(\tau, B_{k_0})\}$. Lastly since $\mathfrak{M}(m) = \mathfrak{M}(m-1) \times \vartheta(\tau, A)$ for any odd integer $m \geq 3$, we asusme that m is even. For $m \geq 6$, $e_m = d_{m-4}$. Therefore $\mathfrak{S}(m)$ is the product of $\mathfrak{M}(m-4)$ and a one-dimensional space spanned by a cusp form of degree -8. Moreover, since $\vartheta(\tau, A)^m = \vartheta(\tau, M)^{\frac{m}{2}} = 1$ at $\tau = \infty$, $\vartheta(\tau, A)^m = 2^{-m}$ at $\tau = 0$, and $\vartheta(\tau, M)^{\frac{m}{2}} = 1$ at $\tau = 0$, we can deduce that $\mathfrak{M}(m)$ is generated by $\vartheta(\tau, A)^m$, $\vartheta(\tau, M)^{\frac{m}{2}}$ and cusp forms in $\mathfrak{S}(m)$. Thus we have completed the proof for $q = 2$.

For $q = 3, 5$ and 11 the proof is analogous under some modifications and we simply point out them.

The dimer sion formula (1) should be replaced by the followings:

$$\begin{cases} d_s = \left[\dfrac{2}{3}s\right]+1 \\ e_1 = 0 \\ e_t = \left[\dfrac{2}{3}t\right]-1, \end{cases} \quad \begin{cases} d_s = 2\left[\dfrac{s}{2}\right]+1 \\ e_1 = 0 \\ e_t = 2\left[\dfrac{t}{2}\right]-2, \end{cases} \quad \begin{cases} d_s = 2s \\ e_1 = 1 \\ e_t = 2t-2 \end{cases}$$

for $q = 3, 5, 11$ respectively where s represents any positive integer and t represents any positive integer ≥ 2.

An example for an even positive 4×4 matrix with level q and determinant q^2 is the following:

$$A = \begin{pmatrix} 2 & 1 & & \\ 1 & 2 & & \\ & & 2 & 1 \\ & & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 10 & 5 \\ 0 & 1 & 5 & 4 \end{pmatrix}$$

for $q = 3, 5$ respectively.

We may use Siegel's theorem for $m = 3$ (resp. 2) if $q = 3$ (resp. 5) and instead of (2) we obtain: for $q = 3$,

$$\begin{aligned} F(\tau, B) &= \frac{252}{13}(G_6(\tau) - G_6(3\tau)) - 504G_6(3\tau) \\ &= 1 + \frac{252}{13}e^{2\pi i\tau} + \cdots\cdot, \end{aligned}$$

where
$$B = \begin{pmatrix} A & & \\ & A & \\ & & A \end{pmatrix},$$

for $q = 5$,

$$\begin{aligned} F(\tau, B) &= \frac{120}{13}(G_4(\tau) - G_4(5\tau)) + 240G_4(5\tau) \\ &= 1 + \frac{120}{13}e^{2\pi i\tau} + \cdots\cdot, \end{aligned}$$

where $B = \begin{pmatrix} A & \\ & A \end{pmatrix}$. Moreover noticing that for even m $\mathfrak{M}(m)$ is spanned by $\vartheta(\tau, A)^m$, $\vartheta(\tau, M)^{\frac{m}{2}}$ and $\mathfrak{S}(m)$ and for odd $m \geq 3$ $\mathfrak{M}(m)$ is spanned by

$\vartheta(\tau, A)^m$, $\vartheta(\tau, M)^{\frac{m-1}{2}}$ $\vartheta(\tau, A)$ and $\mathfrak{S}(m)$, the theorem for $q = 3, 5$ can be proved by induction on m as in the case of $q = 2$. For $q = 11$, using the fact that all modular forms of degree -2 for $\Gamma_0(11)$ is generated by theta functions of degree -2 with level 11, which is proved by Eichler [1] in a more general form, we can prove the theorem only by the dimension formula.

Remark. We proved the theorem for $q = 11$ by using a general result of Eichler but we can also prove this like the other case by extending a theorem of Siegel for the case of the even positive quarternary quadratic forms according to a methos of Maads [2].

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