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SOME RESULTS FOR QUADRATIC ELEMENTS OF A BANACH ALGEBRA

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Abstract. Several properties of some quadratic elements of a unitial Banach algebra are studied. Deddens subspaces are also introduced and discussed.

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1. Introduction. Let $\mathcal{A}$ be a complex Banach algebra with unit e. An element $a \in \mathcal{A}$ is called *quadratic* if it satisfies $a^2 + \lambda_1 a + \lambda_2 e = 0$ for some scalars λ_1 and λ_2. Observe that idempotents and nilpotent elements of order 2 are quadratic elements.

Our main goal in this paper is to study some properties of such elements. See [**1**, **2**, **3**, **4**] for concrete applications.

2. Deddens subspaces. Let $\mathcal{A}$ be a Banach algebra with a unit e. For any two invertible elements $a_1, a_2 \in \mathcal{A}$ put

$$\mathcal{D}_{a_1,a_2} \stackrel{\text{def}}{=} \left\{ x \in \mathcal{A} : \sup_{n\geq 0} \left\| a_1^n x a_2^{-n} \right\| \stackrel{\text{def}}{=} c_x < \infty \right\}.$$

We call the subspaces $\mathcal{D}_{a_1,a_2}$ and $\mathcal{D}_{a_2,a_1}$ the *Deddens subspaces*. Note that, when $a_1 = a_2$ the notion of Deddens subspace coincides with the notion of Deddens algebra, introduced in [**4**].

Our main result in this section is the following theorem.

THEOREM 1. *Let $\mathcal{A}$ be a Banach algebra with unit e. Let p be any idempotent and q a nilpotent of order 2, respectively. We have*
(a) $\mathcal{D}_{e+p,e+q} = \{x \in \mathcal{A} : px = xq\}$,
(b) $\mathcal{D}_{e+q,e+p} = \{x \in \mathcal{A} : qx = xp\}$.

Proof. (a) Let us denote $Intertw\{p, q\} = \{x \in \mathcal{A} : px = xq\}$. The inclusion $Intertw\{p, q\} \subset \mathcal{D}_{e+p,e+q}$ is obvious. To prove the reverse inclusion $\mathcal{D}_{e+p,e+q} \subset Intertw\{p, q\}$, let $x \in \mathcal{D}_{e+p,e+q}$ be any element. Putting

$$c_n = a_1^n x a_2^{-n} \quad (n \geq 0),$$

where $a_1 = e + p,\ a_2 = e + q$, we deduce that

$$\|c_n\| \leq c_x \quad (n \geq 0). \tag{1}$$

We have

$$c_n a_2 = a_1^n x a_2^{-n} a_2 = a_1\left(a_1^{n-1} x a_2^{-n+1}\right) = a_1 c_{n-1};$$

that is

$$c_n a_2 = a_1 c_{n-1} \quad (n \geq 1). \tag{2}$$

From (2) we obtain

$$c_n a_2^n = a_1^n c_0 \quad (n \geq 1)$$

or

$$c_n (e+q)^n = (e+p)^n c_0 = (e + (2^n - 1)p)x \quad (n \geq 1);$$

that is

$$c_n = (e + (2^n - 1)p)\, x\, (e - nq) \quad (n \geq 1).$$

From this equality, we deduce that

$$c_n - x = (2^n - 1)px - nxq - n(2^n - 1)pxq, \tag{3}$$

for all $n \geq 1$. By taking the equality (1) into account, it follows from (3) that

$$\|pxq\| \leq \frac{\|c_n - x\|}{n(2^n - 1)} + \frac{\|px\|}{n} + \frac{\|xq\|}{2^n - 1} \to 0$$

as $n \to \infty$. Hence $pxq = 0$, and therefore

$$c_n - x = (2^n - 1)px - nxq.$$

From this we obtain

$$\|px\| \leq \frac{\|c_n - x\|}{2^n - 1} + \frac{n}{2^n - 1}\|xq\| \to 0$$

as $n \to \infty$. Hence $px = 0$. Therefore $c_n - x = -nxq$, which implies that

$$\|xq\| = \frac{\|c_n - x\|}{n} \to 0 \quad \text{as } n \to \infty$$

and so $xq = 0$. Hence $c_n - x = 0$ $(n \geq 1)$. In particular $c_1 = x$, so that $x = (e+p)\, x\, (e-q)$. Hence, $(e+p)\, x = x\, (e+q)$. Therefore $px = xq$, which means that $x \in Intertw\{p, q\}$, and so $\mathcal{D}_{e+p,e+q} \subset Intertw\{p, q\}$, which completes the proof of (a).

(b) The proof is very similar to that of (a) and is omitted.

Corollary 2. *Let $\mathcal{A}$ be a complex Banach algebra with unit e. Let p be any idempotent and q a nilpotent of order 2, respectively. We have*

$$(\mathcal{D}_{e+p,e+q} \cap \mathcal{D}_{e+q,e+p}) \cap \{p\}' = (\mathcal{D}_{e+p,e+q} \cap \mathcal{D}_{e+q,e+p}) \cap \{q\}'.$$

Here $\{t\}'$ stands for the commutant of t.

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Let $\mathcal{A}$ be a Banach algebra with the idempotent p and with a unit e. Define the set S_p as follows:

$$S_p = \{x \in \mathcal{A} : px(e-p) = 0\}.$$

By analogy with the proof of Theorem 1, we can state directly the following theorem.

THEOREM 3. *Let $\mathcal{A}$ be a Banach algebra with an idempotent p and with a unit e. Then $\mathcal{D}_{e+p,e+p}$ is an algebra and $\mathcal{D}_{e+p,e+p} = S_p$; thus the Deddens algebra $\mathcal{D}_{e+p,e+p}$ coincides with the algebra S_p.*

Proof. It follows from the definition of Deddens subspaces that $\mathcal{D}_{e+p,e+p}$ is an algebra. For the second statement of the theorem, it is easy to check that $(e+p)^{-1} = e - \frac{1}{2}p$. Therefore

$$(e+p)^n = e + (2^n - 1)p \quad (n \geq 0) \tag{4}$$

and

$$(e+p)^{-n} = e + \left(\frac{1}{2^n} - 1\right)p \quad (n \geq 0). \tag{5}$$

By setting

$$c_n = (e+p)^n x (e+p)^{-n} \quad (n \geq 0),$$

where x is any element of $\mathcal{A}$, we obtain

$$c_n(e+p) = (e+p)\left[(e+p)^{n-1} x \left(e - \frac{1}{2}p\right)^{n-1}\right] = (e+p)c_{n-1},$$

for every $n \geq 1$, which implies that

$$c_n(e+p)^n = (e+p)^n c_0 \quad (n \geq 0).$$

By applying equalities (4), (5) we have

$$c_n = x + \left(\frac{1}{2^n} - 1\right)xp + (2^n - 1)px + (2^n - 1)\left(\frac{1}{2^n} - 1\right)pxp, \tag{6}$$

$n = 0, 1, 2, \ldots$. Then, for every $x \in \mathcal{D}_{e+p,e+p}$, it follows that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{1}{(2^n - 1)(\frac{1}{2^n} - 1)}(c_n - x) \\ &= \lim_{n\to\infty} \left(\frac{1}{\frac{1}{2^n} - 1}px + \frac{1}{2^n - 1}xp + pxp\right) \\ &= pxp - px, \end{aligned}$$

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or equivalently

$$px(e-p)=0,$$

i.e., $x \in S_p$.

Conversely, if $x \in S_p$, then again it is clear from the equality (6) that

$$\begin{aligned}\|c_n\| &= \left\| x + \left(\frac{1}{2^n} - 1\right) xp + \frac{2^n-1}{2^n} pxp \right\| \\ &\leq \|x\| + \|xp\| + \|pxp\| \\ &= c_x < +\infty,\end{aligned}$$

for any $n \geq 0$. Hence $x \in \mathcal{D}_{e+p,e+p}$.

Prior to stating two corollaries of Theorem 3, we require some terminology and notation.

Let $\mathcal{A}$ be a complex Banach algebra with unit e. An element $a \in \mathcal{A}$, is said to be a *regular von Neumann element* if there exists $b \in \mathcal{A}$, such that $a = aba$.

It is obvious that the invertible elements of $\mathcal{A}$ are regular von Neumann elements. Also, in the special case in which $\mathcal{A} = \mathcal{B}(H)$, the algebra of bounded linear operators on a complex Hilbert space H, all isometries of $\mathcal{B}(H)$ are regular von Neumann elements of $\mathcal{B}(H)$.

For any non invertible regular von Neumann element a, define

$$q_a = a(e-ab), \quad p_a = ab,$$

where $b \in \mathcal{A}$ and $a = aba$. Clearly $q_a^2 = 0$ and $p_a^2 = p_a$.

COROLLARY 4. *Let $\mathcal{A}$ be a complex Banach algebra with unit e. Then $\mathcal{D}_{e+p_a,e+p_a} = S_{p_a}$.*

Proof. This follows at once from Theorem 3.

Finally, we give one more result on quadratic elements q_a and $p_a^{\perp} \overset{\text{def}}{=} e - p_a$.

PROPOSITION 5. *Let $\mathcal{A}$ be a Banach algebra with unit e. Suppose that $x, y, a \in \mathcal{A}$ are elements such that a is a regular element and*

$$xa - ay = q_a x. \tag{7}$$

If $\sigma(x) = \{0\}$ (that is, x is a quasinilpotent element), then $\sigma(p_a^{\perp} x) = \{0\}$.

Proof. By induction on n, we prove that

$$(p_a^{\perp} x)^n = p_a^{\perp} x^n, \tag{8}$$

for every $n \geq 1$. For $n = 1$, the assertion is obvious. Let $n > 1$ and let $(p_a^{\perp} x)^{n-1} = p_a^{\perp} x^{n-1}$. The regularity of element a implies that $p_a^{\perp} a = 0$. Then by using the condition (7) we have

$$\begin{aligned}(p_a^{\perp} x)^n &= p_a^{\perp} x (p_a^{\perp} x)^{n-1} = p_a^{\perp} x p_a^{\perp} x^{n-1} = p_a^{\perp} x (e-ab) x^{n-1} \\ &= p_a^{\perp} x^n - p_a^{\perp} xabx^{n-1} = p_a^{\perp} x^n - p_a^{\perp} (ay + q_a x) bx^{n-1} \\ &= p_a^{\perp} x^n - p_a^{\perp} a(y + p_a^{\perp} x) bx^{n-1} = p_a^{\perp} x^n.\end{aligned}$$

Thus, the equality (8) is proved. From (8) the assertion of the proposition is obvious. This completes the proof.

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Remarks (*a*). We recall that for any two elements x, a of the Banach algebra $\mathcal{A}$, the well-known "Kleinecke-Shirokov" condition $[x, [x, a]] = 0$ implies the quasinilpotency of the commutator $[x, a] \stackrel{\text{def}}{=} xa - ax$. See [**5**], [**6**]. In particular, the condition

$$[x, a] = x \tag{9}$$

implies that $[x, a]$ is a nilpotent element. It is known that (9) is not a necessary condition for the nilpotency of $[x, a]$. The condition (7) of Proposition 5, in particular, gives such an example. Indeed, the relation(7) implies that $(xa - ay)^2 = 0$. Therefore, when $y = x$, $(xa - ax)^2 = 0$, but clearly $xa - ax = q_a x \neq x$.

(*b*). It should be mentioned that the statement of the type "$\sigma(x) = \{0\} \Rightarrow \sigma(ax) = \{0\}$" is of importance in many problems of Banach algebra theory and operator theory. The Shulman's paper [**6**] is a good reference in this sense. In particular, in [**6**] the following question is raised.

Question. Let elements x, a of Banach algebra satisfy the conditions

$$[x, [x, a]] = 0$$

and $\sigma(x) = \{0\}$. Is it true that $\sigma(ax) = \{0\}$?

3. Reducing subspaces. Let H be a complex Hilbert space and $\mathcal{B}(H)$ the algebra of bounded linear operators on H.

Corollary 6. *$AlgLat\mathcal{Q} = \bigcap_{E \in Lat\mathcal{Q}} \mathcal{D}_{I+P_E, I+P_E}$, where $\mathcal{Q}$ is a subset of $\mathcal{B}(H)$, $Lat\mathcal{Q}$ the lattice of closed subspaces E of H invariant under $\mathcal{Q}$, P_E is the orthogonal projection of H onto E and $AlgLat\mathcal{Q} = \{T \in \mathcal{B}(H) : TE \subseteq E$ for all $E \in Lat\mathcal{Q}\}$.*

Proof. This follows at once from Theorem 3.

We recall that a *reducing subspace* of a bounded linear operator T on H is a common invariant subspace for T and T^*. It is known that a subspace $E \subset H$ is a reducing subspace for T if and only if $TP_E = P_E T$, where P_E is the orthogonal projection of H onto E.

Allan and Zemanek proved in [**2**, Corollary 9] that every quadratic operator on H has a reducing subspace. Our next theorem describes the reducing subspaces of a nilpotent operator on a Hilbert space H in terms of C_Q classes. We first recall the definition of the C_Q class. Let S be a positive linear operator on a Hilbert space H. There are positive real numbers m and M and Q in $\mathcal{B}(H)$ such that $0 < mI \leq S \leq MI$ and $Q = S^{-1/2}$. Then

$$C_Q = \{T \in \mathcal{B}(H) \,:\, QT^nQ = P_H U^n | H, n = 1, 2, \ldots\},$$

where U is a unitary operator on some Hilbert space $K \supset H$. Note that $T \in C_Q$ if and only if T satisfies the condition:

$$(Sh, h) + 2Re(z(I - S)Th, h) + |z|^2((S - 2I)Th, Th) \geq 0,$$

for any $h \in H$ and $z \in \mathbb{C}$, $|z| \leq 1$. The classes C_Q were defined by Langer. See [**7**, p. 55].

Theorem 7. *Let $N \in \mathcal{B}(H)$ be a nilpotent operator. The subspace $E \subset H$ is a reducing subspace of the operator N if and only if $E = T^{k-1}H$, for some operator*

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T belonging to some class C_Q and for some integer $k \geq 2$ satisfying $T^k = (I+N)T(I+N)^{-1}$.

Proof. The first part of the theorem is obvious. Indeed, if $E \subset H$ is a reducing subspace of N, then $P_E N = N P_E$, where P_E is the orthogonal projection of H onto E, and hence, $E = P_E H$, $T = P_E$, $Q = I$ and $k = 2$.

We now prove the "only part" of the theorem.

From the condition $T^k = (I+N)T(I+N)^{-1}$ it is easy to see that

$$T^{k^n}(I+N) = (I+N)T^{k^{n-1}}, \tag{10}$$

for all $n \geq 1$ Since $T \in C_Q$, for some Q, then we have that

$$\|T^{k^n}\| \leq \|Q^{-1}\|^2, \tag{11}$$

and hence, from (10) and (11) by the result of Deddens and Wong [**8**, Lemma 2] we assert that $T^k = T$, see also [**4**, Lemma 2]. Therefore $TN = NT$, and hence, $T^{k-1}N = NT^{k-1}$. On the other hand,

$$T^{2(k-1)} = T^k T^{k-2} = TT^{k-2} = T^{k-1};$$

that is, T^{k-1} is a projection. T^{k-1} is an orthogonal projection, by [**9**], since $T \in C_Q$ and so the equality $T^{k-1}N = NT^{k-1}$ means that the subspace $T^{k-1}H$ reduces N; that is, E reduces N which completes the proof of theorem.

Before passing to the next result, we recall the following definition.

Definition ([**10**]). The operator $T \in \mathcal{B}(H)$ is called *quasidiagonal* if there exists a non-decreasing sequence $\{P_n\}_{n\geq 1}$ of finite-dimensional orthogonal projections, for which $P_n \to I$ (strongly) and $\|TP_n - P_nT\| \to 0$ as $n \to \infty$.

Herrero [**11**] defined the notion of *module of quasidiagonality*:

$$qd(T) = \liminf_{\substack{P\in\mathcal{P} \\ P\to I}} \|TP - PT\|,$$

where $\mathcal{P}$ is an ordered (with respect to natural order) set of all finite-dimensional orthogonal projections in H. It is known [**11**] that T is a quasidiagonal operator if and only if $qd(T) = 0$. The following theorem was proved by Arora and Sahdev in [**12**].

Theorem 8. *Let $T \in \mathcal{B}(H)$, $\ker T^* \neq \{0\}$ and $C = \inf_{\|x\|=1} \|Tx\| > 0$. Then $qd(T) \geq C$.*

According to a result of Allan and Zemanek [**2**, Example 6] there is an operator R on H, with $R^2 = 0$, but having no finite-dimensional reducing subspace. In the remainder of this section, as an illustration of Theorem 8, we give an example (see Example 10 below) of a family $\{T_\alpha\}$ of operators on a Hilbert space H, with no finite-dimensional reducing subspace, converging to the nilpotent operator (with the index of nilpotency2) with finite-dimensional reducing subspace. However, we first prove the following proposition.

Proposition 9. *Let $V, W \in \mathcal{B}(H)$ be operators such that V is a nonunitary isometry, $VW = WV$(i.e., $W \in \{V\}'$) and $\|W\| < 1$. Let us consider the operator $N_{V,W} \overset{\text{def}}{=} V(I - WVV^*)$. Then $qd(N_{V,W}) \geq 1 - \|W\|$.*

Proof. Since $\|W\| < 1$, the operators $I - WVV^*$ and $I - VV^*W^*$ are invertible. Then we have

$$\ker N_{V,W}^* = \ker(I - VV^*W^*)V^* = \ker V^* = (VH)^\perp \neq \{0\}$$

and

$$\begin{aligned} C &= \inf_{\|x\|=1} \|N_{V,W}x\| \\ &= \inf_{\|x\|=1} \|V(I - WVV^*)x\| \\ &= \inf_{\|x\|=1} \|(I - WVV^*)x\| \\ &\geq \inf_{\|x\|=1} \frac{\|x\|}{\|(I - WVV^*)^{-1}\|} \\ &= \frac{1}{\|(I - WVV^*)^{-1}\|} > 0. \end{aligned}$$

Hence, the conditions of Theorem 8 are valid for the operator $N_{V,W}$. Then, by applying Theorem 8, we have

$$\begin{aligned} qd(N_{V,W}) &\geq \frac{1}{\|(I - WVV^*)^{-1}\|} \\ &= \frac{1}{\|\sum_{n\geq 0}(WVV^*)^n\|} \\ &= \frac{1}{\|I + (\sum_{n\geq 1}(W^nVV^*)\|} \\ &= \frac{1}{\|I + (\sum_{n\geq 1} W^n)VV^*\|} \\ &\geq \frac{1}{\sum_{n\geq 0}\|W\|^n} \\ &= 1 - \|W\|, \end{aligned}$$

which completes the proof.

EXAMPLE 10. *Let $N_{V,\alpha} = V(I - \alpha VV^*)$, where α is a scalar, $|\alpha| < 1$* and *$V \in \mathcal{B}(H)$ is an isometry.*

(i) *Each of the operators $N_{V,\alpha}$ $(|\alpha| < 1)$ does not have any finite-dimensional reducing subspace.*

(ii) *$N_{V,\alpha}$ converges to $N_V = V(I - VV^*)$ in the uniform operator topology as $\alpha \to 1^-$.*

(iii) *N_V has a finite-dimensional reducing subspace.*

Proof. (i) Indeed, by Proposition 9 $qd(N_{V,\alpha}) \geq 1 - |\alpha|$, and hence, by the definition of the value $qd(N_{V,\alpha})$, each of the operators $N_{V,\alpha}$, where $|\alpha| < 1$, has no finite-dimensional reducing subspace.

(ii) From the equality

$$N_{V,\alpha}, = \alpha N_V + (1 - \alpha)V$$

it follows that $N_{V,\alpha} \to N_V$ as $\alpha \to 1^-$ in the uniform operator topology. Evidently, $N^2{}_V = 0$.

(iii) For arbitrary fixed $0 \neq x \in \ker V^*$ let $E_x = span\{x, Vx\}$. Then it is easy to verify that $N_V E_x \subset E_x$ and $N_V^* E_x \subset E_x$. Indeed, $N_V x = V(I - VV^*)x = Vx \in E_x$, $N_V Vx = V(I - VV^*)Vx = 0 \in E_x$, and hence, $N_V span\{x, Vx\} \subset span\{x, Vx\}$, i.e., $N_V E_x \subset E_x$. On the other hand, $N_V^* x = (I - VV^*)V^* x = 0 \in E_x$, $N_V^* Vx = (I - VV^*)V^* Vx = (I - VV^*)x = x \in E_x$, and so, $N_V^* E_x \subset E_x$. Hence E_x reduces N_V and $\dim E_x = 2$, which completes the proof.

4. Other properties. In this section we collect some other properties of the operators $N_{V,W}(W \in \{V\}')$.

PROPOSITION 11. *Let α be a scalar and $V, W \in \mathcal{B}(H)$ operators such that V is an isometry and $W \in \{V\}'$. Then the following statements are true.*

(*a*) *$r(N_{V,W}) \leq r(I - W)$, where $r(T)$ stands for the spectral radius of the operator T.*

(*b*) *If $\|W\| < 1$, then $\ker N_{V,W}^n = \ker N_{V,W}$ and $\ker N_{V,W}^{*n} = \ker V^*$, for any integer $n \geq 1$.*

(*c*) $r(N_{V,\alpha}) = |1 - \alpha|$.

(*d*) $\|N_{V,\alpha}\| \geq |1 - \alpha|$.

(*e*) *If $|\alpha|^2 - 2Re\alpha \geq 0$, then $\|N_{V,\alpha}\| = (|\alpha|^2 - 2Re\alpha + 1)^{\frac{1}{2}}$.*

(*f*) *$\omega(N_{V,\alpha}) \leq \frac{|\alpha|}{2} + |1 - \alpha|$, where $\omega(T)$ stands for the numerical radius of the operator T.*

(*g*) *If $0 < \alpha < 1$, then $1 - \alpha \leq \omega(N_{V,\alpha}) \leq 1 - \frac{\alpha}{2}$.*

(*h*) *If $\alpha > 2$ is any real number, then $r(N_{V,\alpha}) = \omega(N_{V,\alpha}) = \|N_{V,\alpha}\| = \alpha - 1$.*

Proof. (a) Since $N_{V,W} = WN_V + (I - W)V$, $WV - VW = 0$ and $N_V V = 0$, we have

$$\begin{aligned} N_{V,W}^2 &= [WN_V + (I - W)V]^2 \\ &= (I - W)WVN_V + (I - W)^2 V^2 \\ &= (I - W)V[WN_V + (I - W)V] \\ &= (I - W)VN_{V,W}, \end{aligned}$$

so that $N_{V,W}^2 = (I - W)VN_{V,W}$. After these simple calculations we conclude that

$$N_{V,W}^n = (I - W)^{n-1} V^{n-1} N_{V,W}, \tag{12}$$

for each $n \geq 1$. Since V is an isometry, by the last equality, it follows that $r(N_{V,W}) \leq r(I - W)$.

(b) From (12), it is clear that

$$N_{V,W}^{*n} = (I - VV^* W^*)(I - W^*)^{n-1} V^{*n}, \tag{13}$$

for each $n \geq 1$. The condition $\|W\| < 1$ ensures invertibility of the operators $(I - W)^{n-1}$ and $(I - VV^* W^*)(I - W^*)^{n-1}$, so that the equalities (12),(13) apply.

(c) If $W = \alpha I$, then (12) directly implies that $r(N_{V,\alpha}) = |1 - \alpha|$.

(c) $\Rightarrow$ (d).

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(e) In fact, for every $x \in H$, $\|x\| = 1$, we have

$$\begin{aligned}\|N_{V,\alpha}x\|^2 &= \|V(I-\alpha VV^*)x\|^2\\ &= \|(I-\alpha VV^*)x)\|^2\\ &= ((I-\alpha VV^*)x, (I-\alpha VV^*)x)\\ &= ((I-\alpha VV^* - \bar{\alpha}VV^* + |\alpha|^2 VV^*x), x)\\ &= ((I-(2Re\alpha)VV^* + |\alpha|^2 VV^*)x, x)\\ &= 1 + (|\alpha|^2 - 2Re\alpha)(VV^*x, x).\end{aligned}$$

Since $|\alpha|^2 - 2Re\alpha \geq 0$, we obtain $\|N_{V,\alpha}\| = (|\alpha|^2 - 2Re\alpha + 1)^{\frac{1}{2}}$ by taking the suprema of both sides over all unit vectors x in H.

(f) It is known [**3**] that the numerical range of the nilpotent operator N_V is a closed circular disc with center 0 and radius $\frac{1}{2}$. (Note that in the work of Tso and Wu [**13**] a more general theorem was proved, describing the numerical range of any quadratic operator on a complex Hilbert space.) Then, taking into account the equality $N_{V,\alpha} = \alpha N_V + (1-\alpha)V$, we get the required inequality.

(c), (f) $\Rightarrow$ (g).

(h) Since $r(N_{V,\alpha}) \leq \omega(N_{V,\alpha}) \leq \|N_{V,\alpha}\|$, from (c) and (e) follows (h). The proof of Proposition 11 is completed.

Now we apply the operator N_V to estimate the angle between subspaces of a Hilbert space H. The angle between subspaces $E_1 \subset H$ and $E_2 \subset H$ is determined as follows:

$$\langle E_1, E_2\rangle \in \left[0, \tfrac{\pi}{2}\right], \quad \cos\langle E_1, E_2\rangle = \sup\left\{\tfrac{|(x,y)|}{\|x\|\|y\|} : x \in E_1, y \in E_2\right\}.$$

From the definition it immediately follows that

$$\cos\langle E_1, E_2\rangle = \sup\left\{\frac{\|P_{E_2}x\|}{\|x\|} : x \in E_1\right\} = \left\|P_{E_2}P_{E_1}\right\|,$$
$$\sin\langle E_1, E_2\rangle = \inf\left\{\frac{\|(I-P_{E_2})x\|}{\|x\|} : x \in E_1\right\} = \left\|\mathcal{P}_{E_1\|E_2}\right\|^{-1},$$

where $P_{E_i}(i = 1, 2)$ are orthogonal projections of H onto $E_i(i = 1, 2)$ and $\mathcal{P}_{E_1\|E_2}$ is the projection onto E_1, parallel to E_2.

Proposition 12. *Let K be an arbitrary subspace of a Hilbert space H and let $V_1, V_2 \in \mathcal{B}(H)$ be isometries. Then*

$$|\cos\langle R(V_1)^\perp, K\rangle - \cos\langle R(V_2)^\perp, K\rangle| \leq \left\|N_{V_1} - N_{V_2}\right\|. \tag{14}$$

Proof. We use the arguments of the reference [**14**]. (See also [**15**].) Indeed,

$$\begin{aligned}\cos\langle R(V_2)^\perp, K\rangle &= \sup_{x\in K}\frac{\|P_{(V_2H)^\perp}x\|}{\|x\|}\\ &= \sup_{x\in K}\frac{\|V_2P_{(V_2H)^\perp}x\|}{\|x\|}\end{aligned}$$

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$$
\begin{aligned}
&= \sup_{x\in K} \frac{\|V_2(I - V_2V_2^*)x\|}{\|x\|} \\
&= \sup_{x\in K} \frac{\|N_{V_2}x\|}{\|x\|} \\
&\le \sup_{x\in K} \frac{\|(N_{V_2} - N_{V_1})x\| + \|N_{V_1}x\|}{\|x\|} \\
&\le \|N_{V_1} - N_{V_2}\| + \sup_{x\in K} \frac{\|N_{V_1}x\|}{\|x\|} \\
&= \|N_{V_1} - N_{V_2}\| + \sup_{x\in K} \frac{\|V_1P_{(V_1H)^\perp}x\|}{\|x\|} \\
&= \|N_{V_1} - N_{V_2}\| + \sup_{x\in K} \frac{\|P_{(V_1H)^\perp}x\|}{\|x\|} \\
&= \|N_{V_1} - N_{V_2}\| + \cos\langle (V_1H)^\perp, K\rangle \\
&= \|N_{V_1} - N_{V_2}\| + \cos\langle R(V_1)^\perp, K\rangle.
\end{aligned}
$$

Similarly, it can be shown that

$$\cos\langle R(V_1)^\perp, K\rangle \le \|N_{V_1} - N_{V_2}\| + \cos\langle R(V_2)^\perp, K\rangle.$$

From these inequalities, we get (14), which completes the proof.

We say that an isometry $V \in \mathcal{B}(H)$ has a *finite defect* if $\dim(VH)^\perp < +\infty$. Let us denote by IFD the set of isometries, with finite defects. Put $\mathcal{N}_{IFD} = \{N_V : V \in IFD\}$, which is a subset of the set of all finite-dimensional operators in H.

Corollary 13. *Let $V \in \mathcal{B}(H)$ be an isometry. Then the following inequalities are valid:*

$$
\begin{aligned}
\inf_{U\in IFD\cap\{V\}'} \cos\langle R(U)^\perp, R(V)\rangle &\le dist\big(N_V, \mathcal{N}_{IFD\cap\{V\}'}\big) \\
&\le 4dist\big(V, IFD\cap\{V\}'\big). \qquad (15)
\end{aligned}
$$

Proof. The simple calculations show that

$$\|N_{V_1} - N_{V_2}\| \le 4\|V_1 - V_2\|, \qquad (16)$$

for every pair of commuting isometries V_1 and V_2. In fact,

$$
\begin{aligned}
\|N_{V_1} - N_{V_2}\| &= \|V_1(I - V_1V_1^*) - V_2(I - V_2V_2^*)\| \\
&= \|V_1 - V_2 + V_2^2V_2^* - V_1^2V_1^*\| \\
&\le \|V_1 - V_2\| + \|V_2^2V_2^* - V_1^2V_2^* + V_1^2V_2^* - V_1^2V_1^*\| \\
&\le \|V_1 - V_2\| + \|(V_2^2 - V_1^2)V_2^*\| + \|V_1^2(V_2^* - V_1^*)\| \\
&\le \|V_1 - V_2\| + \|(V_2 - V_1)(V_2 + V_1)\| + \|V_2^* - V_1^*\| \\
&\le \|V_1 - V_2\| + 2\|V_1 - V_2\| + \|V_1 - V_2\| \\
&= 4\|V_1 - V_2\|.
\end{aligned}
$$

By inequality (14) we have

$$|\cos\langle R(U)^{\perp}, K\rangle - \cos\langle R(V)^{\perp}, K\rangle| \leq \|N_V - N_U\|,$$

for any $U \in IFD \cap \{V\}'$ and $K \subset H$. Then, by choosing $K = VH$ and using (16), from the last inequality we get (15). This completes the proof.

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