INFINITE SERIES OF E-FUNCTIONS

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1. Introductory. In § 2 a number of infinite series of E-functions are summed by expressing the E-functions as Barnes integrals and interchanging the order of summation and integration.

The Barnes integral employed is

$$E(p\,;\,\alpha_r:\,q\,;\,\rho_s:\,z)=\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\,\Pi\,\Gamma(\alpha_r-\zeta)}{\Pi\,\Gamma(\rho_s-\zeta)}z^{\zeta}\,d\zeta,\qquad\text{......................(1)}$$

where $|\operatorname{amp} z|<\pi$ and the integral is taken up the η-axis, with loops, if necessary, to ensure that the origin lies to the left of the contour and the points $\alpha_1, \alpha_2, \ldots, \alpha_p$ to the right of the contour. Zero and negative integral values of the α's and ρ's are excluded, and the α's must not differ by integral values. When $p<q+1$ the contour is bent to the left at each end.

The three following formulae, (2) due to Whipple, (3) to Dougall and (4) to Kummer, will be required.

If $R(\alpha-2\beta-2\gamma)>-2$,

$$F\binom{\alpha,\ \ 1+\frac{1}{2}\alpha,\ \ \beta,\ \ \gamma;\ \ -1}{\frac{1}{2}\alpha,\ \alpha-\beta+1,\ \alpha-\gamma+1}=\frac{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\gamma+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta-\gamma+1)}.\qquad\text{...............(2)}$$

If $R(\alpha-\beta-\gamma-\delta)>-1$,

$$F\binom{\alpha,\ \ 1+\frac{1}{2}\alpha,\ \ \beta,\ \ \gamma,\ \ \delta\ \ ;\ \ 1}{\frac{1}{2}\alpha,\ \alpha-\beta+1,\ \alpha-\gamma+1,\ \alpha-\delta+1}$$

$$=\frac{\Gamma(\alpha-\beta+1)\Gamma(\alpha-\gamma+1)\Gamma(\alpha-\delta+1)\Gamma(\alpha-\beta-\gamma-\delta+1)}{\Gamma(\alpha+1)\Gamma(\alpha-\beta-\gamma+1)\Gamma(\alpha-\gamma-\delta+1)\Gamma(\alpha-\delta-\beta+1)}.\qquad\text{......(3)}$$

If $R(\beta)<1$,

$$F\binom{\alpha,\ \beta\ \ ;\ -1}{\alpha-\beta+1}=\frac{\Gamma(\alpha-\beta+1)\Gamma(\frac{1}{2}\alpha+1)}{\Gamma(\alpha+1)\Gamma(\frac{1}{2}\alpha-\beta+1)}.\qquad\text{..........................(4)}$$

2. Infinite Series. The first summation is

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(2z)^n}E\binom{p\,;\,\alpha_r+n:\,2z}{q\,;\,\rho_s+n}=E(p\,;\,\alpha_r:\,q\,;\,\rho_s:\,z),\qquad\text{.....................(5)}$$

where $|\operatorname{amp} z|<\pi$.

To prove it, substitute from (1) on the left and get

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,(2z)^n}\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\,\Pi\,\Gamma(\alpha_r+n-\zeta)}{\Pi\,\Gamma(\rho_s+n-\zeta)}(2z)^{\zeta}\,d\zeta.$$

Here replace ζ by $\zeta+n$ and interchange the order of summation and integration, so getting

$$\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\,\Pi\,\Gamma(\alpha_r-\zeta)}{\Pi\,\Gamma(\rho_s-\zeta)}(2z)^{\zeta}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\zeta\,;\,n)\,d\zeta.$$

Now the sum of the last series is $(1+1)^{-\zeta}$. Hence, from (1), the result follows.

In the same way it can be shown that, if $|\operatorname{amp} z| < \pi$, $|\lambda - 1| < 1$,

$$\sum_{n=0}^{\infty} \frac{(1-\lambda)^n}{n!\,(\lambda z)^n} E\begin{pmatrix} p\,;\, \alpha_r + n \,:\, \lambda z \\ q\,;\, \rho_s + n \end{pmatrix} = E(p\,;\,\alpha_r : q\,;\,\rho_s : z). \qquad \ldots\ldots(6)$$

Likewise, if $|l/k| < 1$, $|\operatorname{amp} k| < \pi$, $|\operatorname{amp}(k+l)| < \pi$,

$$\sum_{n=0}^{\infty} (-1)^n \frac{l^n}{n!\,k^n} E(\alpha_1 + n, \alpha_2, \ldots, \alpha_p : q\,;\,\rho_s : k) = \left(\frac{k}{k+l}\right)^{\alpha_1} E(p\,;\,\alpha_r : q\,;\,\rho_s : k+l). \quad \ldots(7)$$

For the expression on the left is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\,\Pi\,\Gamma(\alpha_r - \zeta)}{\Pi\,\Gamma(\rho_s - \zeta)} k^{\zeta} F\left(\alpha_1 - \zeta\,;\,;\, -\frac{l}{k}\right) d\zeta,$$

and the sum of the series is $(1 + l/k)^{\zeta - \alpha_1}$.

Note. A special case of formula (7), with $p = 2$, $q = 0$, $\alpha_1 = 1$, was given by B. R. Bhonsle, *Bulletin of the Calcutta Mathematical Society*, **48** (1956), 97.

The next summation is

$$\sum_{n=0}^{\infty} \frac{(\rho_1 - \sigma\,;\, n)}{(\rho_1\,;\, n) n!} z^{-n} E\begin{pmatrix} p\,;\, \alpha_r + n & : z \\ q\,;\, \sigma + n, \rho_2 + n, \ldots, \rho_q + n & \end{pmatrix} = \frac{\Gamma(\rho_1)}{\Gamma(\sigma)} E\begin{pmatrix} p\,;\, \alpha_r : z \\ q\,;\, \rho_s \end{pmatrix}, \quad \ldots\ldots(8)$$

where $|\operatorname{amp} z| < \pi$, $R(\sigma) > 0$.

Here the series on the left is equal to

$$\sum_{n=0}^{\infty} \frac{(\rho_1 - \sigma\,;\, n)}{(\rho_1\,;\, n) n!} \frac{1}{2\pi i} \int \frac{\Gamma(\zeta + n)\,\Pi\,\Gamma(\alpha_r - \zeta)}{\Gamma(\sigma - \zeta) \prod_{s=2}^{q} \Gamma(\rho_s - \zeta)} z^{\zeta}\, d\zeta$$

$$= \frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\,\Pi\,\Gamma(\alpha_r - \zeta)}{\Gamma(\sigma - \zeta) \prod_{s=2}^{q} \Gamma(\rho_s - \zeta)} z^{\zeta} F\begin{pmatrix} \rho_1 - \sigma, \zeta\,;\, 1 \\ \rho_1 \end{pmatrix} d\zeta,$$

and, on applying Gauss's theorem, the result is obtained.

Again, if $|\operatorname{amp} z| < \pi$, $R(2\rho_1 - l) > 0$,

$$\sum_{n=0}^{\infty} \frac{(l+2n)\Gamma(l+n)\Gamma(\rho_1 - l)}{\Gamma(\rho_1 + n)\Gamma(\rho_1 - l - n)\, n!} z^{-n}\, E\begin{pmatrix} p\,;\, \alpha_r + n & : z \\ l + 2n + 1, \rho_2 + n, \ldots, \rho_q + n & \end{pmatrix}$$

$$= E(p\,;\,\alpha_r : q\,;\,\rho_s : z). \qquad \ldots\ldots\ldots(9)$$

On proceeding as before, and noting that

$$l + 2n = l(\tfrac{1}{2}l + 1\,;\, n)/(\tfrac{1}{2}l\,;\, n),$$

it is found that the series is equal to

$$\frac{\Gamma(l+1)}{\Gamma(\rho_1)} \frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\,\Pi\,\Gamma(\alpha_r - \zeta)}{\Gamma(l + 1 - \zeta) \prod_{s=2}^{q} \Gamma(\rho_s - \zeta)} z^{\zeta} F\begin{pmatrix} l, \tfrac{1}{2}l + 1, l - \rho_1 + 1, \zeta\,;\, -1 \\ \tfrac{1}{2}l, \rho_1, l + 1 - \zeta \end{pmatrix} d\zeta.$$

From this, on applying (2) and (1), the expression on the right is obtained.

Similarly, if $|\operatorname{amp} z| < \pi$, $R(\rho_1 - l) > 0$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(l + \rho_1 + n)(l+1\,;\, n)(l + \rho_1 + 2n)}{\Gamma(\rho_1 + n)\, n!\, z^n} E\begin{pmatrix} p\,;\, \alpha_r + n & : z \\ l + \rho_1 + 1 + 2n, \rho_2 + n, \ldots, \rho_q + n & \end{pmatrix}$$

$$= E(p\,;\,\alpha_r : q\,;\,\rho_s : z). \qquad \ldots\ldots\ldots(10)$$

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Next if $|\operatorname{amp} z| < \pi$, $R(\rho_1 - k) > 0$,

$$\sum_{n=0}^{\infty} \frac{(l+2n)\Gamma(l+n)(l-\rho_1+1\,;\,n)(k\,;\,n)\Gamma(\rho_1-k)}{\Gamma(\rho_1+n)\Gamma(l-k+1+n)\,n!\,z^n} E\left(\begin{matrix} l-k+1+n,\ \alpha_1+n,\ \ldots,\ \alpha_p+n & : z \\ l+2n+1,\ \rho_1-k+n,\ \rho_2+n,\ \ldots,\ \rho_q+n & \end{matrix}\right)$$
$$= E(p\,;\,\alpha_r : q\,;\,\rho_s\,;\,z). \quad \ldots\ldots\ldots(11)$$

For the series is equal to

$$\frac{\Gamma(l+1)\Gamma(\rho_1-k)}{\Gamma(\rho_1)\Gamma(l-k+1)} \frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\Gamma(l-k+1-\zeta)\,\Pi\,\Gamma(\alpha_r-\zeta)}{\Gamma(l+1-\zeta)\Gamma(\rho_1-k-\zeta)\prod_{s=2}^{q}\Gamma(\rho_s-\zeta)} z^{\zeta} F\left(\begin{matrix} l,\ \tfrac{1}{2}l+1,\ l-\rho_1+1,\ k,\ \zeta\ ;\ 1 \\ \tfrac{1}{2}l,\ \rho_1,\ l-k+1,\ l+1-\zeta \end{matrix}\right) d\zeta,$$

and, on applying (3), the result is obtained.

Finally, if $|\operatorname{amp} z| < \pi$,

$$\sum_{n=0}^{\infty} (-1)^n \frac{(k\,;\,n)}{n!\,z^n} E\left(\begin{matrix} \tfrac{1}{2}k+1+n,\ \alpha_1+n,\ \ldots,\ \alpha_p+n\ : & z \\ k+1+2n,\ \rho_1+n,\ \ldots,\ \rho_q+n & \end{matrix}\right)$$
$$= \frac{\Gamma(\tfrac{1}{2}k+1)}{\Gamma(k+1)} E(p\,;\,\alpha_r\,;\,q\,;\,\rho_s\,;\,z). \quad \ldots\ldots(12)$$

For the expression on the left is equal to

$$\frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\Gamma(\tfrac{1}{2}k+1-\zeta)\,\Pi\,\Gamma(\alpha_r-\zeta)}{\Gamma(k+1-\zeta)\,\Pi\,\Gamma(\rho_s-\zeta)} z^{\zeta} F\left(\begin{matrix} \zeta,\ k & ;\ -1 \\ k+1-\zeta & \end{matrix}\right) d\zeta,$$

and the result follows on applying (4).

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