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PAPER

Symmetry actions and brackets for adjoint-symmetries. II: Physical examples

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Abstract

Symmetries and adjoint-symmetries are two fundamental (coordinate-free) structures of PDE systems. Recent work has developed several new algebraic aspects of adjoint-symmetries: three fundamental actions of symmetries on adjoint-symmetries; a Lie bracket on the set of adjoint-symmetries given by the range of a symmetry action; a generalised Noether (pre-symplectic) operator constructed from any non-variational adjoint-symmetry. These results are illustrated here by considering five examples of physically interesting nonlinear PDE systems – nonlinear reaction-diffusion equations, Navier-Stokes equations for compressible viscous fluid flow, surface-gravity water wave equations, coupled solitary wave equations and a nonlinear acoustic equation.

1. Introduction

Symmetries and conservation laws are fundamental intrinsic (coordinate-free) structures of a PDE system [13, 23, 24]. From an algebraic viewpoint, the infinitesimal symmetries of a PDE are the solutions of the linearisation (Frechet derivative) equation holding on the space of solutions to the PDE. Solutions of the adjoint linearisation equation, holding on the space of solutions to the PDE, are called adjoint-symmetries [5, 25, 26] and provide a direct link to conservation laws. In particular, adjoint-symmetries that satisfy a certain variational condition represent multipliers which yield conservation laws [2, 5, 6, 20].

A recent study [3] has developed several new algebraic aspects of adjoint-symmetries:

- three fundamental actions of symmetries on adjoint-symmetries
- a Lie bracket on the set of adjoint-symmetries given by the range of a symmetry action
- a generalised Noether (pre-symplectic) operator constructed from any adjoint-symmetry that is not a multiplier

These results have some clear applications for PDE systems. Firstly, the symmetry actions on adjoint-symmetries can be used to produce a new adjoint-symmetry (and hence possibly a conservation law) from a known adjoint-symmetry and a known symmetry, while the Lie brackets on adjoint-symmetries allow a pair of known adjoint-symmetries to generate a new adjoint-symmetry (and hence possibly a conservation law), just as a pair of known symmetries can generate a new symmetry from their commutator. Secondly, for evolution PDEs, adjoint-symmetries can encode a Hamiltonian structure through the existence of a symplectic structure constructed from the Noether operator and a Hamiltonian functional given by a conservation law. Thirdly, a Lie bracket on adjoint-symmetries provides a corresponding

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bracket structure for conservation laws, which is a broad generalisation of a Poisson bracket applicable to non-Hamiltonian systems.

The present paper will illustrate these main results by considering five examples of physically interesting nonlinear PDE systems:

(1) coupled nonlinear reaction-diffusion equations

(2) Navier-Stokes equations for compressible viscous fluid flow

(3) Boussinesq system for surface-gravity water waves

(4) coupled solitary wave equations

(5) nonlinear acoustic equation

PDE systems (1), (2) and (4) will be considered in one spatial dimension; PDE systems (3) and (5) will be considered respectively in two and three spatial dimensions.

In each example, first, the Lie point symmetries and the low-order adjoint-symmetries will be summarised. Second, the three actions of the Lie point symmetries on the adjoint-symmetries will be presented, and the corresponding adjoint-symmetry commutator brackets will be obtained. Third, in examples (1) and (2), a correspondence between symmetries and adjoint-symmetries will shown to exist in the absence of any local variational structure (Hamiltonian or Lagrangian) for dissipative PDE systems. Fourth, a Noether (pre-symplectic) operator will be shown to arise directly from the symmetry actions in examples (3) to (5), using an adjoint-symmetry that is not a multiplier. In examples (3) and (4), this operator yields a symplectic 2-form and a corresponding Hamiltonian structure. In example (5), the Noether operator yields a Lagrangian structure. These latter examples will illustrate how variational structures are naturally encoded in the adjoint-symmetries of non-dissipative PDE systems.

All of the symmetries and adjoint-symmetries in the examples are obtained by solving the determining equations (2.2) and (2.3) through a standard method (see [2, 23]).

The rest of the paper is organised as follows. A summary of the symmetry actions, bracket structures and Noether operator is provided in Section 2. Sections 3–7 contain the five examples. Concluding remarks are given in Section 8.

2. Summary of symmetry actions and brackets for adjoint-symmetries

The mathematical setting will be calculus in jet space [23], which is summarised in the appendix of [3]. Partial derivatives and total derivatives are denoted using a coordinate notation. The Frechet derivative will be denoted by $'$. Adjoints of total derivatives and linear operators will be denoted by *. Prolongations will be denoted as pr. The transpose of a column/row vector and a matrix will be denoted by t. Hereafter, a 'symmetry' will refer to an infinitesimal symmetry in evolutionary form.

As explained in [3], some technical conditions related to local solvability, involutivity, and existence of a solved form for leading derivatives will be assumed on PDE systems, which are called regular systems [2]. These conditions hold for essentially all PDE systems of interest in physical applications. (See [7] for additional discussion.)

A general treatment of symmetries relevant for the present work can be found in [2, 13, 23].

2.1. Determining equations for symmetries and adjoint-symmetries

To begin, the algebraic formulation of determining equations for symmetries and adjoint-symmetries will be stated for a general (regular) PDE system of order N consisting of M equations

$$G^A(x, u^{(N)}) = 0, \quad A = 1, \dots, M \tag{2.1}$$

with independent variables x^i, $i = 1, \dots, n$, and dependent variables u^α, $\alpha = 1, \dots, m$. $\mathcal{E}_G$ will denote the solution space of the PDE system. The coordinate space $(x^i, u^\alpha, u^\alpha_j, \dots)$ is called the jet space $J^{(\infty)}$.

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The determining equation for symmetries is given by

$$G'(P)^A|_{\mathcal{E}_G} = 0, \tag{2.2}$$

where $P^\alpha(x, u^{(k)})$ is a set of functions representing the components of the symmetry in evolutionary form. Geometrically speaking, a symmetry is a vector field $\mathbf{X}_P = P^\alpha \partial_{u^\alpha}$ whose prolongation is tangent to $\mathcal{E}_G$ in $J^{(\infty)}$.

The adjoint of the symmetry equation (2.2) is the determining equation for adjoint-symmetries

$$G'^*(Q)_\alpha|_{\mathcal{E}_G} = 0, \tag{2.3}$$

where $Q_A(x, u^{(k)})$ is a set of functions representing the components of the adjoint-symmetry. Its geometrical meaning is that the 1-form $Q_A \mathbf{d}G^A$ in $J^{(\infty)}$ functionally vanishes on $\mathcal{E}_G$, as discussed in [9].

Off of the solution space $\mathcal{E}_G$, these determining equations are respectively given by

$$G'(P)^A = R_P(G)^A \tag{2.4}$$

and

$$G'^*(Q)_\alpha = R_Q(G)_\alpha, \tag{2.5}$$

where $R_P = (R_P)^{A\,I}_B D_I$ and $R_Q = (R_Q)^I_{\alpha\,B} D_I$ are some linear differential operator in total derivatives whose coefficients $(R_P)^{A\,I}_B$ and $(R_Q)^I_{\alpha\,B}$ are functions that are non-singular on $\mathcal{E}_G$.

Recall that a multiplier is a set of functions $\Lambda_A(x, u^{(k)})$ that are non-singular on $\mathcal{E}_G$ and satisfy $\Lambda_A G^A = D_i \Psi^i$ off of $\mathcal{E}_G$, for some vector function Ψ^i in $J^{(\infty)}$. This total divergence condition is equivalent to

$$0 = E_{u^\alpha}(\Lambda_A G^A) = \Lambda'^*(G)_\alpha + G'^*(\Lambda)_\alpha, \tag{2.6}$$

whereby Λ_A is an adjoint-symmetry,

$$G'^*(\Lambda)_\alpha|_{\mathcal{E}_G} = 0. \tag{2.7}$$

These equations (2.4), (2.5), (2.6) play a key role in formulating the actions of symmetries on adjoint-symmetries.

2.2. Actions of symmetries on adjoint-symmetries

There are two basic different actions of symmetries on adjoint-symmetries [3, 10]. One action arises geometrically from applying the Lie derivative with respect to a symmetry $\mathbf{X}_P$ to the determining equation for adjoint-symmetries, which yields

$$Q_A \xrightarrow{\mathbf{X}_P} Q'(P)_A + R^*_P(Q)_A. \tag{2.8}$$

The other action comes from the adjoint relationship between the determining equation for infinitesimal symmetries and adjoint-symmetries, yielding

$$Q_A \xrightarrow{\mathbf{X}_P} R^*_P(Q)_A - R^*_Q(P)_A. \tag{2.9}$$

Under this action, adjoint-symmetries are mapped into conservation law multipliers.

For adjoint-symmetries that are conservation law multipliers, these two actions coincide with the better known action of symmetries on multipliers [1, 4, 8]. Furthermore, the difference of the two actions produces a third action

$$Q_A \xrightarrow{\mathbf{X}_P} Q'(P)_A + R^*_Q(P)_A, \tag{2.10}$$

which has the property that it vanishes on multipliers, as seen from the multiplier determining equation (2.6).

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2.3. Dual actions and a Noether operator

For any symmetry action $Q_A \longrightarrow S_P(Q)_A$, there is a dual action

$$S_Q(P)_A := S_P(Q)_A \tag{2.11}$$

that maps symmetries into adjoint-symmetries. The symmetry action (2.10) is distinguished from the other two by the property that its dual action can be expressed as a linear operator in total derivatives:

$$\mathcal{J} := S_Q = Q' + R^*_Q. \tag{2.12}$$

This operator (2.12) maps symmetries into adjoint-symmetries and thus is a Noether (pre-symplectic) operator. Its formal inverse defines a pre-Hamiltonian (inverse Noether) operator which maps adjoint-symmetries into symmetries.

2.4. Lie Bracket for adjoint-symmetries

A dual symmetry action S_Q can be used to construct an associated Lie bracket on the subspace of adjoint-symmetries given by the range of the action. This yields a homomorphism from the Lie algebra of symmetries into a Lie algebra of adjoint-symmetries.

Let Symm_G and $\mathrm{AdjSymm}_G$ denote the linear spaces of symmetries and adjoint-symmetries. The Lie bracket is given by

$${}^Q[Q_1, Q_2]_A := S_Q([S_Q^{-1}Q_1, S_Q^{-1}Q_2])_A \tag{2.13}$$

on the linear space $S_Q(\mathrm{Symm}_G) \subseteq \mathrm{AdjSymm}_G$. Note that, since S_Q^{-1} is well-defined only modulo $\ker(S_Q)$, the condition that $\ker(S_Q)$ is an ideal is necessary and sufficient for the bracket to be well-defined. This condition will select a set of adjoint-symmetries Q_A that can be used in constructing the bracket. If there is more than one such adjoint-symmetry (up to scaling), then a natural choice will be to select a Q_A such that $\mathrm{ran}(S_Q)$ is maximal in $\mathrm{AdjSymm}_G$.

An alternative way to have the bracket be well-defined arises when the symmetry Lie algebra contains a scaling symmetry. If $\ker(S_Q)$ can be characterised as a subspace by its scaling weight, then the symmetry Lie algebra possesses an extra structure of a direct-sum decomposition as a linear space

$$\mathrm{Symm}_G = \ker(S_Q) \oplus \mathrm{coker}(S_Q) \tag{2.14}$$

with $\mathrm{coker}(S_Q)$ being defined by having a distinct scaling weight. In this situation, S_Q^{-1} can be defined as belonging to the subspace $\mathrm{coker}(S_Q)$, and hence, the bracket will be well-defined. More generally, a scaling decomposition (2.14) in which both $\ker(S_Q)$ and $\mathrm{coker}(S_Q)$ are each a direct sum of scaling homogeneous subspaces that have no scaling weights in common is sufficient.

2.5. Results for evolution PDEs

The preceding general results have a further development for evolution PDEs

$$u^\alpha_t = g^\alpha(x, u, \partial_x u, \ldots, \partial^N_x u), \tag{2.15}$$

where x now denotes the spatial independent variables x^i, $i = 1, \ldots, n$, while t is the time variable. Note that, for such a PDE system,

$$G^\alpha(t, x, u^{(N)}) = u^\alpha_t - g^\alpha(x, u, \partial_x u, \ldots, \partial^N_x u) \tag{2.16}$$

with $A = \alpha$ for the indices.

On the solution space $\mathcal{E}_G$, all t-derivatives of u^α can be eliminated in any expression through substituting the equation (2.15) and its spatial derivatives. Consequently, symmetries consist of a set of functions $P^\alpha(t, x, u, \partial_x u, \ldots, \partial^k_x u)$ satisfying

$$\partial_t P^\alpha + P'(g)^\alpha - g'(P)^\alpha = \partial_t P^\alpha + [g, P]^\alpha = 0 \tag{2.17}$$

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which is the symmetry determining equation (2.2) in simplified form off of $\mathcal{E}_G$. Hence,

$$R_P = P'. \tag{2.18}$$

Likewise, adjoint-symmetries consist of a set of functions $Q_\alpha(t, x, u, \partial_x u, \ldots, \partial_x^k u)$ satisfying the adjoint equation

$$-(\partial_t Q_\alpha + Q'(g)_\alpha + g'^*(Q)_\alpha) = 0 \tag{2.19}$$

which is a simplified form of the determining equation (2.19) off of $\mathcal{E}_G$. Thus,

$$R_Q = -Q'. \tag{2.20}$$

An equivalent formulation is given by

$$\partial_t Q_\alpha + \{Q, g\}^*_\alpha = 0 \tag{2.21}$$

in terms of the anti-commutator $\{A, B\} = A'(B) + B'(A)$, where $\{A, B\}^* = A'^*(B) + B'^*(A)$.

The well-known necessary and sufficient condition for an adjoint-symmetry to be a conservation law multiplier is that its Frechet derivative is self-adjoint

$$Q' = Q'^*, \tag{2.22}$$

which follows directly from equations (2.20) and (2.6). Self-adjointness (2.22) is equivalent to the property that Q_α is a variational derivative (gradient)

$$\Lambda_\alpha = E_{u^\alpha}(\Phi) \tag{2.23}$$

for some function $\Phi(x, u^{(k)})$, $k \geq 0$, where E_{u^α} is the Euler operator with respect to u^α.

The relations (2.18) and (2.20) give simplified expressions for the symmetry actions (2.8) and (2.9):

$$Q_\alpha \xrightarrow{\mathbf{X}_P} Q'(P)_\alpha + P'^*(Q)_\alpha, \tag{2.24}$$

$$Q_\alpha \xrightarrow{\mathbf{X}_P} Q'^*(P)_\alpha + P'^*(Q)_\alpha, = E_{u^\alpha}(P^\beta Q_\beta), \tag{2.25}$$

which coincide if Q_α is a conservation law multiplier. The symmetry action (2.10) is given by

$$Q_\alpha \xrightarrow{\mathbf{X}_P} Q'(P)_\alpha - Q'^*(P)_\alpha, \tag{2.26}$$

which is trivial if Q_α is a conservation law multiplier.

For the sequel, indices will be omitted for simplicity of notation wherever it is convenient.

2.6. Symplectic 2-form

The Noether operator defined by the symmetry action (2.26) is simply

$$\mathcal{J} = Q' - Q'^* = -\mathcal{J}^*, \tag{2.27}$$

which is skew. It gives rise to a 2-form on the linear space of symmetries:

$$\boldsymbol{\omega}_Q(P_1, P_2) = \int_{\mathbb{R}^n} (P_1^\alpha Q'(P_2)_\alpha - P_2^\alpha Q'(P_1)_\alpha)\, d^n x. \tag{2.28}$$

This 2-form is symplectic, namely $\mathbf{d}\boldsymbol{\omega}_Q = 0$, as proven in [3].

The formal inverse of the Noether operator (2.27) defines a pre-Hamiltonian (inverse Noether) operator $\mathcal{J}^{-1}$ which maps adjoint-symmetries into symmetries. It also formally yields a Poisson bracket defined by

$$\{F_1, F_2\}_{\mathcal{J}^{-1}} := \int_{\mathbb{R}^n} (\delta F_1/\delta u)\mathcal{J}^{-1}(\delta F_2/\delta u)\, d^n x \tag{2.29}$$

for functionals $F = \int_{\mathbb{R}^n} f(x, u^{(k)})\, d^n x$, where $\delta/\delta u$ denotes the variational derivative, namely $\delta F/\delta u^\alpha = E_{u^\alpha}(f)$. In particular, the Jacobi identity for this bracket holds as a consequence of closure of the symplectic 2-form (see [3], and also [23] for a related general result).

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3. Reaction-diffusion system

Consider a coupled system of mass-conserving reaction-diffusion equations with quadratic nonlinearities

$$u_t = \kappa_1 u_{xx} + \alpha u^p v, \quad v_t = \kappa_2 v_{xx} - \alpha u^p v, \tag{3.1}$$

where $\kappa_1 > 0$, $\kappa_2 > 0$ are the diffusivity coefficients; α is a reaction coefficient; $p > 0$ is an interaction power. This evolution system is a simplified model for two interacting reactive chemicals or ions that are diffusing in a solute, or two proteins in a cell with an activator-inhibitor interaction [19, 22], with densities $u(t, x)$ and $v(t, x)$. Here, the equilibrium concentrations are $u = v = 0$. Note that more general reactivities $\pm(\alpha_1 u - \beta_1 v)^p(\alpha_2 u - \beta_2 v)$ can be expressed in the form (3.1) through a linear transformation on (u, v) if the coefficient matrix $\begin{pmatrix} \alpha_1 & -\beta_1 \\ \alpha_2 & -\beta_2 \end{pmatrix}$ has a negative determinant.

Symmetries (in evolutionary form) $P^u\partial_u + P^v\partial_v$ are determined by the equations

$$(D_t P^u - \kappa_1 D_x^2 P^u - \alpha(pu^{p-1}vP^u + u^p P^v)|_{\mathcal{E}_G} = 0, \tag{3.2a}$$

$$(D_t P^v - \kappa_2 D_x^2 P^v + \alpha(pu^{p-1}vP^u + u^p P^v)|_{\mathcal{E}_G} = 0, \tag{3.2b}$$

where $\mathcal{E}_G$ denotes the solution space of the reaction-diffusion system (3.1). Adjoint-symmetries (Q^u, Q^v) are determined by the adjoint equations

$$(-D_t Q^u - \kappa_1 D_x^2 Q^u + \alpha p u^{p-1} v(Q^v - Q^u))|_{\mathcal{E}_G} = 0, \tag{3.3a}$$

$$(-D_t Q^v - \kappa_2 D_x^2 Q^v + \alpha u^p (Q^v - Q^u))|_{\mathcal{E}_G} = 0. \tag{3.3b}$$

Note that, in the general notation (2.1) for PDE systems, here

$$G = (G^u, G^v) = (u_t - \kappa_1 u_{xx} - \alpha u^p v, v_t - \kappa_2 v_{xx} + \alpha u^p v). \tag{3.4}$$

A basis for the linear space of Lie point symmetries, with $P = (P^u, P^v)$, consists of

$$P_1 = (u_t, v_t), \quad P_2 = (u_x, v_x), \quad P_3 = \left(u + ptu_t + \frac{1}{2}pxu_x, v + ptv_t + \frac{1}{2}pxv_x\right), \tag{3.5}$$

which represent generators for a time translation, a space translation, and a scaling. Their algebra is given by the non-zero commutators

$$[P_1, P_3] = -pP_1, \quad [P_2, P_3] = -\frac{1}{2}pP_2. \tag{3.6}$$

A basis of the linear space of adjoint-symmetries, $Q = (Q^u, Q^v)$, is given by

$$Q_1 = (1, 1), \quad Q_2 = (x, x), \tag{3.7}$$

which are also multipliers for conservation laws of mass $\mathcal{M} = \int_{\mathbb{R}} (u + v)\, dx$ and centre of mass $\mathcal{X} = \int_{\mathbb{R}} x(u + v)\, dx$.

Consequently, (cf Section 2.2) the third symmetry action (2.26) is trivial, while the other two symmetry actions (2.24) and (2.25) are given by the linear operator

$$S_P(Q) = (E_u(PQ^{\mathrm{t}}), E_v(PQ^{\mathrm{t}})). \tag{3.8}$$

This action is summarised in Table 1. Note that, for evaluating the symmetry actions, all t-derivatives of u and v are replaced through equations (3.1).

3.1. Adjoint-symmetry bracket

The adjoint-symmetry bracket (2.13) arising from this symmetry action is defined via the dual operator

$$S_Q(P) = (E_u(PQ^{\mathrm{t}}), E_v(PQ^{\mathrm{t}})). \tag{3.9}$$

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Table 1. *Reaction-diffusion system: symmetry action (3.8) on adjoint-symmetries*

	P_1	P_2	P_3
Q_1	0	0	$(1-\frac{1}{2}p)Q_1$
Q_2	0	$-Q_1$	$(1-p)Q_2$

To obtain the maximal domain, namely the whole linear space span(Q_1, Q_2), one can choose $Q = Q_2$, whence

$$S_{Q_2}(P) = (E_u(x(P^u + P^v)), E_v(x(P^u + P^v))). \tag{3.10}$$

Thereby, one has ker $(S_{Q_2}) = \text{span}(P_1)$, which is an ideal, and ran$(S_{Q_2}^{-1}) = \text{span}(P_2, P_3)$ modulo ker (S_{Q_2}). From Table 1, one then obtains

$$S_{Q_2}^{-1}(Q_1) = -P_2, \quad S_{Q_2}^{-1}(Q_2) = \frac{1}{1-p}P_3, \tag{3.11}$$

and thus the adjoint-symmetry bracket can be directly computed by

$$^{Q_2}[Q_1, Q_2] = S_{Q_2}\left(\left[-P_2, \frac{1}{1-p}P_3\right]\right) = \frac{p}{2(1-p)}S_{Q_2}(P_2) = \frac{p}{2(p-1)}Q_1 \tag{3.12}$$

through the symmetry commutator (3.6).

This bracket (3.12), shown in Table 2, is a non-trivial Lie bracket. It is isomorphic to the symmetry subalgebra $\mathcal{A} = \text{span}(P_2, P_3)$, which is generated by space translation and scaling. This correspondence between symmetries and adjoint-symmetries exists in the absence of any local variational structure (Hamiltonian or Lagrangian) for the reaction-diffusion equations (3.1).

Since the third symmetry action is trivial, both the corresponding Noether operator (2.27) and symplectic 2-form (2.28) are trivial. This is expected, as reaction-diffusion systems are inherently dissipative (namely, the linearised system is parabolic).

4. Navier-Stokes equations

Consider the Navier-Stokes equations for compressible fluid flow [11, 16] with fluid velocity $u(t, x)$ and the density $\rho(t, x)$ in one spatial dimension

$$u_t + uu_x = (1/\rho)(-p_x + \mu u_{xx}), \quad \rho_t + (u\rho)_x = 0, \tag{4.1}$$

where $\mu \neq 0$ is the viscosity coefficient. Here the pressure will be specified by a general polytropic equation of state

$$p(\rho) = k\rho^q, \quad q \neq 0 \tag{4.2}$$

with coefficient $k > 0$. All of the parameters will be treated as being arbitrary. In the general notation (2.1) for PDE systems,

$$G = (G^u, G^\rho) = (u_t + uu_x + (1/\rho)(p_x - \mu u_{xx}), \rho_t + (u\rho)_x). \tag{4.3}$$

The determining equations for symmetries $P^u\partial_u + P^\rho\partial_\rho$ (in evolutionary form) are given by

$$(D_tP^u + D_x(uP^u) + qD_x((p/\rho^2)P^\rho) + \mu(1/\rho)^2u_{xx}P^\rho - \mu(1/\rho)D_x^2P^u)|_{\mathcal{E}_G} = 0, \tag{4.4a}$$

$$(D_tP^\rho + D_x(uP^\rho + \rho P^u))|_{\mathcal{E}_G} = 0, \tag{4.4b}$$

where $\mathcal{E}_G$ denotes the solution space of system (4.1)–(4.2). The adjoint of these equations yields the determining equations for adjoint-symmetries (Q^u, Q^ρ):

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Table 2. *Reaction-diffusion system: adjoint-symmetry bracket*

	Q_1	Q_2
Q_1	0	$\frac{p}{2(p-1)}Q_1$
Q_2		0

$$(-D_tQ^u - uD_xQ^u - \rho D_xQ^\rho - \mu D_x^2((1/\rho)Q^u))|_{\mathcal{E}_G} = 0, \tag{4.5a}$$

$$(-Q_tP^\rho - uD_xQ^\rho - q(p/\rho^2)D_xQ^u + \mu(1/\rho)^2u_{xx}Q^u)|_{\mathcal{E}_G} = 0. \tag{4.5b}$$

A basis for the linear space of Lie point symmetries, with $P = (P^u, P^\rho)$, consists of generators for a time translation, a space translation, a Galilean boost, and a scaling:

$$P_1 = (u_t, \rho_t), \quad P_2 = (u_x, \rho_x), \quad P_3 = (1 - tu_x, -t\rho_x),$$
$$P_4 = \left(\frac{1-q}{1+q}u - \frac{2q}{1+q}tu_t - xu_x, -\frac{2}{1+q}\rho - \frac{2q}{1+q}t\rho_t - x\rho_x\right). \tag{4.6}$$

Their algebra is given by the non-zero commutators

$$[P_1, P_3] = P_2, \quad [P_1, P_4] = \frac{2q}{1+q}P_1, \quad [P_2, P_4] = P_2, \quad [P_3, P_4] = \frac{1-q}{1+q}P_3. \tag{4.7}$$

A basis of the linear space of adjoint-symmetries, $Q = (Q^u, Q^\rho)$, is given by

$$Q_1 = (0, 1), \quad Q_2 = (\rho, u), \quad Q_3 = (t\rho, tu - x). \tag{4.8}$$

They are multipliers for conservation laws of mass $\mathcal{M} = \int_{\mathbb{R}} \rho\, dx$, momentum $\mathcal{P} = \int_{\mathbb{R}} \rho u\, dx$, and Galilean momentum $\mathcal{G} = \int_{\mathbb{R}} \rho(tu - x)\, dx = t\mathcal{P} - \mathcal{X}(t)$ which is related to the motion of the centre of mass $\mathcal{X}(t) = \int_{\mathbb{R}} x\rho\, dx$.

The third symmetry action (2.26) is trivial (cf Section 2.2), while the other two symmetry actions (2.25), (2.24) are given by the linear operator (3.8) with $v = \rho$,

$$S_P(Q) = (E_u(PQ^t), E_\rho(PQ^t)). \tag{4.9}$$

This action is summarised in Table 3. For evaluating the symmetry actions, all t-derivatives of u and ρ are replaced through equations (4.1).

4.1. Adjoint-symmetry bracket

To obtain a maximal domain on which an adjoint-symmetry bracket (2.13) can be defined via the symmetry action (4.9), one seeks a maximal range for the dual operator

$$S_Q(P) = (E_u(PQ^t), E_\rho(PQ^t)). \tag{4.10}$$

From Table 3, it is clear that the maximal range will be the whole linear space of adjoint-symmetries, which is obtained for the choice $Q = Q_3 + c_2Q_2 + c_1Q_1$, where one has

$$\ker(S_{Q_3+c_2Q_2+c_1Q_1}) = \text{span}(P_3 - c_2P_2). \tag{4.11}$$

The subalgebra (4.11) is not an ideal, since $[P_1, P_3 - c_2P_2] = P_2$, and consequently, the resulting adjoint-symmetry bracket will depend on how $\text{coker}(S_{Q_3+c_2Q_2+c_1Q_1})$ is chosen in $\text{span}(P_1, P_2, P_3, P_4)$. However, the scaling symmetry P_4 can be utilised (cf Section 2.4) to fix a canonical choice of $\text{coker}(S_{Q_3+c_2Q_2+c_1Q_1})$ as follows. From the symmetry commutators (4.7), observe that $\text{span}(P_3)$ has a different scaling weight compared to $\text{span}(P_1)$ and $\text{span}(P_2)$. Also observe that $\text{span}(Q_1)$, $\text{span}(Q_2)$, and $\text{span}(Q_3)$ have different scaling weights. Hence, one can take $c_2 = c_1 = 0$, whereby $\ker(S_{Q_3}) = \text{span}(P_3)$

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Table 3. *Navier-Stokes equations: symmetry action (4.9) on adjoint-symmetries*

	P_1	P_2	P_3	P_4
Q_1	0	0	0	$\frac{q-1}{q+1}Q_1$
Q_2	0	0	Q_1	0
Q_3	$-Q_2$	Q_1	0	$\frac{2q}{q+1}Q_3$

Table 4. *Navier-Stokes equations: adjoint-symmetry bracket*

	Q_1	Q_2	Q_3
Q_1	0	0	$\frac{q+1}{2q}Q_1$
Q_2		0	Q_2
Q_3			0

and $\mathrm{coker}(S_{Q_3}) = \mathrm{span}(P_4) \oplus \mathrm{span}(P_1) \oplus \mathrm{span}(P_2)$ provides a well-defined decomposition (2.14) of the symmetry Lie algebra under scaling. This determines $\mathrm{ran}(S_{Q_3}^{-1}) = \mathrm{span}(P_1, P_2, P_4)$.

From Table 3, one now obtains

$$S_{Q_3}^{-1}(Q_1) = P_2, \quad S_{Q_3}^{-1}(Q_2) = -P_1, \quad S_{Q_3}^{-1}(Q_3) = \frac{q+1}{2q}P_4. \tag{4.12}$$

Hence, the adjoint-symmetry bracket (2.13) can be directly computed by

$${}^{Q_3}[Q_1, Q_2] = S_{Q_3}([P_2, -P_1]) = S_{Q_3}(0) = 0, \tag{4.13a}$$

$${}^{Q_3}[Q_1, Q_3] = S_{Q_3}\left(\left[P_2, \frac{q+1}{2q}P_4\right]\right) = \frac{q+1}{2q}S_{Q_3}(P_2) = \frac{q+1}{2q}Q_1, \tag{4.13b}$$

$${}^{Q_3}[Q_2, Q_3] = S_{Q_3}([-P_1, \frac{q+1}{2q}P_4]) = -S_{Q_3}(P_1) = Q_2, \tag{4.13c}$$

through the symmetry commutator (4.7).

This bracket (3.12), shown in Table 4, is a non-trivial Lie bracket on the whole linear space $\mathrm{span}(Q_1, Q_2, Q_3)$ of adjoint-symmetries. In particular, from the inverse of the dual operator (4.12), one sees that this Lie bracket structure is isomorphic to the symmetry subalgebra $\mathcal{A} = \mathrm{span}(P_1, P_2, P_4)$, which is generated by time translation, space translation, and scaling.

Remarkably, this correspondence between symmetries and adjoint-symmetries exists despite the lack of any local variational structure (Hamiltonian or Lagrangian) for the Navier-Stokes equations (4.1).

Finally, since the third symmetry action is trivial, both the corresponding Noether operator (2.27) and symplectic 2-form (2.28) are trivial. This is expected, as viscous fluid flow is inherently dissipative.

5. Boussinesq system

Long wavelength, small amplitude surface water waves are described by a system of coupled Boussinesq equations [14, 15]

$$\vec{v}_t + \nabla h + \vec{v}\cdot\nabla\vec{v} - \Delta\vec{v}_t = 0, \quad h_t + \nabla\cdot\vec{v} + \nabla\cdot(h\vec{v}) - \Delta h_t = 0, \tag{5.1}$$

where, up to scaling of variables, $h(t, x, y)$ is the water elevation and $\vec{v}(t, x, y)$ is the horizontal velocity of the water. For irrotational flow, one has $\vec{v} = \nabla u$, which gives the coupled equations

$$u_t + h + \frac{1}{2}|\nabla u|^2 - \Delta u_t = 0, \quad h_t + \Delta u + \nabla\cdot(h\nabla u) - \Delta h_t = 0 \tag{5.2}$$

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for $u(t, x, y)$ and $h(t, x, y)$. This system is denoted as $G = (G^u, G^h)$ in the general notation (2.1) for PDE systems.

Symmetries (in evolutionary form) $P^u\partial_u + P^h\partial_h$ are determined by the equations

$$(D_t(1 - D_x^2 - D_y^2)P^u + u_xD_xP^u + u_yD_yP^u + P^h)|_{\mathcal{E}_G} = 0, \tag{5.3a}$$

$$(D_t(1 - D_x^2 - D_y^2)P^h + D_x(u_xP^h + hD_xP^u) + D_y(u_yP^h + hD_yP^u) + (D_x^2 + D_y^2)P^u)|_{\mathcal{E}_G} = 0, \tag{5.3b}$$

where $\mathcal{E}_G$ denotes the solution space of the system (5.2). The linear space of Lie point symmetries, with $P = (P^u, P^h)$, is generated by a shift, a time translation, space translations, a rotation, and a scaling:

$$\begin{aligned} &P_1 = (1, 0), \quad P_2 = (u_t, h_t), \quad P_3 = (u_x, h_x), \quad P_4 = (u_y, h_y), \\ &P_5 = (xu_y - yu_x, xh_y - yh_x), \quad P_6 = (u + t(u_t - 2), 2h + th_t + 2). \end{aligned} \tag{5.4}$$

Their algebra has the non-zero commutators

$$[P_1, P_6] = P_1, \quad [P_2, P_6] = 2P_1 - P_2, \quad [P_3, P_5] = -P_4, \quad [P_4, P_5] = P_3. \tag{5.5}$$

Adjoint-symmetries $Q = (Q^u, Q^h)$ are determined by the adjoint of the symmetry equations

$$((D_x^2 + D_y^2 - 1)D_tQ^u - D_x(u_xQ^u) - D_y(u_yQ^u) + (D_x^2 + D_y^2)Q^h + D_x(hD_xQ^h) + D_y(hD_yQ^h))|_{\mathcal{E}_G} = 0, \tag{5.6a}$$

$$((D_x^2 + D_y^2 - 1)D_tQ^h - u_xD_xQ^h - u_yD_yQ^h + Q^u)|_{\mathcal{E}_G} = 0. \tag{5.6b}$$

The linear space of low-order adjoint-symmetries is given by the basis

$$\begin{aligned} &Q_1 = (1, 0), \quad Q_2 = (h_t, -u_t), \quad Q_3 = (h_x, -u_x), \quad Q_4 = (h_y, -u_y), \\ &Q_5 = (xh_y - yh_x, yu_x - xu_y), \quad Q_6 = (-2h - th_t - 2), u + t(u_t - 2)). \end{aligned} \tag{5.7}$$

The first five of these adjoint-symmetries are also multipliers for conservation laws of (up to an overall factor) mass, x- and y-momenta, angular momentum, and energy:

$$\mathcal{M} = \int_{\mathbb{R}^2} h\, dx\, dy, \tag{5.8a}$$

$$\mathcal{P}^x = \int_{\mathbb{R}^2} (u_xh_{xx} + u_yh_{xy} + uh_x)\, dx\, dy, \quad \mathcal{P}^y = \int_{\mathbb{R}^2} (u_xh_{xy} + u_yh_{yy} + uh_y)\, dx\, dy, \tag{5.8b}$$

$$\mathcal{I} = \int_{\mathbb{R}^2} (y(u_xh_{xx} + u_yh_{xy} + uh_x) - x(u_xh_{xy} + u_yh_{yy} + uh_y))\, dx\, dy, \tag{5.8c}$$

$$\mathcal{E} = \int_{\mathbb{R}^2} \frac{1}{2}(h^2 + h(u_x^2 + u_y^2))\, dx\, dy. \tag{5.8d}$$

The sixth adjoint-symmetry is not a multiplier. Consequently, (cf Section 2.2) all three symmetry actions (2.8), (2.9), (2.10) on adjoint-symmetries are different.

Moreover, the first and second actions differ only on the non-multiplier Q_6. They are computed by use of:

$$\begin{aligned} &R_{P_1} = 0, \quad R_{P_2} = \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix}, \quad R_{P_3} = \begin{pmatrix} D_x & 0 \\ 0 & D_x \end{pmatrix}, \quad R_{P_4} = \begin{pmatrix} D_y & 0 \\ 0 & D_y \end{pmatrix}, \\ &R_{P_5} = \begin{pmatrix} xD_y - yD_x & 0 \\ 0 & xD_y - yD_x \end{pmatrix}, \quad R_{P_6} = \begin{pmatrix} tD_t + 2 & 0 \\ 0 & tD_t + 3 \end{pmatrix}, \end{aligned} \tag{5.9}$$

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Table 5. *Boussinesq system: symmetry action (2.8) on adjoint-symmetries*

	P_1	P_2	P_3	P_4	P_5	P_6
Q_1	0	0	0	0	0	$2Q_1$
Q_2	0	0	0	0	0	$4Q_2-2Q_1$
Q_3	0	0	0	0	Q_4	$3Q_3$
Q_4	0	0	0	0	$-Q_3$	$3Q_4$
Q_5	0	0	$-Q_4$	Q_3	0	$3Q_5$
Q_6	Q_1	$2Q_1-Q_2$	0	0	0	$3Q_6$

Table 6. *Boussinesq system: symmetry action (2.9) on adjoint-symmetries*

	P_1	P_2	P_3	P_4	P_5	P_6
Q_1	0	0	0	0	0	$2Q_1$
Q_2	0	0	0	0	0	$4Q_2-2Q_1$
Q_3	0	0	0	0	Q_4	$3Q_3$
Q_4	0	0	0	0	$-Q_3$	$3Q_4$
Q_5	0	0	$-Q_4$	Q_3	0	$3Q_5$
Q_6	$-2Q_1$	$2Q_1-4Q_2$	$-3Q_3$	$-3Q_4$	$-3Q_5$	0

and

$$R_{Q_1}=0,\quad R_{Q_2}=\begin{pmatrix}0 & D_t\\ -D_t & 0\end{pmatrix},\quad R_{Q_3}=\begin{pmatrix}0 & D_x\\ -D_x & 0\end{pmatrix},\quad R_{Q_4}=\begin{pmatrix}0 & D_y\\ -D_y & 0\end{pmatrix},$$
$$R_{Q_5}=\begin{pmatrix}0 & xD_y-yD_x\\ yD_x-xD_y & 0\end{pmatrix},\quad R_{Q_6}=\begin{pmatrix}0 & tD_t+3\\ -tD_t-2 & 0\end{pmatrix},\tag{5.10}$$

which are readily derived from the expressions (5.4) and (5.7). This leads to the results shown in Tables 5 and 6.

Using the first action (2.8), one can obtain an adjoint-symmetry bracket on a maximal domain which is given by having a maximal range for the dual action (2.11). The kernel of the dual action is thereby desired to have a minimal dimension, and it also must be an ideal in the symmetry algebra. From the symmetry commutators (5.5), the only 1-dimensional ideal is span(P_1), while the 2-dimensional ideals consist of $\{P_1, P_2\}$, $\{P_3, P_4\}$. One can find the kernel of the dual action $S_Q(P)$ for $Q=\sum_{i=1..6} c_iQ_i$ from Table 5. The cases having dimension at most 2 consist of ker $(S_Q)=$ span$(P_5-\frac{c_4}{c_5}P_4-\frac{c_3}{c_5}P_3)$ if $c_5\neq 0$, and ker $(S_Q)=$ span(P_3,P_4) if $c_5=0$, $c_6\neq 0$. Only the latter case is an ideal. Then, the range of $S_{Q^v}(P^v)$ turns out to be contained in span$(Q_1, Q_2, Q_4-\frac{c_4}{c_3}Q_3, Q_6+\frac{{c_3}^2+{c_4}^2}{c_3c_6}Q_3)$, which will be the maximal domain for the adjoint-symmetry bracket. This yields the result shown in Table 7.

For the second action (2.9), one can show from Table 5 that none of the cases in which the kernel of the dual action $S_Q(P)$ for $Q=\sum_{i=1..6} c_iQ_i$ is at most 2-dimensional are ideals. The case having minimal dimension such that the kernel is an ideal turns out to be ker $(S_Q)=$ span(P_1,P_2,P_3,P_4), which arises when $c_5=c_6=0$. The maximal domain of the resulting adjoint-symmetry bracket is then 2 dimensional.

Finally, the third symmetry action (2.9), which is shown in Table 8, is non-trivial only on the non-multiplier Q_6. This action directly yields a bracket on the whole space span$(Q_1,Q_2,Q_3,Q_4,Q_5,Q_6)$, which is isomorphic to symmetry algebra.

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Table 7. *Boussinesq system: adjoint-symmetry bracket from symmetry action (2.8) with* $Q = Q_6 + c_4Q_4 + c_3Q_3 + c_2Q_2 + c_1Q_1$, *where* $Q_{4'} = Q_6 + \frac{{c_3}^2+{c_4}^2}{c_3}Q_3$, $Q_{3'} = Q_4 - \frac{c_4}{c_3}Q_3$, $Q_{2'} = Q_2$, $Q_{1'} = Q_1$

	$Q_{1'}$	$Q_{2'}$	$Q_{3'}$	$Q_{4'}$
$Q_{1'}$	0	0	0	$\frac{1}{3}Q_{1'}$
$Q_{2'}$		0	0	$\frac{2}{3}Q_{1'} - \frac{1}{3}Q_{2'}$
$Q_{3'}$			0	0
$Q_{4'}$				0

Table 8. *Boussinesq system: symmetry action (2.10) on the non-multiplier adjoint-symmetry*

	P_1	P_2	P_3	P_4	P_5	P_6
Q_6	$3Q_1$	$3Q_2$	$3Q_3$	$3Q_4$	$3Q_5$	$3Q_6$

5.1. Symplectic 2-form and Hamiltonian operator

The third symmetry action encodes a Noether operator $\mathcal{J}$ (cf Section 2.3) for the Boussinesq system (5.2). Specifically, one has

$$\mathcal{J} = \frac{1}{3}(Q_6' + R^*_{Q_6}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{5.11}$$

which is obtained from

$$Q'_6 = \begin{pmatrix} 0 & -2 - tD_t \\ 1 + tD_t & \end{pmatrix}, \quad R^*_{Q_6} = \begin{pmatrix} 0 & tD_t - 1 \\ -tD_t + 2 & 0 \end{pmatrix}. \tag{5.12}$$

The factor $\frac{1}{3}$ has been inserted here as a convenient normalisation for the sequel.

There is a bilinear form (2.28) associated with the Noether operator (5.11):

$$\boldsymbol{\omega}_{Q_6}(P, \tilde{P}) = \int \tilde{P}\mathcal{J}(P)^{\mathrm{t}}\, dx = \int (\tilde{P}^h P^u - \tilde{P}^u P^h)\, dx, \tag{5.13}$$

where $P^u\partial_u + P^h\partial_h$ and $\tilde{P}^u\partial_u + \tilde{P}^h\partial_h$ are any pair of symmetries. This bilinear form is skew and closed (cf Section 2.6), whence it defines a symplectic 2-form. In particular, $\mathcal{J}$ is a symplectic operator.

The inverse of $\mathcal{J}$ defines a Hamiltonian operator

$$\mathcal{H} = \mathcal{J}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.14}$$

This indicates that the Boussinesq system (5.2) has a Hamiltonian formulation. If the time-derivative terms in each equation in the system are combined on one side, then

$$(u_t - \Delta u_t, h_t - \Delta h_t)^{\mathrm{t}} = \mathcal{H}(\delta\mathcal{E}/\delta u, \delta\mathcal{E}/\delta h)^{\mathrm{t}} \tag{5.15}$$

yields a Hamiltonian formulation in terms of the conserved energy, $\mathcal{E}$. This formulation can be expressed in the equivalent form

$$\begin{pmatrix} u \\ h \end{pmatrix}_t = (1 - \Delta)^{-1}\mathcal{H}\begin{pmatrix} \delta\mathcal{E}/\delta u \\ \delta\mathcal{E}/\delta h \end{pmatrix}, \tag{5.16}$$

where $(1 - \Delta)^{-1}\mathcal{H}$ is also a Hamiltonian operator.

Thus, the symmetry action (6.17) in Table 8 involving the non-multiplier adjoint-symmetry directly encodes the Hamiltonian structure of the Boussinesq system (5.2). Moreover, because the Hamiltonian operator (5.14) is algebraic, it directly yields a Lagrangian structure:

$$\mathcal{H}\begin{pmatrix}G^u\\ G^h\end{pmatrix}=\begin{pmatrix}E_u(L)\\ E_h(L)\end{pmatrix},\quad L=(h-\Delta h)u_t+\frac{1}{2}(h^2+h(u_x^2+u_y^2)),\tag{5.17}$$

where G^u and G^h denote the respective PDEs in the Boussinesq system (5.2). Here, E_u and E_h are the Euler operators with respect to u and h.

6. Coupled solitary wave equations

The near-resonant interaction of weakly nonlinear solitary waves can be described by a coupled system of KdV equations [17]. Consider, in particular, the nonlinearly coupled equations

$$u_t+(uv)_x+u_{xxx}=0,\quad v_t+uu_x+\kappa v_{xxx}=0,\tag{6.1}$$

after scaling of the variables, where $\kappa\neq 0$ is a relative dispersion parameter. Here $u(t,x)$ and $v(t,x)$ are the amplitudes of the two waves. This system models [12, 21] the energy transfer between Rossby waves in equatorial latitudes and mid-latitudes in the atmosphere.

Symmetries (in evolutionary form) $P^u\partial_u+P^v\partial_v$ are determined by the equations

$$(D_tP^u+D_x(uP^v+vP^u)+D_x^3P^u)|_{\mathcal{E}_G}=0,\tag{6.2a}$$

$$(D_tP^v+D_x(uP^u)+\kappa D_x^3P^v)|_{\mathcal{E}_G}=0,\tag{6.2b}$$

where $\mathcal{E}_G$ denotes the solution space of the system (6.1). Adjoint-symmetries (Q^u,Q^v) are determined by the adjoint equations

$$(-D_tQ^u-vD_xQ^u-uD_xQ^v-D_x^3Q^u)|_{\mathcal{E}_G}=0,\tag{6.3a}$$

$$(-D_tQ^v-uD_xQ^u-\kappa D_x^3Q^v)|_{\mathcal{E}_G}=0.\tag{6.3b}$$

In the general notation (2.1) for PDE systems, here $G=(G^u,G^v)$ denotes the two respective equations (6.1).

The linear space of Lie point symmetries of this system (6.1), with $P=(P^u,P^v)$, is generated by a time translation, a space translation, and a scaling:

$$P_1=(u_t,v_t),\quad P_2=(u_x,v_x),\quad P_3=(2u+xu_x+3tu_t,2v+xv_x+3tv_t).\tag{6.4}$$

Their algebra has the non-zero commutators

$$[P_1,P_3]=-3P_1,\quad [P_2,P_3]=-P_2.\tag{6.5}$$

The linear space of low-order adjoint-symmetries, $Q=(Q^u,Q^v)$, is given by the basis

$$Q_1=(1,0),\quad Q_2=(0,1),\quad Q_3=(u,v),\quad Q_4=(uv+u_{xx},\frac{1}{2}u^2+\kappa v_{xx}).\tag{6.6}$$

These adjoint-symmetries are also multipliers for conservation laws of (up to an overall factor) the mass $\mathcal{M}^u=\int_{\mathbb{R}}u\,dx$ and $\mathcal{M}^v=\int_{\mathbb{R}}v\,dx$ for each wave, the combined momentum of the waves $\mathcal{P}=\int_{\mathbb{R}}(u^2+v^2)\,dx$, and the net energy of the waves $\mathcal{E}=\int_{\mathbb{R}}\frac{1}{2}(u_x^2+\kappa v_x^2-vu^2)\,dx$.

One sees that the third symmetry action (2.26) is trivial (cf Section 2.2), while the other two symmetry actions (2.24) and (2.25) are given by the linear operator (3.8). This action is summarised in Table 9. For evaluating the symmetry actions, all t-derivatives of u and v are replaced through equations (6.1).

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Table 9. *Coupled solitary wave equations: symmetry action (3.8) on adjoint-symmetries*

	P_1	P_2	P_3
Q_1	0	0	Q_1
Q_2	0	0	Q_2
Q_3	0	0	$3Q_3$
Q_4	0	0	$5Q_4$

6.1. A nonlocal adjoint-symmetry and associated adjoint-symmetry brackets

From the symmetry action shown in Table 9, the dual action has $\ker S_Q(P) = \mathrm{span}(P_1, P_2)$ for any choice of adjoint-symmetry Q. Hence, the cokernel, which is given by $\mathrm{span}(P_3)$, is only 1-dimensional. Since the dimension of the cokernel gives the dimension of the maximal domain on which a corresponding adjoint-symmetry bracket can be defined, the resulting bracket is trivial.

However, the situation changes when one considers the possibility of nonlocal adjoint-symmetries arising from potentials obtained via the mass conservation laws. These potentials are given by $u = U_x$ and $v = V_x$, and they satisfy the coupled system

$$U_t + U_xV_x + U_{xxx} = 0, \quad V_t + \frac{1}{2}U_x^2 + \kappa V_{xxx} = 0. \tag{6.7}$$

Adjoint-symmetries (Q^U, Q^V) of this system are determined by the equations

$$(-D_tQ^U - D_x(V_xQ^U + U_xQ^V) - D_x^3Q^U)|_{\mathcal{E}_G} = 0, \tag{6.8a}$$

$$(-D_tQ^V - D_x(U_xQ^U) - \kappa D_x^3Q^V)|_{\mathcal{E}_G} = 0. \tag{6.8b}$$

Note the relation

$$(Q^U, Q^V) = -D_x(Q^u, Q^v) \tag{6.9}$$

holds directly from the adjoint-symmetry equations (6.3) and (6.8).

The linear space of low-order adjoint-symmetries (Q^U, Q^V) is generated by three adjoint-symmetries, two of which are inherited from the adjoint-symmetries Q_3 and Q_4 of the original system (6.1) for u, v, through the relation (6.9). The other low-order adjoint-symmetry is given by

$$(Q^U, Q^V) = -(2U_x + xU_{xx} + 3tU_{tx}, 2V_x + xV_{xx} + 3tV_{tx}). \tag{6.10}$$

Applying the inverse relation

$$(Q^u, Q^v) = -D_x^{-1}(Q^U, Q^V), \tag{6.11}$$

one obtains the nonlocal adjoint-symmetry

$$Q_5 = (U + xU_x + 3tU_t, V + xV_x + 3tV_t) = \left(U + xu - 3t(uv + \kappa u_{xx}), V + xv - 3t\left(\frac{1}{2}u^2 + v_{xx}\right)\right) \tag{6.12}$$

admitted by the system (6.1) for u, v. One can straightforwardly show that this adjoint-symmetry is not a multiplier.

When the first symmetry action, as given by the linear operator (3.8), is applied to Q_5, one obtains

$$S_{P_1}(Q_5) = 5Q_4, \quad S_{P_2}(Q_5) = -3Q_3, \quad S_{P_3}(Q_5) = 0, \tag{6.13}$$

by using the variational derivative relations $E_u = -D_x^{-1}E_U$ and $E_v = -D_x^{-1}E_V$. Consequently, if one chooses

$$Q = Q_5 + c_2Q_2 + c_1Q_1 := Q_{5'} \tag{6.14}$$

Table 10. Coupled solitary wave equations: adjoint-symmetry bracket from symmetry action (3.8) with $Q = Q_5 + c_2Q_2 + c_1Q_1$

	$c_1Q_1 + c_2Q_2$	Q_3	Q_4
$c_1Q_1 + c_2Q_2$	0	Q_3	$3Q_4$
Q_3		0	0
Q_4			0

Table 11. Coupled solitary wave equations: symmetry action (6.17) on the nonlocal adjoint-symmetry (6.12)

	P_1	P_2	P_3
Q_5	$-2Q_4$	$2Q_3$	$2Q_5$

Table 12. Coupled solitary wave equations: adjoint-symmetry bracket from symmetry action (6.17) with $Q = Q_5$

	Q_3	Q_4	Q_5
Q_3	0	0	$-\frac{1}{2}Q_3$
Q_4		0	$-\frac{3}{2}Q_4$
Q_5			0

with at least one of c_1, c_2 being non-zero, then $\ker S_Q(P)$ is empty, and so the cokernel consists of the whole linear space of Lie point symmetries, $\mathrm{span}(P_1, P_2, P_3)$. This choice of Q produces a maximal domain for defining the adjoint-symmetry bracket (2.13).

From equation (6.13) and Table 9, one obtains

$$S_{P_1}(Q_{5'}) = S_{Q_{5'}}(P_1) = 5Q_4, \quad S_{P_2}(Q_{5'}) = S_{Q_{5'}}(P_2) = -3Q_3,$$
$$S_{P_3}(Q_{5'}) = S_{Q_{5'}}(P_3) = c_1Q_1 + c_2Q_2, \tag{6.15}$$

which yields

$$S^{-1}_{Q_{5'}}(c_1Q_1 + c_2Q_2) = P_3, \quad S^{-1}_{Q_{5'}}(Q_3) = -\frac{1}{3}P_2, \quad S^{-1}_{Q_{5'}}(Q_4) = \frac{1}{5}P_1. \tag{6.16}$$

The resulting adjoint-symmetry bracket (2.13) on the linear subspace $\mathrm{span}(c_1Q_1 + c_2Q_2, Q_3, Q_4)$ is shown in Table 10. This bracket is a Lie bracket which is isomorphic to the symmetry algebra (6.5).

Since Q_5 is not a multiplier, the third symmetry action (2.26) is now non-trivial. In terms of components $Q = (Q^u, Q^v)$ and $P = (P^u, P^v)$, the form of this symmetry action is given by the linear operator

$$S_P(Q) = \big(\mathrm{pr}\mathbf{X}_f(Q^u) - E_u(Qf^{\mathrm{t}}), \mathrm{pr}\mathbf{X}_f(Q^v) - E_v(Qf^{\mathrm{t}})\big)\big|_{f=P} \tag{6.17}$$

with $\mathbf{X}_f = f^u(t,x)\partial_u + f^v(t,x)\partial_v$, where $\mathrm{pr}\mathbf{X}$, E_u, E_v are regarded as operators in total derivatives when $f = (f^u(t,x), f^v(t,x))$ is replaced by $P = (P^u, P^v)$. For $Q = Q_5$, the resulting action is summarised in Table 11.

The range of this action is the linear subspace of adjoint-symmetries $\mathrm{span}(Q_3, Q_4, Q_5)$, which provides a maximal domain for the adjoint-symmetry bracket (2.13) with $Q = Q_5$, as shown in Table 12. The resulting bracket is a Lie bracket which is isomorphic to the symmetry algebra (6.5).

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6.2. Symplectic 2-form and Hamiltonian operator

The symmetry action (6.17) constructed in terms of the nonlocal adjoint-symmetry (6.12) encodes a Noether operator $\mathcal{J}$ (cf Section 2.3) for the coupled KdV equations (6.1). Specifically, one has $\mathcal{J}(P) = S_P(Q_5) = \big(\mathrm{pr}\mathbf{X}_f Q_5^u - E_u(Q_5 f^{\mathrm{t}}), \mathrm{pr}\mathbf{X}_f Q_5^v - E_v(Q_5 f^{\mathrm{t}})\big)\big|_{f=P}$, where

$$\begin{aligned}(\mathrm{pr}\mathbf{X}_f Q_5^u, \mathrm{pr}\mathbf{X}_f Q_5^v)|_{f=P} \\ = \big(D_x^{-1}P^u + xP^u - 3t(vP^u + uP^v + \kappa D_x^2 P^u), D_x^{-1}P^v + xP^v - 3t(uP^u + D_x^2 P^v)\big)\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}(E_u(Q_5 f^{\mathrm{t}}), E_v(Q_5 f^{\mathrm{t}}))|_{f=P} \\ = \big(-D_x^{-1}P^u + xP^u - 3t(vP^u + uP^v + \kappa D_x^2 P^u), -D_x^{-1}P^v + xP^v - 3t(uP^u + D_x^2 P^v)\big).\end{aligned} \tag{6.19}$$

This yields, after scaling by a convenient normalisation factor $\frac{1}{2}$,

$$\mathcal{J} = \begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix} \tag{6.20}$$

which actually is a symplectic operator. In particular, there is a bilinear form (2.28) associated with this operator,

$$\boldsymbol{\omega}_{Q_5}(P, \tilde{P}) = \int \tilde{P}\mathcal{J}(P)^{\mathrm{t}}\, dx = \int (\tilde{P}^u D_x^{-1} P^u + \tilde{P}^v D_x^{-1} P^v)\, dx, \tag{6.21}$$

where $P^u\partial_u + P^v\partial_v$ and $\tilde{P}^u\partial_u + \tilde{P}^v\partial_v$ are any pair of symmetries, and t denotes the transpose. Modulo boundary terms, this bilinear form is skew and closed (cf Section 2.6), and hence it defines a symplectic 2-form.

The inverse of $\mathcal{J}$ defines a Hamiltonian operator

$$\mathcal{H} = \mathcal{J}^{-1} = \begin{pmatrix} D_x & 0 \\ 0 & D_x \end{pmatrix}. \tag{6.22}$$

As a consequence, the coupled KdV equations (6.1) have the Hamiltonian formulation

$$(u_t, v_t)^{\mathrm{t}} = \mathcal{H}(\delta\mathcal{E}/\delta u, \delta\mathcal{E}/\delta v)^{\mathrm{t}} \tag{6.23}$$

in terms of the conserved energy, $\mathcal{E}$. From this formulation, one has $u_t = D_x(\delta\mathcal{E}/\delta u)$ and $v_t = D_x(\delta\mathcal{E}/\delta v)$, which are analogous to the well-known first Hamiltonian structure of the KdV equation.

The symmetry action (6.17) involving the nonlocal adjoint-symmetry (6.12) thereby directly encodes the Hamiltonian structure of the coupled KdV equations (6.1).

7. Nonlinear acoustic equation

Nonlinear and dissipative effects in the propagation of sound waves in a compressible medium (like gases, liquids, or human tissue) [18] can be modelled by the wave equation $p_{tt} - \beta(p^2)_{tt} - \alpha p_{ttt} = \Delta p$, known as Westervelt's equation [27], where $\alpha > 0$ is the damping coefficient and $\beta > 0$ is the nonlinearity coefficient which arises from the equation of state $\rho \approx p - \beta p^2 - \alpha p_t$. Here, $p(t, x, y, z)$ is the pressure fluctuation, $\rho(t, x, y, z)$ is the density, and units have been chosen so that the sound speed for small amplitude waves (i.e. the linear approximation) is $c = 1$.

For the situation of spherical waves, Westervelt's equation becomes

$$p_{tt} - \beta(p^2)_{tt} - \alpha p_{ttt} = p_{rr} + \frac{2}{r}p_r \tag{7.1}$$

for $p(t, r)$, where r is radial variable.

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The determining equation for symmetries (in evolutionary form) $P\partial_p$ is given by

$$(D_t^2(P+2\beta pP)-\alpha D_t^3P-D_r^2P+\frac{2}{r}D_rP)|_{\mathcal{E}_G}=0, \tag{7.2}$$

where $\mathcal{E}_G$ denotes the solution space of equation (7.1). Lie point symmetries are comprised by a time translation and two generalised scalings:

$$P_1=p_t, \quad P_2=2p+2tp_t+3rp_r-\frac{1}{\beta}, \tag{7.3}$$

and, when $\alpha=0$,

$$P_3=tp_t+rp_r. \tag{7.4}$$

The non-zero commutators in the symmetry algebra consist of

$$[P_1,P_2]=-2P_1, \quad [P_1,P_3]=-P_1. \tag{7.5}$$

Adjoint-symmetries Q are determined by the adjoint of equation (7.2):

$$(D_t^2Q+2\beta pD_t^2Q-D_r^2Q-\frac{2}{r}D_rQ+\frac{2}{r^2}Q)|_{\mathcal{E}_G}=0. \tag{7.6}$$

The linear space of low-order adjoint-symmetries is given by the basis

$$Q_1=r, \quad Q_2=r^2 \quad Q_3=t\,r, \quad Q_4=t\,r^2. \tag{7.7}$$

These adjoint-symmetries are also multipliers for conservation laws of the following quantities (up to an overall factor):

$$\mathcal{I}=\int_0^\infty((1-2\beta p)p_t-\alpha p_{tt})\,r^2\,dr=\frac{d}{dt}\mathcal{M}(t), \quad \mathcal{X}=t\mathcal{I}-\mathcal{M}(t), \tag{7.8}$$

$$\mathcal{I}^r=\int_0^\infty((1-2\beta p)p_t-\alpha p_{tt})\,r\,dr=\frac{d}{dt}\mathcal{M}^r(t), \quad \mathcal{X}^r=t\mathcal{I}^r-\mathcal{M}^r(t), \tag{7.9}$$

where, from the equation of state, $\mathcal{M}(t)\approx\int_0^\infty\rho\,r^2\,dr$ and $\mathcal{M}^r(t)\approx\int_0^\infty\rho\,r\,dr$. The latter two integrals have the respective physical meanings of the total mass and a weighted total mass with the weighting factor being $\frac{1}{r}$. Hence, conservation of the quantity $\mathcal{I}$ corresponds to $\mathcal{M}(t)$ having a constant rate of change, while conservation of χ gives $\mathcal{M}(t)=t\mathcal{I}-\mathcal{M}(0)$. Conservation of $\mathcal{I}^r$ and χ^r are their radially weighted counterparts.

Because the symmetry space is two dimensional if $\alpha\neq 0$, the maximal domain on which an adjoint-symmetry bracket can be defined is a 2-dimensional subspace of the four-dimensional space of adjoint-symmetries. In the case $\alpha=0$, the maximal domain is at most 3-dimensional.

However, an adjoint-symmetry bracket on a much larger space can be found by considering the adjoint-symmetries and symmetries in a potential system. Potentials can be introduced in several different ways through each of the four conservation laws. The most useful potential arises through two layers as follows, using the conservation law for transverse momentum.

First, in the standard way [13], put

$$(1-2\beta p)p_t-\alpha p_{tt}=r^{-2}u_r, \tag{7.10a}$$

$$p_r=r^{-2}u_t, \tag{7.10b}$$

which turns the wave equation (7.1) into an identity satisfied by the potential $u(t,r)$. Next, introduce a further potential $v(t,r)$ through equation (7.10b), by putting

$$p=r^{-2}v_t, \quad r^{-2}u=(r^{-2}v)_r. \tag{7.11}$$

Then, equation (7.10a) yields $v_{tt}-2\beta r^{-2}v_tv_{tt}-\alpha v_{ttt}=u_r$, which gives a second layer potential system

$$v_r=2r^{-1}v+u, \quad (v_t-\beta r^{-2}v_t^2-\alpha v_{tt})_t=u_r, \tag{7.12}$$

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or equivalently

$$(v_t - \beta r^{-2} v_t^2 - \alpha v_{tt})_t = \left(v_r - \frac{2}{r} v \right)_r . \tag{7.13}$$

7.1. Adjoint-symmetry brackets arising from a potential system

Symmetries (in evolutionary form) $P^v \partial_v$ of the potential equation (7.13) are determined by the equation

$$(D_t^2 P^v - 2\beta r^{-2} D_t(v_t D_t P^v) - \alpha D_t^3 P^v - D_r^2 P^v + 2D_r(r^{-1} P^v))|_{\mathcal{E}_{G^v}} = 0, \tag{7.14}$$

where $\mathcal{E}_{G^v}$ denotes the solution space of equation (7.13). Adjoint-symmetries Q^v are determined by the adjoint equation

$$(D_t^2 Q^v - 2\beta r^{-2} D_t(v_t D_t Q^v) + \alpha D_t^3 Q^v - D_r^2 Q^v - 2r^{-1} D_r Q^v))|_{\mathcal{E}_{G^v}} = 0. \tag{7.15}$$

The linear space of Lie point symmetries of the potential equation (7.13) is generated by two shifts, in addition to the three point symmetries that are inherited from the acoustic wave equation (7.1) via the relation

$$P = r^{-2} D_t P^v. \tag{7.16}$$

In particular, these symmetries are given by

$$P_1^v = r^2, \quad P_2^v = r, \quad P_3^v = v_t, \quad P_4^v = 2tv_t + 3rv_r - 6v - \frac{1}{\beta} t\, r^2, \tag{7.17}$$

and, when $\alpha = 0$,

$$P_5^v = tv_t + rv_r - 3v. \tag{7.18}$$

Their algebra has the non-zero commutators

$$\begin{aligned} &[P_1^v, P_5^v] = -P_1^v, \quad [P_2^v, P_4^v] = -3P_2^v, \quad [P_2^v, P_5^v] = -2P_2^v, \\ &[P_3^v, P_4^v] = \frac{1}{\beta} P_1^v - 2P_3^v, \quad [P_3^v, P_5^v] = -P_3^v. \end{aligned} \tag{7.19}$$

Note that the shifts P_1^v and P_2^v project onto trivial symmetries $P = 0$ of the acoustic wave equation (7.1).

The linear space of low-order adjoint-symmetries is given by the basis

$$Q_1^v = 1, \quad Q_2^v = r^{-1}, \tag{7.20}$$

and, when $\alpha = 0$,

$$Q_3^v = r^{-2} v_t, \quad Q_4^v = r^{-2}(3tv_t - 9v) + 7r^{-1} v_r - 4\frac{1}{\beta} t, \tag{7.21a}$$

$$Q_5^v = r^{-1} v_r - \frac{1}{\beta} t. \tag{7.21b}$$

One can easily check that the adjoint-symmetry (7.21b) is not a conservation law multiplier. The two adjoint-symmetries (7.21a) are multipliers for conservation of energy

$$\mathcal{E} = \int_0^\infty \left(\frac{1}{2} p^2 - \frac{2}{3} \beta p^3 + \frac{1}{2} r^{-4} v_r^2 - r^{-6} v^2 \right) r^2\, dr, \tag{7.22}$$

and a dilation energy

$$\begin{aligned} \mathcal{W} = \int_0^\infty \Bigg(& t \left(\frac{11}{2} p^2 - 2\beta p^3 - 4\frac{1}{\beta} p + \frac{3}{2} r^{-2} v_r^2 - 3r^{-6} v^2 \right) \\ & + 7r^{-1}(p - \beta p^2) - \left(9p - 9\beta p^2 - 4\frac{1}{\beta} \right) v \Bigg)\, r^2\, dr, \end{aligned} \tag{7.23}$$

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Table 13. *Acoustic potential equation: symmetry action (2.8) on adjoint-symmetries*

	P^v_1	P^v_2	P^v_3	P^v_4	P^v_5
Q^v_1	0	0	0	$-5Q^v_1$	$-3Q^v_1$
Q^v_2	0	0	0	$-2Q^v_2$	$-2Q^v_2$
Q^v_3	0	0	0	$-\frac{1}{\beta}Q^v_1 - 3Q^v_3$	$-3Q^v_3$
Q^v_4	$5Q^v_1$	$-2Q^v_2$	$4\frac{1}{\beta}Q^v_1 - 3Q^v_3$	$-5Q^v_4$	$-4Q^v_4$
Q^v_5	$2Q^v_1$	Q^v_2	$\frac{1}{\beta}Q^v_1$	$-5Q^v_5$	$-4Q^v_5$

Table 14. *Acoustic potential equation: symmetry action (2.9) on adjoint-symmetries*

	P^v_1	P^v_2	P^v_3	P^v_4	P^v_5
Q^v_1	0	0	0	$-5Q^v_1$	$-3Q^v_1$
Q^v_2	0	0	0	$-2Q^v_2$	$-2Q^v_2$
Q^v_3	0	0	0	$-\frac{1}{\beta}Q^v_1 - 3Q^v_3$	$-3Q^v_3$
Q^v_4	$5Q^v_1$	$-2Q^v_2$	$4\frac{1}{\beta}Q^v_1 - 3Q^v_3$	$-5Q^v_4$	$-4Q^v_4$
Q^v_5	$-Q^v_1$	$-2Q^v_2$	$\frac{1}{\beta}Q^v_1 - 3Q^v_3$	$-2Q^v_4$	$-Q^v_4$

Table 15. *Acoustic potential equation: symmetry action (2.10) on the non-multiplier adjoint-symmetry*

	P^v_1	P^v_2	P^v_3	P^v_4	P^v_5
Q^v_5	$3Q^v_1$	$3Q^v_2$	$3Q^v_3$	$2Q^v_4 - 5Q^v_5$	$Q^v_4 - 4Q^v_5$

respectively. Finally, the first two adjoint-symmetries (7.20) are inherited from Q_3 and Q_4 via the relation

$$Q^v = -r^{-2}D_t Q \tag{7.24}$$

which follows directly from the adjoint-symmetry determining equations (7.6) and (7.15). These inherited adjoint-symmetries are thus the respective multipliers for conservation of $\mathcal{X}$ and $\mathcal{X}^r$. (Note that $\mathcal{P}$ and $\mathcal{P}^r$ are locally trivially when they are evaluated on $\mathcal{E}_{G^v}$ in terms of the potential v.)

The three actions of the symmetries (7.17)–(7.18) on the adjoint-symmetries (7.20)–(7.21) give rise to adjoint-symmetry brackets at the level of the potential equation (7.13). This structure will not be preserved under projection (7.16) and (7.24) back to the acoustic wave equation, since two of the potential symmetries are lost, in contrast to the example of the coupled solitary wave equations considered in the previous section.

The first and second actions (2.8), (2.9) here differ only on the non-multiplier Q_5. They are computed from the symmetry and adjoint-symmetry expressions by use of:

$$R_{P^v_1} = R_{P^v_2} = 0, \quad R_{P^v_3} = D_t, \quad R_{P^v_4} = 2tD_t + 3rD_r, \quad R_{P^v_5} = tD_t + rD_r - 1, \tag{7.25}$$

and

$$R_{Q^v_1} = R_{Q^v_2} = 0, \quad R_{Q^v_3} = r^{-}2D_t, \quad R_{Q^v_4} = r^{-2}(3tD_t + 7rD_r + 5), \quad R_{Q^v_5} = r^{-2}(rD_r + 2). \tag{7.26}$$

This leads to the results shown in Tables 13 and 14.

The third action (2.10), which is shown in Table 15, is non-trivial only on the non-multiplier Q_5.

Now, the adjoint-symmetry brackets arising from the dual actions (2.11) will be considered. The maximal domain for defining a bracket is given by having a maximal range for the dual action, and, therefore, a minimal dimension for the kernel which also must be an ideal in the symmetry algebra. From

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Table 16. *Acoustic potential equation: adjoint-symmetry bracket from symmetry action (2.8) with* $Q^v = Q^v_5 - \frac{2}{5}Q^v_4 + c_3Q^v_3 + c_2Q^v_2 + c_1Q^v_1$, *where* $Q^v_{4'} = \lambda(Q^v_5 - \frac{2}{5}Q^v_4)$, $Q^v_{3'} = \lambda Q^v_3$, $Q^v_{2'} = \lambda Q^v_2$, $Q^v_{1'} = \lambda(Q^v_1 - 2\beta Q^v_3)$, $\lambda = c_3 + 2\beta c_1$

	$Q^v_{1'}$	$Q^v_{2'}$	$Q^v_{3'}$	$Q^v_{4'}$
$Q^v_{1'}$	0	$-\frac{4\beta}{5}Q^v_{2'}$	$\frac{3}{5}Q^v_{1'}$	$-\frac{3c_3}{5}Q^v_{1'} + \frac{4\beta c_2}{5}Q^v_{2'}$
$Q^v_{2'}$		0	$\frac{2}{5}Q^v_{2'}$	$\frac{c_3+2\beta c_1}{5}Q^v_{2'}$
$Q^v_{3'}$			0	$\frac{\beta c_1 - c_3}{5\beta}Q^v_{1'} + \frac{2c_2}{5}Q^v_{2'}$
$Q^v_{4'}$				0

Table 17. *Acoustic potential equation: adjoint-symmetry bracket from symmetry action (2.8) with* $Q^v = Q^v_5 + \frac{1}{2}Q^v_4 + c_3Q^v_3 + c_2Q^v_2 + c_1Q^v_1$, *where* $Q^v_{4'} = Q^v_5 + \frac{1}{2}Q^v_4$, $Q^v_{3'} = Q^v_3$, $Q^v_{2'} = c_2Q^v_2$, $Q^v_{1'} = Q^v_1$

	$Q^v_{1'}$	$Q^v_{2'}$	$Q^v_{3'}$	$Q^v_{4'}$
$Q^v_{1'}$	0	$\frac{5}{2}Q^v_{1'}$	0	$-Q^v_{1'}$
$Q^v_{2'}$		0	$-\frac{2}{\beta}Q^v_{1'} + \frac{3}{2}Q^v_{3'}$	$\frac{4c_3+5\beta c_1}{2\beta}Q^v_{1'} - \frac{3c_3}{2}Q^v_{3'}$
$Q^v_{3'}$			0	$\frac{1}{\beta}Q^v_{1'} + Q^v_{3'}$
$Q^v_{4'}$				0

the symmetry commutators (7.19), the ideals with smallest dimension consist of span(P^v_1), span(P^v_2), which are 1-dimensional.

For the first symmetry action, shown in Table 13, the dual action $S_{Q^v}(P^v)$ given by $Q^v = \sum_{i=1..5} c_iQ^v_i$ will have span(P^v_1) as the kernel if $c_5 = 1$, $c_4 = -\frac{2}{5}$, and span(P^v_2) as the kernel if $c_5 = 1$, $c_4 = \frac{1}{2}$. In both cases, the range of $S_{Q^v}(P^v)$ is contained in span($Q^v_1, Q^v_2, Q^v_3, Q^v_5 + c_4Q^v_4$), which will be the maximal domain for the resulting adjoint-symmetry bracket. This yields the brackets shown in Tables 16 and 17.

For the second symmetry action, shown in Table 14, the dual action $S_{Q^v}(P^v)$ given by $Q^v = \sum_{i=1..5} c_iQ^v_i$ never has span(P^v_1) or span(P^v_2) as the kernel. In particular, in the first case, one needs $c_5 = 1$, $c_4 = \frac{1}{5}$, but then the kernel also contains span($P^v_5 - \frac{3}{5}P^v_4 - \frac{c_3}{3}P^v_3 - \frac{c_2}{3}P^v_2$); and in the second case, one needs $c_5 = 1$, $c_4 = -1$, whereby the kernel then also contains span($P^v_3 - \frac{1}{2\beta}P^v_1, P^v_5 - P^v_4 + \frac{c_3+2\beta c_1}{6\beta}P^v_1$). In both cases, the kernel is at least 2-dimensional, and hence, the maximal domain on which an adjoint-symmetry bracket can be defined is at most 3-dimensional, which is no larger than the domain arising at the level of the acoustic wave equation.

Finally, for the third symmetry action, which appears in Table 15, the dual action with $Q^v = Q^v_5$ has an empty kernel, whence an adjoint-symmetry bracket is obtained on the whole space of adjoint-symmetries span($Q^v_1, Q^v_2, Q^v_3, Q^v_4, Q^v_5$). This bracket is shown in Table 18, and one sees that it is isomorphic to symmetry algebra through the correspondence $Q^v_1 \leftrightarrow P^v_1$, $Q^v_2 \leftrightarrow P^v_2$, $Q^v_3 \leftrightarrow P^v_3$, $Q^v_4 \leftrightarrow 4P^v_4 - 5P^v_5$, $Q^v_5 \leftrightarrow P^v_4 - 2P^v_5$.

7.2. Noether operator and variational structure

The third symmetry action (2.10) encodes a Noether operator $\mathcal{J}$ for the acoustic potential equation (7.12) with $\alpha = 0$, namely when there is no damping.

This operator is given by $\mathcal{J}^v = Q^{v\prime}_5 + R^*_{Q^v_5} = 3r^{-2}$, since one has

$$Q^{v\prime}_5 = r^{-1}D_r, \quad R^*_{Q^v_5} = r^{-2}(3 - rD_r). \tag{7.27}$$

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Table 18. *Acoustic potential equation: adjoint-symmetry bracket from symmetry action (2.10) with $Q^v = Q^v_5$*

	Q^v_1	Q^v_2	Q^v_3	Q^v_4	Q^v_5
Q^v_1	0	0	0	$\frac{5}{3}Q^v_1$	$\frac{2}{3}Q^v_1$
Q^v_2		0	0	$-\frac{2}{3}Q^v_2$	$\frac{1}{3}Q^v_2$
Q^v_3			0	$\frac{4}{3\beta}Q^v_1 - Q^v_3$	$\frac{1}{3\beta}Q^v_1$
Q^v_4				0	0
Q^v_5					0

For convenience, a factor $\frac{1}{3}$ will be inserted in hereafter so that

$$\mathcal{J}^v = r^{-2}. \tag{7.28}$$

The Noether operator (7.28) provides a mapping of symmetries into adjoint-symmetries:

$$\mathcal{J}^v(P^v) = r^{-2}P^v = Q^v. \tag{7.29}$$

Specifically, in terms of the respective symmetry basis (7.17)–(7.18) and adjoint-symmetry basis (7.20)–(7.21), one sees that $\mathcal{J}^v(P^v_1) = Q^v_1$, $\mathcal{J}^v(P^v_2) = Q^v_2$, $\mathcal{J}^v(P^v_3) = Q^v_3$, $\mathcal{J}^v(P^v_4) = \frac{2}{3}Q^v_4 - \frac{5}{3}Q^v_5$, $\mathcal{J}^v(P^v_5) = \frac{1}{3}Q^v_4 - \frac{4}{3}Q^v_5$.

As the Noether operator is algebraic, it yields a Lagrangian structure:

$$\mathcal{J}^v(G^v) = E_v(L), \tag{7.30}$$

where G^v denotes the potential equation (7.12) with $\alpha = 0$, and where the Lagrangian is straightforwardly found to be

$$L = \frac{1}{2}r^{-2}(v_r^2 - v_t^2) + r^{-4}\left(\frac{1}{3}\beta v_t^3 - v^2\right). \tag{7.31}$$

Here, E_v is the Euler operator with respect to v.

The preceding structure can be lifted to the acoustic wave equation (7.1) through the relations (7.24) and

$$r^{-2}D_tG^v = G, \tag{7.32}$$

where G denotes the PDE (7.1) with $\alpha = 0$. These relations imply that

$$\mathcal{J} = -r^2D_t^{-1}\mathcal{J}^v r^2 D_t^{-1} = -r^2D_t^{-2} \tag{7.33}$$

defines a Noether operator for the acoustic wave equation (7.1) with no damping, $\alpha = 0$. In particular, using the form of the potential $p = r^{-2}v_t$, one has

$$\mathcal{J}(P_1) = -r^2D_t^{-2}(p_t) = -v, \tag{7.34a}$$

$$\mathcal{J}(P_2) = -r^2D_t^{-2}\left(2p + 2tp_t + 3rp_r - \frac{1}{\beta}\right) = \frac{1}{\beta}t^2r^2 - 2tv_t - 3r\partial_t^{-1}v_r + 8\partial_t^{-1}v, \tag{7.34b}$$

$$\mathcal{J}(P_3) = -r^2D_t^{-2}(tp_t + rp_r) = v - \partial_t^{-1}(tv) - r\partial_t^{-1}v_r + 2\partial_t^{-1}v, \tag{7.34c}$$

all of which can be verified to be nonlocal adjoint-symmetries Q.

The Lagrangian structure becomes

$$\mathcal{J}(G) = E_p(L) \tag{7.35}$$

through the variational derivative relation $\delta/\delta v = -r^{-2}D_t\delta/\delta p$, where L is a nonlocal expression (7.31) in terms of p:

$$L = r^2\left(\frac{1}{2}\left(\partial_t^{-1}p_r\right)^2 - \frac{1}{2}p^2 + \frac{1}{3}\beta p^3\right) - (\partial_t^{-1}p)^2. \tag{7.36}$$

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Thus, the symmetry action in Table 15 involving the non-multiplier adjoint-symmetry directly encodes a variational structure for the undamped acoustic wave equation.

7.3. Hamiltonian structure

The variational structure can also be lifted to the level of the first-layer potential system (7.10). Denote the PDEs in this system by $G^1 = p_r - r^{-2}u_t$ and $G^2 = w_r - r^2(p - \beta p^2)_t$, which satisfy the relation

$$D_r(r^2G^1) + D_tG^2 = -D_tG^v = -r^2G. \tag{7.37}$$

The determining equations for an adjoint-symmetry (Q^1, Q^2) of the potential system $(G^1, G^2) = 0$ are given by

$$(-D_xQ^1 + r^2(1-2\beta p)D_tQ^2)|_{\mathcal{E}_{(G^1,G^2)}} = 0, \quad (-D_xQ^2 + r^{-2}D_tQ^1)|_{\mathcal{E}_{(G^1,G^2)}} = 0. \tag{7.38}$$

From the PDE relation (7.37), one can show that

$$Q^1 = -r^2D_rD_t^{-1}Q^v = r^2D_r(r^{-2}Q), \quad Q^2 = -Q^v = r^2D_tQ. \tag{7.39}$$

Similarly, a symmetry $P\partial_p + P^u\partial_u$ of the potential system satisfies

$$P = r^{-2}D_tP^v, \quad P^u = r^2D_r(r^{-2}P^v). \tag{7.40}$$

Now consider the non-multiplier adjoint-symmetry (7.21b). The corresponding adjoint-symmetry (Q^1, Q^2) of the potential system $(G^1, G^2) = 0$ is given by

$$\begin{aligned} Q^1 &= -r^2D_rD_t^{-1}\left(r^{-1}v_r - \frac{1}{\beta}t\right) = -(\partial_t^{-1}u + r^3(p - \beta p^2)), \\ Q^2 &= -\left(r^{-1}v_r - \frac{1}{\beta}t\right) = -(r^{-1}u + 2\partial_t^{-1}p) + \frac{1}{\beta}t, \end{aligned} \tag{7.41}$$

with the use of equations (7.10) and (7.11) for the potentials. A straightforward computation then yields

$$Q^{1\prime}(P, P^u) = -(D_t^{-1}P^u + r^3(1-2\beta p)P), \quad Q^{2\prime}(P, P^u) = -(r^{-1}P^u + 2D_t^{-1}P), \tag{7.42}$$

and

$$R^*_{Q^1}(P, P^u) = r^3(1-2\beta p)P, \quad R^*_{Q^2}(P, P^u) = D_t^{-1}P + r^{-1}P^u. \tag{7.43}$$

Hence, one obtains

$$\begin{pmatrix} Q^{1\prime}(P, P^u) + R^*_{Q^1}(P, P^u) \\ Q^{2\prime}(P, P^u) + R^*_{Q^2}(P, P^u) \end{pmatrix} = \begin{pmatrix} -D_t^{-1}P^u \\ -D_t^{-1}P \end{pmatrix} = \mathcal{J}\begin{pmatrix} P \\ P^u \end{pmatrix}, \tag{7.44}$$

giving the Noether operator

$$\mathcal{J} = -\begin{pmatrix} 0 & D_t^{-1} \\ D_t^{-1} & 0 \end{pmatrix}. \tag{7.45}$$

The equations in the potential system have the following variational structure in terms of the inverse of the Noether operator (7.45),

$$\mathcal{J}^{-1} = -\begin{pmatrix} 0 & D_t \\ D_t & 0 \end{pmatrix}. \tag{7.46}$$

One sees that

$$(p_r, u_r)^{\mathrm{t}} = -\mathcal{J}^{-1}(\partial_pE, \partial_uE)^{\mathrm{t}}, \tag{7.47}$$

where

$$E = r^2\left(\frac{1}{2}p^2 - \frac{1}{3}\beta p^3\right) + r^{-2}\frac{1}{2}u^2. \tag{7.48}$$

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This structure is, formally, a Hamiltonian formulation in which $\mathcal{J}^{-1}$ and E respectively play the roles of the Hamiltonian operator and the Hamiltonian density, where r represents the 'time' coordinate for the evolution and t represents the 'spatial' coordinate with respect to which $\mathcal{J}^{-1}$ is skew-adjoint. Also, E is in fact the density of the energy integral (7.22) expressed in terms of p and u: $\mathcal{E} = \int_0^\infty \big(r^2(\frac{1}{2}p^2 - \frac{1}{3}\beta p^3) + r^{-2}\frac{1}{2}u^2\big)\, dr$.

8. Concluding remarks

For general PDE systems, the general results developed in [3] on the basic algebraic structure surrounding adjoint-symmetries and symmetry actions are very rich. In particular, as shown by the examples of physically interesting PDE systems in the present work, the adjoint-symmetry bracket constitutes a homomorphism of a Lie (sub) algebra of symmetries into a Lie algebra of adjoint-symmetries, which can hold, surprisingly, even for dissipative systems with no variational structure. Moreover, whenever a PDE system possesses a non-multiplier adjoint-symmetry, there a dual symmetry action that yields a Noether operator which leads to existence of a variational structure (Hamiltonian or Lagrangian).

Thus, the adjoint-symmetries of a given PDE system carry useful information about important aspects of the system. Further developments and exploration of more examples will be an interesting problem for future work.

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Conflicts of interest. None

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