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ISOMETRIC STABILITY PROPERTY OF CERTAIN BANACH SPACES

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ABSTRACT. Let E be one of the spaces $C(K)$ and L_1, F be an arbitrary Banach space, $p > 1$, and (X, σ) be a space with a finite measure. We prove that E is isometric to a subspace of the Lebesgue-Bochner space $L_p(X; F)$ only if E is isometric to a subspace of F. Moreover, every isometry T from E into $L_p(X; F)$ has the form $Te(x) = h(x)U(x)e$, $e \in E$, where $h: X \longrightarrow R$ is a measurable function and, for every $x \in X$, $U(x)$ is an isometry from E to F.

1. **Introduction.** Let E and F be Banach spaces, $p \geq 1$, (X, σ) be a finite measure space, and $L_p(X, F)$ be the Lebesgue-Bochner space of (equivalence classes of) strongly measurable functions $f: X \longmapsto F$ with

$$\|f\|^p = \int_X \|f(x)\|^p \, d\sigma(x) < \infty.$$

We show that if $E = C(K)$ (with K being a compact metric space) or $E = L_1$ then the space E can be isometric to a subspace of $L_p(X, F)$ with $p > 1$ only if E is isometric to a subspace of F. The isomorphic version of this result has been proved for the spaces $E = c_0$ (S. Kwapien [6] and J. Bourgain [1]), $E = l_1$ (G. Pisier [8]) and $E = l_\infty$ (J. Mendoza [7], see also [2] and [3] for generalizations to Köthe spaces of vector valued functions). E. Saab and P. Saab [10] have proved the isomorphic version for the space $E = L_1$ under the additional assumption that F is a dual space. For any $1 \leq p \leq q \leq r$, the space L_q is isometric to a subspace of $L_p(X, L_r)$ (see [9]). On the other hand, if $r > 2$, $r \neq q$, $q \neq 2$ the space L_q is not isomorphic to a subspace of L_r. Thus, both isometric and isomorphic versions fail to be true in the case where $E = L_q$, $q > 1$. (The space L_2 is isometric to a subspace of L_p for any $p > 0$, so it is isometric to a subspace of $L_p(X, F)$ for any space F.)

Besides proving the above mentioned result for the spaces $C(K)$ and L_1, we completely characterize isometric embeddings of these spaces into L_p-spaces of vector valued functions.

Denote by $I(E, F)$ the set of isometries from E to F. A mapping $U: X \longmapsto I(E, F)$ is called *strongly measurable* if, for each $e \in E$, the function $\|U(x)e\|$ is measurable on X. If the set $I(E, F)$ is non-empty then, obviously, for every strongly measurable mapping

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$U: X \mapsto I(E, F)$ and every function $h: X \mapsto R$ with $\|h\|_{L_p(X)} = 1$, the operator $T: E \mapsto L_p(X, F)$ defined by

$$Te(x) = h(x)U(x)e, \quad e \in E \tag{1}$$

is an isometry. We prove below that, for $E = C(K)$ or $E = L_1$ and for an arbitrary space F, every isometry from E to $L_p(X, F)$ has the form (1).The question if all isometries from E to $L_p(X, F)$ have the form (1) makes sense and has some applications even if $E = F$. For example, if $E = F = L_q$, $q > 1, p > 1$, then every isometry from L_q to $L_p(X, L_q)$ has the form (1) if and only if $p \neq q$, $q \neq 2$, $q \notin (p, 2)$. This result was proved and applied to the description of isometries of Lebesgue-Bochner spaces in the paper [4].

As a consequence of the characterization of isometries from $E = C(K)$ or $E = L_1$ to $L_p(X, F)$ we obtain the following result on random operators. Suppose that (X, σ) is a probability space, let $p > 1$, $L(E, F)$ be the space of linear operators from E to F and consider a random operator $S: X \mapsto L(E, F)$ which is an isometry in average, *i.e.* S is strongly measurable and, for every $e \in E$,

$$\|e\|^p = \int_X \|S(x)e\|^p \, d\sigma(x).$$

Then the random operator S is an isometry (up to a constant) with probability 1 (namely, operators $S(x)$ are isometries multiplied by constants for almost all $x \in X$).

In fact, if we define an operator $T: E \mapsto L_p(X, F)$ by $Te(x) = S(x)e$ then T is an isometry and (1) implies the desired result. In the case $E = C(K)$, the result about random operators has been proved before [5] under the assumption that the operators $S(x)$ are bounded.

2. **Main results.**

The proofs are based on the following simple fact.

LEMMA 1. *Let E, F be Banach spaces, $p > 1$, (X, σ) be a finite measure space and T be an isometry from E to $L_p(X, F)$. If e, f are elements from E with $\|e\| = \|f\| = 1$ and having a common tangent functional (i.e. there exists $x^* \in E^*$ with $\|x^*\| = 1$, $x^*(e) = x^*(f) = 1$) then, for almost all (with respect to σ) $x \in X$, we have*

(i) $\|Te(x)\| = \|Tf(x)\|$,

(ii) for every $\alpha > 0$, $\|Te(x) + \alpha Tf(x)\| = \|Te(x)\| + \alpha \|Tf(x)\|$.

PROOF. Obviously, $\|e + \alpha f\| = 1 + \alpha$ for every $\alpha > 0$, and we have

$$\begin{aligned}(1+\alpha)^p = \|e + \alpha f\|^p &= \int_X \|Te(x) + \alpha Tf(x)\|^p \, d\sigma(x) \\ &\leq \int_X \big(\|Te(x)\| + \alpha \|Tf(x)\|\big)^p \, d\sigma(x).\end{aligned} \tag{2}$$

For $\alpha = 0$, (2) turns into an equality. Therefore, we get a correct inequality if we take in both sides of (2) the right-hand derivatives at the point $\alpha = 0$ and apply Hölder's inequality:

$$\begin{aligned}1 &\leq \int_X \|Te(x)\|^{p-1} \|Tf(x)\| \, d\sigma(x) \\ &\leq \Big(\int_X \|Te(x)\|^p \, d\sigma(x)\Big)^{\frac{p-1}{p}} \Big(\int_X \|Tf(x)\|^p \, d\sigma(x)\Big)^{1/p} = 1.\end{aligned}$$

From the conditions for equality in Hölder's inequality, we conclude that, for almost all (with respect to σ) $x \in X$, $\|Te(x)\| = c\|Tf(x)\|$ where c is a constant. Further,

$$1 = \int_X \|Te(x)\|^p \, d\sigma(x) = \int_X c^p \|Tf(x)\|^p \, d\sigma(x) = c^p$$

and, hence, $c = 1$.

It is clear now that (2) is, in fact, an equality. Hence, for almost all $x \in X$, we have

$$\|Te(x) + \alpha Tf(x)\| = \|Te(x)\| + \alpha\|Tf(x)\| \tag{3}$$

for every $\alpha > 0$ and the proof is complete. ∎

Now we are able to prove the main result.

THEOREM 1. *Let $p > 1$, K be a compact metric space, $(X, \sigma), (Y, \nu)$ be spaces with finite measures, and F be an arbitrary Banach space. Let E be either the space $L_1 = L_1(Y, \nu)$ or any subspace of $C(K)$ containing the function $1(k) \equiv 1$. Then*

(i) If E is isometric to a subspace of $L_p(X; F)$ then E is isometric to a subspace of F and the set $I(E, F)$ is non-empty.

(ii) If T is an isometry from E into $L_p(X; F)$ then there exist a measurable function $h: X \longrightarrow R$ and a strongly measurable mapping $U: X \longrightarrow I(E, F)$ such that $Te(x) = h(x)U(x)e$ for every $e \in E$.

PROOF. We start with the case $E = L_1$. Any two functions e and f from L_1 with disjoint supports in Y have a common tangent functional so we can apply Lemma 1 to any pair of normalized functions with disjoint supports.

Decompose the set Y into two parts $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = \emptyset$, $\nu(Y_i) > 0$, $i = 1, 2$. Fix a function $e_0 \in L_1(Y_1)$, $\|e_0\| = 1$ and put $h(x) = \|Te_0(x)\|$, $x \in X$.

Let $f_k, k \in N$ be a sequence of linearly independent functions with supports in Y_2 such that their linear span is dense in $L_1(Y_2)$. Denote by D the set of linear combinations of functions f_k with rational coefficients. Given fixed representatives Tf_k from the corresponding equivalence classes of functions from the space $L_p(X; F)$, define an operator $T(x): D \longmapsto F$ for every $x \in X$ by $T(x)(\sum \lambda_i f_i) = \sum \lambda_i Tf_i(x)$, $\lambda_i \in Q$.

It follows from the statement (i) of Lemma 1 and the fact that D is countable that there exists a set $X_0 \subset X$ with $\sigma(X \setminus X_0) = 0$ such that, for every $x \in X_0$ and every $f \in D$,

$$\|T(x)f\| = \|Tf(x)\| = \|Te_0(x)\| \, \|f\| = h(x)\|f\|.$$

Hence, for every $x \in X_0$, either $h(x) = 0$ or the operator $U_2(x) = T(x)/h(x)$ is an isometry from D to F. The operators $U_2(x)$ can be uniquely extended to isometries on the whole space $L_1(Y_2)$.

In fact, given $a \in L_1(Y_2)$ and a sequence $a_k \longrightarrow a$, $a_k \in D$, put

$$U_2(x)(a) = \lim_{k \to \infty} U_2(x)(a_k).$$

Further, for any $a \in L_1(Y_2)$,

$$\begin{aligned} \|a_k - a\|^p &= \int_X \|Ta_k(x) - Ta(x)\|^p \, d\sigma(x) \\ &= \int_{h(x)=0} \|Ta(x)\|^p \, d\sigma(x) + \int_{h(x)\neq 0} \|h(x)U_2(x)a_k - Ta(x)\|^p \, d\sigma(x) \longrightarrow 0 \end{aligned} \tag{4}$$

as $k \longrightarrow \infty$. Therefore, $Ta(x) = 0$ for almost all $x \in X$ with $h(x) = 0$, and $Ta(x) = h(x)U_2(x)a$ for almost all $x \in X$ (if $h(x) = 0$ we put $U_2(x) = 0$.)

Similarly, for almost all $x \in X$, we find isometries $U_1(x)$ from $L_1(Y_1)$ to F such that $Tb(x) = h(x)U_1(x)b$ for every $b \in L_1(Y_2)$.

Consider an arbitrary function $f \in L_1(Y)$. This function can be uniquely represented as a sum $f = f_1 + f_2$ of functions $f_1 \in L_1(Y_1)$ and $f_2 \in L_1(Y_2)$. For all $x \in X$ with $h(x) \neq 0$, define operators $U(x)$ from $L_1(Y)$ to F by $U(x)f = U_1(x)f_1 + U_2(x)f_2$. By the statement (ii) of Lemma 1, for almost every x with $h(x) \neq 0$, $\|U(x)f\| = \big(1/h(x)\big)\|Tf_1(x) + Tf_2(x)\| = \|U_1(x)f_1\| + \|U_2(x)f_2\| = \|f_1\| + \|f_2\| = \|f\|$.

Thus, operators $U(x)$ are isometries for almost all $x \in X$ with $h(x) \neq 0$. In particular, the set $I(L_1, F)$ is non-empty. Fix any $U \in I(L_1, F)$ and put $U(x) = U$ for every x with $h(x) = 0$.

To prove the second statement of Theorem 1 note that, for every $f \in L_1(Y)$, we have $Tf(x) = Tf_1(x) + Tf_2(x) = h(x)\big(U_1(x)f_1 + U_2(x)f_2\big) = h(x)U(x)f$ for almost all $x \in X$, so Tf and $h(x)U(x)f$ are equal elements of the space $L_p(X, F)$.

Now let E be a subspace of $C(K)$ containing the function $1(k) \equiv 1$. Any function $e \in C(K)$ has a common tangent functional either with the function 1 or with the function -1. Setting, correspondingly, $f = 1$ or $f = -1$ in Lemma 1 we obtain that, for an arbitrary $e \in E$, $\|Te(x)\| = \|T1(x)\| \, \|e\|$ for almost all $x \in X$. Let $h(x) = \|T1(x)\|$.

Let e_k, $k \in N$ be a sequence of linearly independent functions from E such that their linear span is dense in E and denote by D the set of linear combinations of functions e_k with rational coefficients. Given fixed representatives Te_k from the corresponding equivalence classes of functions from the space $L_p(X; F)$, define an operator $T(x)$ on D for every $x \in X$ by $T(x)(\sum \lambda_i e_i) = \sum \lambda_i Te_i(x)$, $\lambda_i \in R$.

Since the set D is countable there exists a set $X_0 \subset X$ with $\sigma(X \setminus X_0) = 0$ such that, for every $x \in X_0$ and every $e \in D$, $\|T(x)e\| = \|Te(x)\| = h(x)\|e\|$. Hence, for every $x \in X_0$, either $h(x) = 0$ or the operator $U(x) = T(x)/h(x)$ is an isometry from D to F.

The operators $U(x)$ can be uniquely extended to isometries on the whole space E, therefore, we have proved the first statement of Theorem 1. Now an argument similar to (4) proves the second statement. ■

REMARK. If $p = 1$ the statement of Theorem 1 is not true. For instance, the two-dimensional space l_∞^2 is isometric to a subspace of $L_1([0, 1])$. Thus, for any Banach space F, l_∞^2 is isometric to a subspace of $L_1([0, 1]; F)$, and Theorem 1 would have implied that l_∞^2 is isometric to a subspace of any Banach space.

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