ON ANTI-COMMUTATIVE ALGEBRAS WITH AN INVARIANT FORM

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1. Introduction. In this paper we consider anti-commutative algebras with an invariant form, that is, an algebra A over a field F such that

(1) $$xy = -yx \qquad \text{for all } x, y \text{ in } A$$

and A possesses a symmetric bilinear form $f(x, y)$ such that

(2) $$f(xy, z) = f(x, yz) \qquad \text{for all } x, y, z \text{ in } A.$$

Lie and Malcev algebras **(2, 3)** are examples of such algebras and we shall consider generalizations of these algebras obtained by introducing commutation, $x \circ y = xy - yx$, as a new multiplicative operation in the non-commutative Jordan algebras of **(1)**. Thus if $\mathfrak{A}$ is such a Jordan algebra we form the anti-commutative algebra $\mathfrak{A}^-$ which is the same vector space $\mathfrak{A}$ but with commutation $x \circ y$ as multiplication. If $\mathfrak{C}$ is the centre of $\mathfrak{A}^-$, that is, $\mathfrak{C}$ is the set of elements x in $\mathfrak{A}^-$ such that $x \circ y = 0$ for all $y \in \mathfrak{A}^-$, then we consider $\mathfrak{A}^0 = \mathfrak{A}^-/\mathfrak{C}$ and use this algebra to construct more simple Jordan and anti-commutative algebras. Finally these results are used to prove the following theorem.

THEOREM. *If A is a finite-dimensional anti-commutative algebra with an invariant form $f(x, y)$ over a field F of characteristic not 2, then there exists a non-commutative Jordan algebra $\mathfrak{B}$ with identity element 1 such that $\mathfrak{B}^-/1F$ is isomorphic to A. Furthermore if $f(x, y)$ is non-degenerate and the mapping $x \to R_x$, where R_x is right multiplication by x, is injective, then $\mathfrak{B}$ is simple.*

2. Basic properties. We shall assume that all algebras discussed are finite dimensional and for any algebra A we let

$$(x, y, z) = xy \cdot z - x \cdot yz \qquad \text{for any } x, y, z \in A.$$

The Jordan algebras of **(1)** are constructed as follows. Let A be an anti-commutative algebra with an invariant form $f(\alpha, \beta)$ and let $\mathfrak{A}(A, f, s, t)$ denote the set of matrices

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix},$$

where $\alpha, \beta \in A$ and $a, b \in F$. For these matrices define equality, addition,

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and multiplication by elements in F in the obvious manner. Next define multiplication of two such matrices by

$$\begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}\begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} = \begin{pmatrix} ac + f(\alpha, \delta) & a\gamma + d\alpha + t\beta\delta \\ c\beta + b\delta + s\alpha\gamma & bd + f(\beta, \gamma) \end{pmatrix},$$

where $f(\alpha, \beta)$ denotes the invariant form on A and $s, t \in F$. Thus

$$\mathfrak{A} \equiv \mathfrak{A}(A, f, s, t)$$

becomes an algebra with the following properties **(1)**: (i) $\mathfrak{A}$ is a flexible quadratic algebra with identity element 1, that is, for all $x, y \in \mathfrak{A}$, $(x, y, x) = 0$ and $x^2 - (a + b)x + [ab - f(\alpha, \beta)]1 = 0$ for all

$$x = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} \in \mathfrak{A}.$$

Thus $(x^2, y, x) = 0$ and so $\mathfrak{A}$ is a non-commutative Jordan algebra. (ii) $\mathfrak{A}$ is simple if and only if $f(\alpha, \beta)$ is non-degenerate on A, where $f(\alpha, \beta)$ being non-degenerate on A means that $f(\alpha, \beta) = 0$ for all $\beta \in A$ implies $\alpha = 0$.

Next introduce commutation $x \circ y$ as a new multiplication in $\mathfrak{A}$ and form the anti-commutative algebra $\mathfrak{A}^-$ which is the same vector space as $\mathfrak{A}$, but if

$$x = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}, \qquad y = \begin{pmatrix} c & \gamma \\ \delta & d \end{pmatrix} \in \mathfrak{A}^-,$$

we now have the multiplication in $\mathfrak{A}^-$ given by

$$(3) \quad x \circ y = \begin{pmatrix} f(\alpha, \delta) - f(\gamma, \beta) & (d - c)\alpha + (a - b)\gamma + 2t\beta\delta \\ (b - a)\delta + (c - d)\beta + 2s\alpha\gamma & f(\gamma, \beta) - f(\alpha, \delta) \end{pmatrix}.$$

Now the identity matrix 1 is such that for every $x \in \mathfrak{A}^-$, $1 \circ x = 0$ and it is easy to see from (3) that scalar multiples of 1 are the only such elements. Thus $1F$ is the centre $\mathfrak{C}$ of $\mathfrak{A}^-$ and we form the quotient algebra $\mathfrak{A}^0 = \mathfrak{A}^-/\mathfrak{C}$. If F is of characteristic not 2, then every $\bar{x} = x + \mathfrak{C}$ of $\mathfrak{A}^0$ has the form

$$\bar{x} = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix} + \mathfrak{C} = \begin{pmatrix} (a - b)/2 & \alpha \\ \beta & -(a - b)/2 \end{pmatrix} + \mathfrak{C}$$
$$= \begin{pmatrix} x_0 & \alpha \\ \beta & -x_0 \end{pmatrix} + \mathfrak{C}$$

for $x_0 \equiv (a - b)/2 \in F$. Identify x with $\bar{x}$ in $\mathfrak{A}^0$ and note that the multiplication in $\mathfrak{A}^0$ is now given by

$$(4) \qquad x \circ y = \begin{pmatrix} f(\alpha, \delta) - f(\gamma, \beta) & -2y_0\alpha + 2x_0\gamma + 2t\beta\delta \\ -2x_0\delta + 2y_0\beta + 2s\alpha\gamma & f(\gamma, \beta) - f(\alpha, \delta) \end{pmatrix},$$

where $$y = \begin{pmatrix} y_0 & \gamma \\ \delta & -y_0 \end{pmatrix} \text{ and } x = \begin{pmatrix} x_0 & \alpha \\ \beta & -x_0 \end{pmatrix}$$

are now in $\mathfrak{A}^0$. We assume for the remainder of the paper that the characteristic of F is not 2.

Next we construct a basis for $\mathfrak{A}^0$ and the corresponding multiplication table. Let $\{e_1, \ldots, e_n\}$ be a basis of A over F and set

$$E = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_i = \begin{pmatrix} 0 & e_i \\ 0 & 0 \end{pmatrix}, \qquad E_i' = \begin{pmatrix} 0 & 0 \\ e_i & 0 \end{pmatrix}, \tag{5}$$

$i = 1, \ldots, n$. Clearly these elements form a basis for $\mathfrak{A}^0$ and we have the following relations:

$$\begin{aligned} &E \circ E_i = 2E_i, \qquad E \circ E_i' = -2E_i', \\ &E_i \circ E_j = 2s\begin{pmatrix} 0 & 0 \\ e_i e_j & 0 \end{pmatrix}, \qquad E_i' \circ E_j' = 2t\begin{pmatrix} 0 & e_i e_j \\ 0 & 0 \end{pmatrix}, \\ &E_i \circ E_j' = f(e_i, e_j)E. \end{aligned} \tag{6}$$

We now prove the following theorem.

THEOREM 1. *$\mathfrak{A}^0$ is simple if and only if $f(\alpha, \beta)$ is non-degenerate on A.*

Proof. Assume that $f(\alpha, \beta)$ is non-degenerate on A and $\mathfrak{B}$ is a non-zero ideal containing the non-zero element $b = b_0 E + \sum b_i E_i + \sum b_i' E_i'$. Then, using (6),

$$\begin{aligned} E \circ b &= \sum b_i E \circ E_i + \sum b_i' E \circ E_i' \\ &= 2\sum b_i E_i - 2\sum b_i' E_i' \in \mathfrak{B}, \end{aligned}$$

and, therefore,

$$E \circ (E \circ b) = 4\sum b_i E_i + 4\sum b_i' E_i' \in \mathfrak{B}.$$

Thus $4b - E \circ (E \circ b) = 4b_0 E \in \mathfrak{B}$ and if $b_0 \neq 0$, $E \in \mathfrak{B}$, and from (6), $\mathfrak{B} = \mathfrak{A}^0$. We shall now show that $\mathfrak{B}$ always contains an element with the coefficient of E not equal to zero. Suppose $b = \sum b_i E_i + \sum b_i' E_i' \in \mathfrak{B}$ and assume some $b_k \neq 0$. Let E_j' be as in (5); then from (6)

$$b \circ E_j' = \left(\sum_i b_i f(e_i, e_j)\right)E + \sum_i 2t b_i' \begin{pmatrix} 0 & e_i e_j \\ 0 & 0 \end{pmatrix}$$

is in $\mathfrak{B}$. Now there exists an E_j' such that $\sum_i b_i f(e_i, e_j) \neq 0$. Otherwise we would have, for all $j = 1, \ldots, n$,

$$0 = \sum_i b_i f(e_i, e_j) = f\left(\sum_i b_i e_i, e_j\right) = f(\gamma, e_j).$$

Since $\gamma = \sum_i b_i e_i \neq 0$, this equation implies that $f(\alpha, \beta)$ is degenerate, a contradiction.

Conversely suppose $\mathfrak{A}^0$ is simple and let $N = \{\alpha \in A : f(\alpha, \beta) = 0$ for all $\beta \in A\}$. Since $f(\alpha, \beta)$ is an invariant form, N is an ideal of A and if we set

$$\mathfrak{B} = \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} : \alpha, \beta \in N \right\}$$

and use (4) we see that $\mathfrak{B}$ is an ideal of $\mathfrak{A}^0$. But $E \notin \mathfrak{B}$ and since $\mathfrak{A}^0$ is simple, we must have $\mathfrak{B} = 0$, and therefore $N = 0$, which means that $f(\alpha, \beta)$ is non-degenerate in A.

If $f(\alpha, \beta)$ is a non-degenerate invariant form so that a dual basis $\{e_i, \ldots, e_n\}$ can be chosen for A satisfying $f(e_i, e_j) = \delta_{ij}$ (Kronecker delta), then we obtain a rather natural multiplication table for $\mathfrak{A}^0$. First since $f(e_i, e_j) = \delta_{ij}$, $E_i \circ E_j' = \delta_{ij}E$. To find the remaining relations we shall determine a multiplication table for A relative to $\{e_1, \ldots, e_n\}$. Let $e_i e_j = \sum_k a(i, j, k)e_k$, where $a(i, j, k) \in F$, then

$$f(e_i e_j, e_k) = \sum_m a(i, j, m) f(e_m, e_k) = a(i, j, k).$$

This formula implies that $a(i, j, k)$ is a skew-symmetric function for $i, j, k = 1, \ldots, n$. Conversely, if A is an n-dimensional vector space with basis $\{e_1, \ldots, e_n\}$ and $a(i, j, k)$ is a skew-symmetric function for $i, j, k = 1, \ldots, n$, then $e_i e_j = \sum_k a(i, j, k)e_k$ makes A into an anti-commutative algebra when this multiplication is extended to all of A. Furthermore, if $\alpha = \sum a_i e_i$, $\beta = \sum b_i e_i$, and $f(\alpha, \beta) = \sum_{i=1}^n a_i b_i$, then $f(\alpha, \beta)$ is a non-degenerate invariant form. For it clearly suffices to show that $f(e_i e_j, e_k) = f(e_i, e_j e_k)$ and we have

$$f(e_i e_j, e_k) = \sum_m a(i, j, m) f(e_m, e_k) = a(i, j, k) = a(j, k, i) = f(e_i, e_j e_k).$$

Thus the multiplication table of A, and therefore $\mathfrak{A}^0$, is completely determined by the invariant form $f(\alpha, \beta)$ or equivalently the corresponding skew-symmetric function $a(i, j, k)$.

3. Invariant forms for $\mathfrak{A}^0$. Since $\mathfrak{A}^0$ is constructed from an anti-commutative algebra with an invariant form, it is natural to ask if $\mathfrak{A}^0$ has an invariant form. We shall show that if $f(\alpha, \beta) = \text{trace}\, R_\alpha R_\beta$ is an invariant form for A, then $F(x, y) = \text{trace}\, R_x^0 R_y^0$ is an invariant form for $\mathfrak{A}^0$, where R_x^0 is defined by $zR_x^0 = z \circ x$.

First we note that if $g(x, y)$ is a symmetric bilinear form for any anti-commutative algebra A, then $g(x, y)$ is an invariant form if and only if $g(x, xy) = 0$ for all $x, y \in A$. For if $g(x, y)$ is invariant, then $g(x, xy) = g(x^2, y) = 0$ and for the converse linearize the identity $g(x, xy) = 0$.

Next we determine a matrix for R_x^0 on $\mathfrak{A}^0$. Let A have the basis $\{e_1, \ldots, e_n\}$ and let

$$x = \begin{pmatrix} x_0 & \alpha \\ \beta & -x_0 \end{pmatrix} \in \mathfrak{A}^0,$$

where $\alpha = \sum a_i e_i$, $\beta = \sum b_i e_i$, and $e_k \alpha = \sum_i p_{ki} e_i$, $e_k \beta = \sum_i q_{ki} e_i$ for $k = 1, \ldots, n$. Then from (4) and (5),

$$ER_x^0 = \begin{pmatrix} 0 & 2\alpha \\ -2\beta & 0 \end{pmatrix} = 2 \sum a_i E_i - 2 \sum b_i E_i',$$

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$$E_i R_x^{\ 0} = \begin{pmatrix} f(e_i, \beta) & -2x_0\, e_i \\ 2se_i\, \alpha & -f(e_i, \beta) \end{pmatrix}$$
$$= f(e_i, \beta)E - 2x_0 E_i + 2s \textstyle\sum_j p_{ij}\, E_j',$$
$$E_i' R_x^{\ 0} = \begin{pmatrix} -f(e_i, \alpha) & 2te_i\, \beta \\ 2x_0\, e_i & f(e_i, \alpha) \end{pmatrix}$$
$$= -f(e_i, \alpha)E + 2t \textstyle\sum_j q_{ij}\, E_j + 2x_0\, E_i'.$$

Thus R_x^0 has the matrix

$$M(x) = \begin{bmatrix} 0 & 2a_1 \cdots 2a_n & -2b_1 \cdots -2b_n \\ f(e_1, \beta) & -2x_0 \; 0 \cdots 0 & 2sp_{11} \cdots 2sp_{1n} \\ \vdots & \vdots & \vdots \\ f(e_n, \beta) & 0 \; 0 \cdots -2x_0 & 2sp_{n1} \cdots 2sp_{nn} \\ -f(e_1, \alpha) & 2tq_{11} \cdots 2tq_{1n} & 2x_0 \; 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ -f(e_n, \alpha) & 2tq_{n1} \cdots 2tq_{nn} & 0 \; 0 \cdots 0 \; 2x_0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2a_1 \cdots 2a_n & -2b_1 \cdots -2b_n \\ f(e_1, \beta) & & \\ \vdots & -2x_0 I & 2s\bar{R}_\alpha \\ f(e_n, \beta) & & \\ -f(e_1, \alpha) & & \\ \vdots & 2t\bar{R}_\beta & 2x_0 I \\ -f(e_n, \alpha) & & \end{bmatrix}$$

where I is the $n \times n$ identity matrix and $\bar{R}_\pi$ denotes the matrix for right multiplication by π in A relative to the basis $\{e_1, \ldots, e_n\}$ of A. Now if

$$y = \begin{pmatrix} y_0 & \gamma \\ \delta & -y_0 \end{pmatrix} \in \mathfrak{A}^0,$$

where $\gamma = \sum c_i e_i$ and $\delta = \sum d_i e_i$, then R_y^0 has the matrix

$$M(y) = \begin{bmatrix} 0 & 2c_1 \cdots 2c_n & -2d_1 \cdots -2d_n \\ f(e_1, \delta) & & \\ \vdots & -2y_0 I & 2s\bar{R}_\gamma \\ f(e_n, \delta) & & \\ -f(e_1, \gamma) & & \\ \vdots & 2t\bar{R}_\delta & 2y_0 I \\ -f(e_n, \gamma) & & \end{bmatrix}.$$

Next $R_x{}^0 R_y{}^0$ has as its matrix the product

$$M(x)M(y) = \begin{bmatrix} a_{11} & & * \\ & A_{22} & \\ * & & A_{33} \end{bmatrix},$$

where

$$a_{11} = 2(f(\alpha, \delta) + f(\beta, \gamma)),$$
$$A_{22} = 2[f(e_i, \beta)c_j] + 4x_0y_0 I + 4st\bar{R}_\alpha\bar{R}_\delta,$$
$$A_{33} = 2[f(e_i, \alpha)d_j] + 4x_0y_0 I + 4st\bar{R}_\beta\bar{R}_\gamma,$$

where $[f(e_i, \beta)c_j]$ and $[f(e_i, \alpha)d_j]$ are $n \times n$ matrices with the (i, j)-element as indicated. Therefore

$$\begin{aligned} F(x, y) &= \text{trace } R_x{}^0 R_y{}^0 \\ &= 2(f(\alpha, \delta) + (f(\beta, \gamma)) + 2 \textstyle\sum_i f(e_i, \beta)c_i + 4nx_0y_0 + 4st \,\text{trace}\bar{R}_\alpha\bar{R}_\delta \\ &\qquad + 2 \textstyle\sum_i f(e_i, \alpha)d_i + 4nx_0y_0 + 4st \text{ trace } \bar{R}_\beta\bar{R}_\gamma \\ &= 4[f(\alpha, \delta) + f(\beta, \gamma) + st \,\text{trace}(R_\alpha R_\delta + R_\beta R_\gamma) + 2nx_0y_0]. \end{aligned}$$

Now assume that the invariant form $f(\alpha, \beta)$ is given by $f(\alpha, \beta) = \mu$ trace $R_\alpha R_\beta$, where μ is some element in F; then

$$F(x, y) = 4[(\mu + st) \,\text{trace}(R_\alpha R_\delta + R_\beta R_\gamma) + 2nx_0y_0]. \tag{7}$$

If μ satisfies $\mu + st - n\mu = 0$, we shall show that for all $x, y \in \mathfrak{A}^0$,

$$F(x, x \circ y) = 0,$$

and therefore $F(x, y)$ is an invariant form on $\mathfrak{A}^0$. If $R(\pi)$ also denotes R_π, using (4) and (7) we have

$$\begin{aligned} F(x, x \circ y) &= 4[(\mu + st) \text{ trace } R_\alpha R(-2x_0\delta + 2y_0\beta + 2s\alpha\gamma) \\ &\qquad + (\mu + st) \text{ trace } R_\beta R(-2y_0\alpha + 2x_0\gamma + 2t\beta\delta) \\ &\qquad\qquad + 2nx_0(\mu \text{ trace } R_\alpha R_\delta - \mu \text{ trace } R_\gamma R_\beta)] \\ &= 4[(-2x_0(\mu + st) + 2n\mu x_0) \text{ trace } R_\alpha R_\delta \\ &\qquad + (2y_0(\mu + st) - 2y_0(\mu + st)) \text{ trace } R_\alpha R_\beta \\ &\qquad + 2s(\mu + st) \text{ trace } R_\alpha R_{\alpha\gamma} + 2t(\mu + st) \text{ trace } R_\beta R_{\beta\delta} \\ &\qquad\qquad + (2x_0(\mu + st) - 2n\mu x_0) \text{ trace } R_\gamma R_\beta] \\ &= 0 \end{aligned} \tag{8}$$

also using $\mu + st - n\mu = 0$ and that $f(\alpha, \beta) = \mu$ trace $R_\alpha R_\beta$ is invariant.

Next suppose the anti-commutative algebra A has an invariant form $g(\alpha, \beta) = \lambda$ trace $R_\alpha R_\beta$, $\lambda \neq 0$, where λ need not satisfy $\lambda + st - n\lambda = 0$; then the bilinear form

$$f(\alpha, \beta) = \mu/\lambda g(\alpha, \beta) = \mu \text{ trace } R_\alpha R_\beta$$

is also an invariant form for A which is non-degenerate if and only if $g(\alpha, \beta)$ is non-degenerate. So from the start of the construction we can assume that

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if $f(\alpha, \beta) = \mu$ trace $R_\alpha R_\beta$, then $\mu + st - n\mu = 0$, and call such a bilinear form *normalized*. This proves part of the following theorem.

THEOREM 2. *If* $f(\alpha, \beta) = \mu$ trace $R_\alpha R_\beta$ *is a normalized invariant form on* A, *then* $F(x, y) =$ trace $R_x{}^0 R_y{}^0$ *is an invariant form on* $\mathfrak{A}^0$. *Conversely if* $F(x, y) =$ trace $R_x{}^0 R_y{}^0$ *is an invariant form on* $\mathfrak{A}^0$ *and* $f(\alpha, \beta) = \mu$ trace $R_\alpha R_\beta$ (*not necessarily normalized*) *satisfies* $(\mu + st)(s + t) \neq 0$, *then* $f(\alpha, \beta)$ *is an invariant form on* A. *Furthermore, when* $\mu + st \neq 0$, $F(x, y)$ *is non-degenerate if and only if* $f(\alpha, \beta)$ *is non-degenerate.*

Proof. First we show that trace $R_\alpha R_{\alpha\beta} = 0$ for all $\alpha, \beta \in A$ and therefore $f(\alpha, \beta)$ is an invariant form. Since $F(x, y)$ is assumed invariant, $F(x, x \circ y) = 0$ and from (8) with $x_0 = 0$, $\alpha = \beta$, and $\gamma = \delta$ we obtain

$$(\mu + st)(s + t) \text{ trace } R_\alpha R_{\alpha\gamma} = 0,$$

which implies that $f(\alpha, \beta)$ is an invariant form. Next let

$$y = \begin{pmatrix} y_0 & \gamma \\ \delta & -y_0 \end{pmatrix} \in \mathfrak{A}^0$$

be such that for all

$$x = \begin{pmatrix} x_0 & \alpha \\ \beta & -x_0 \end{pmatrix} \in \mathfrak{A}^0,$$

$$0 = F(x, y) = 4[(\mu + st) \text{ trace}(R_\alpha R_\delta + R_\beta R_\gamma) + 2nx_0y_0].$$

Choose $x_0 = 0$, $\beta = 0$; then $f(\alpha, \delta) = 0$ for all $\alpha \in A$ and since $f(\alpha, \beta)$ is assumed non-degenerate, $\delta = 0$. Similarly, $x_0 = 0$ implies that $\gamma = 0$ and we finally have $0 = 8nx_0y_0$ and with $x_0 = 1$, $y_0 = 0$, so that $y = 0$ and therefore $F(x, y)$ is non-degenerate on $\mathfrak{A}^0$. Conversely, suppose $F(x, y)$ is non-degenerate on $\mathfrak{A}^0$ and assume that for some $\delta \in A$, $f(\alpha, \delta) = \mu$ trace $R_\alpha R_\delta = 0$ for all $\alpha \in A$. Let

$$y = \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix} \in \mathfrak{A}^0;$$

then for all other $x \in \mathfrak{A}^0$, $F(x, y) = 0$ and therefore $y = 0$, which implies that $f(\alpha, \beta)$ is non-degenerate.

Theorem 2 can be used to obtain a family of simple non-commutative Jordan algebras and a corresponding family of simple anti-commutative algebras based on a given anti-commutative algebra A: From the algebra A construct $\mathfrak{A}_1 = \mathfrak{A}(A, f, s, t)$, where $f(\alpha, \beta) = \mu$ trace $R_\alpha R_\beta$ is a normalized non-degenerate invariant form for A. From $\mathfrak{A}_1$ form $\mathfrak{A}_1{}^0$ and construct $\mathfrak{A}_2 = \mathfrak{A}(\mathfrak{A}_1{}^0, F_1, s_1, t_1)$, where $F_1(x, y) = \mu_1$ trace $R_x{}^0 R_y{}^0$ is a normalized non-degenerate invariant form on $\mathfrak{A}_1{}^0$. From $\mathfrak{A}_2$ form $\mathfrak{A}_2{}^0$, construct $\mathfrak{A}_3 = \mathfrak{A}(\mathfrak{A}_2{}^0, F_2, s_2, t_2)$ and continue this process. Now if A is the three-dimensional Lie algebra with the outer product as multiplication, $f(\alpha, \beta)$ the ordinary inner product (which equals $-1/2$ trace $R_\alpha R_\beta$) and $s = 1$, $t = -1$, then $\mathfrak{A} = \mathfrak{A}(A, f, 1, -1)$ is the split

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Cayley-Dickson algebra and $\mathfrak{A}^0$ the corresponding Malcev algebra. Thus starting with the above Lie algebra as the base algebra A with s_i and t_i arbitrary, we obtain a family of non-commutative Jordan algebras $\{\mathfrak{A}_i\}$ which are natural generalizations of the Cayley-Dickson algebra **(4)**, and the corresponding family of anti-commutative algebras $\{\mathfrak{A}_i^0\}$ are generalizations of the seven-dimensional Malcev algebra.

4. Proof of the theorem in the introduction. Let A be an anti-commutative algebra over a field F of characteristic not 2 with an invariant form $f(\alpha, \beta)$. Construct $\mathfrak{A} = \mathfrak{A}(A, f, 1/2, 1/2)$ and let

$$\mathfrak{B} = \left\{ \begin{pmatrix} a & \alpha \\ \alpha & a \end{pmatrix} : \alpha \in A, a \in F \right\};$$

then $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ with multiplication

$$xy = \begin{pmatrix} ab + f(\alpha, \beta) & a\beta + b\alpha + \tfrac{1}{2}\alpha\beta \\ a\beta + b\alpha + \tfrac{1}{2}\alpha\beta & ab + f(\alpha, \beta) \end{pmatrix}, \tag{9}$$

where

$$x = \begin{pmatrix} a & \alpha \\ \alpha & a \end{pmatrix}, \qquad y = \begin{pmatrix} b & \beta \\ \beta & b \end{pmatrix} \in \mathfrak{B}.$$

Note that $\mathfrak{B} = 1F \oplus \tilde{\mathfrak{B}}$ as a vector space sum and form $\mathfrak{B}^-$ and $\mathfrak{B}^-/1F$; then $\mathfrak{B}^-/1F$ is isomorphic to A. For let

$$x = \begin{pmatrix} a & \alpha \\ \alpha & a \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} + a1 \in \mathfrak{B};$$

then

$$\bar{x} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} + 1F \in \mathfrak{B}^-/1F$$

and the mapping $\bar{x} \to \alpha$ can easily be shown to be an isomorphism of $\mathfrak{B}^-/1F$ onto A by noting that it is linear and

$$\bar{x} \circ \bar{y} = \begin{pmatrix} 0 & \alpha\beta \\ \alpha\beta & 0 \end{pmatrix} + 1F.$$

Next suppose $f(\alpha, \beta)$ is non-degenerate and the mapping $\alpha \to R_\alpha$ is injective. Suppose $\mathfrak{J}$ is an ideal of $\mathfrak{B}$ containing the element

$$x = \begin{pmatrix} a & \alpha \\ \alpha & a \end{pmatrix};$$

then for

$$y = \begin{pmatrix} -a & \alpha \\ \alpha & -a \end{pmatrix} \in \mathfrak{B},$$

we see from (9) that

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$$xy = \begin{pmatrix} f(\alpha, \alpha) - a^2 & 0 \\ 0 & f(\alpha, \alpha) - a^2 \end{pmatrix} \in \mathfrak{J}.$$

Thus if there exists an $x \in \mathfrak{J}$ with $f(\alpha, \alpha) \neq a^2$, then the identity $1 \in \mathfrak{J}$ and therefore $\mathfrak{J} = \mathfrak{B}$. So now assume that every $x \in \mathfrak{J}$ has the property $f(\alpha, \alpha) = a^2$. For any

$$y = \begin{pmatrix} b & \beta \\ \beta & b \end{pmatrix} \in \mathfrak{B},$$

$$\tfrac{1}{2}(xy + yx) = \begin{pmatrix} ab + f(\alpha, \beta) & a\beta + b\alpha \\ a\beta + b\alpha & ab + f(\alpha, \beta) \end{pmatrix} \in \mathfrak{J}.$$

Therefore, using the assumption for $\mathfrak{J}$, we have $(ab + f(\alpha, \beta))^2 = f(a\beta + b\alpha, a\beta + b\alpha)$ and obtain

$$f(\alpha, \beta)^2 = a^2 f(\beta, \beta) \qquad \text{for all } \beta \in A.$$

A linearization of this identity yields

$$f(\alpha, \beta) f(\alpha, \gamma) = a^2 f(\beta, \gamma) \qquad \text{for all } \beta, \gamma \in A. \tag{10}$$

Thus if x is a non-zero element of $\mathfrak{J}$ and $a = 0$ we have from (10), $f(\alpha, \beta) f(\alpha, \gamma) = 0$ for every $\beta, \gamma \in A$. Since $x \neq 0$ and $a = 0$, $\alpha \neq 0$, therefore there exists $\beta \in A$ with $f(\alpha, \beta) \neq 0$ and from this $f(\alpha, \gamma) = 0$ for all $\gamma \in A$. This implies $\alpha = 0$, a contradiction; so we assume that $x \in \mathfrak{J}$ is such that $a \neq 0$.

Next suppose that $\delta \in A$ is such that $f(\alpha, \delta) = 0$; then since $a \neq 0$, we see from (10) that $f(\beta, \delta) = 0$ for all $\beta \in A$ and therefore $\delta = 0$. However, for any $\gamma \in A$, $\delta = \alpha\gamma$ is such that $f(\alpha, \delta) = f(\alpha, \alpha\gamma) = 0$ and therefore $0 = \gamma\alpha = \gamma R_\alpha$ for every $\gamma \in A$. This implies that $R_\alpha = 0$ and since $\alpha \to R_\alpha$ is injective, $\alpha = 0$. Thus

$$x = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in \mathfrak{J}$$

with $a \neq 0$ and so $\mathfrak{J} = \mathfrak{B}$.

Finally we note that if A is any anti-commutative algebra, the bilinear form $f(\alpha, \beta) \equiv 0$ for every $\alpha, \beta \in A$ is an invariant form on A. Thus the first part of the proof shows that any anti-commutative algebra A is isomorphic to $\mathfrak{B}^-/1F$, where $\mathfrak{B} = 1F \oplus \tilde{\mathfrak{B}}$ is the non-commutative Jordan algebra constructed above.

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