Can. J. Math., Vol. XXXVIII, No. 3, 1986, pp. 552-562

STRUCTURAL PROPERTIES OF A NEW CLASS OF CM-LATTICES

JOHNNY A. JOHNSON AND GERALD R. SHERETTE

1. Introduction. In this paper we introduce and study a class of multiplicative lattices called q-lattices. A q-lattice is a principally generated multiplicative lattice in which each principal element is compact. One of our main objectives is to characterize principal elements in these lattices (We note that Noether lattices and r-lattices are q-lattices [**1**, Theorem 2.1] and so our results apply to these two types of lattices). Among other things we determine necessary and sufficient conditions for globalizing local results in q-lattices. We then apply localization to establish some properties of principal elements in general q-lattices. Conditions equivalent to an element being principal are known for several different classes of multiplicative lattices. For example, Bogart [**2**] showed that if the lattice is modular, weak principal is equivalent to principal; Johnson and Lediaev pointed out that for Noether lattices, meet principal is equivalent to principal [**5**]; and, in an r-lattice, an element is principal if and only if it is compact and weak meet principal [**6**]. Notice that modularity is required for each of the instances above. In this paper, we obtain results similar to those cited above for our class of q-lattices, the members of which are not required to be modular.

We begin by showing in Section 2 that for q-lattices, in the local setting, the concepts of principal, weak meet principal and completely join irreducible are all equivalent (Theorem 2.5). In Section 3 a method for localization of q-lattices at prime elements is discussed. In Section 4, a globalization theorem (Theorem 4.1) is proved and used to obtain some conditions equivalent to an element being principal in a general q-lattice (Theorem 4.2). Finally in Section 5 we give an example of a q-lattice that is not an r-lattice.

2. Preliminaries. In this section we introduce our terminology and formally state a few observations that are required later. The main result of this section (Theorem 2.5) gives, for the local case anyway, several conditions equivalent to an element being principal in a q-lattice.

By a multiplicative lattice, **L**, we shall mean a complete lattice on which there is defined a commutative, associative, infinitely join distributive

Received March 31, 1983 and in revised form July 23, 1984. The first author was partially supported by a University of Houston Faculty Development Leave Grant.

multiplication such that the greatest element, I, of $\mathbf{L}$ is an identity for multiplication. An element E of $\mathbf{L}$ is called

1) weak meet principal if $(A:E)E = A \wedge E$ for all A in $\mathbf{L}$

2) weak join principal if $(AE):E = A \vee (0:E)$ for all A in $\mathbf{L}$

3) meet principal if $(A \wedge (B:E))E = (AE) \wedge B$ for all A, B in $\mathbf{L}$

4) join principal if $(A \vee (BE)):E = (A:E) \vee B$ for all A, B in $\mathbf{L}$.

If E satisfies both 1) and 2), E is said to be weak principal; E is principal if it satisfies both 3) and 4). An element $M \neq I$ is said to be maximal if I is a cover for M. An element B of a complete lattice L is called completely join irreducible if whenever

$$\{A_\alpha | \alpha \in \Lambda\} \subseteq \mathbf{L} \quad \text{and} \quad B = \vee\{A_\alpha | \alpha \in \Lambda\},$$

then $B = A_\alpha$ for some $\alpha \in \Lambda$ (where Λ is an arbitrary indexing set). Also if the indexing set is clear, we shall write $\{A_\alpha\}$ for $\{A_\alpha | \alpha \in \Lambda\}$ and $\vee A_\alpha$ for $\vee\{A_\alpha | \alpha \in \Lambda\}$. For terminology and notation not covered here, the reader is referred to [**3**] and [**4**].

We begin our discussion by noting a fact that will be very useful later, namely, in a q-lattice the product of compact elements is again compact. This can be shown easily using the distributivity of multiplication and the following characterization of compact elements.

LEMMA 2.1. *Let A be a member of the q-lattice $\mathbf{L}$. Then A is compact if and only if A is the join of a finite set of principal elements.*

Notice that the greatest element of a multiplicative lattice is always principal and since, in a q-lattice, principal elements are compact, any q-lattice has its greatest element compact. This observation will allow us to make use of the following lemma.

LEMMA 2.2. *Let $\mathbf{L}$ be a multiplicative lattice with compact greatest element, I, and suppose $A \in \mathbf{L}$ is not equal to I. Then there exists a maximal element M of $\mathbf{L}$ such that $A \leqq M$.*

Proof. This follows easily from Zorn's Lemma.

For the same setting as in Lemma 2.2, D. D. Anderson proved a result which we restate here for future reference ([**1**], Theorem 1.3).

LEMMA 2.3. *Let $\mathbf{L}$ be a multiplicative lattice with compact greatest element, I, and let $A \in \mathbf{L}$ be weak principal. If $A = \vee\{A_\alpha | \alpha \in \Lambda\}$ for some arbitrary Λ, then there exists a finite set of indices $\{\alpha_1, \dots, \alpha_n\} \subseteq \Lambda$ such that*

$$A = A_{\alpha_1} \vee \dots \vee A_{\alpha_n}.$$

Since a q-lattice is principally generated, applying Lemma 2.3 shows that any weak principal element of a q-lattice can be expressed as a finite join of principal elements. By Lemma 2.1 such an element is compact. Thus we have proved the following result.

LEMMA 2.4. *Let* $\mathbf{L}$ *be a q-lattice and* $A \in \mathbf{L}$ *be weak principal. Then* A *is compact.*

We conclude this section with a result about q-lattices in the local case but first let us make clear what is meant here by the phrase "the local case". A complete lattice $\mathbf{L}$ with greatest element I is said to be *totally quasi-local* if $\mathbf{L}$ has a unique maximal element M and M has the property that, for all $A \in \mathbf{L}$, $A \neq I$ implies $A \leqq M$.

THEOREM 2.5. *Let* $\mathbf{L}$ *be a totally quasi-local q-lattice. For each* $A \in \mathbf{L}$, *the following are equivalent*:

1) A *is principal*
2) A *is meet principal*
3) A *is weak principal*
4) A *is weak meet principal*
5) A *is completely join irreducible.*

Proof. Clearly $1 \Rightarrow 2 \Rightarrow 4$, and $1 \Rightarrow 3 \Rightarrow 4$. Also, since $\mathbf{L}$ is principally generated, $5 \Rightarrow 1$. Thus to complete the proof we need only show $4 \Rightarrow 5$.

Suppose A is weak meet principal and

$$A = \vee\{A_\alpha | \alpha \in \Lambda\}$$

for some $\{A_\alpha | \alpha \in \Lambda\} \subseteq \mathbf{L}$. If $A = 0$, A is trivially join irreducible, so assume $A > 0$. For each $\alpha \in \Lambda$,

$$(A_\alpha : A)A = A \wedge A_\alpha = A_\alpha.$$

Since $A > 0$ and $\mathbf{L}$ is principally generated, we can find $B \neq 0$ such that B is principal and $B \leqq A$. Then

$$\begin{aligned} B = A \wedge B &= (B:A)A \\ &= (B:A)(\vee A_\alpha) \\ &= (B:A)(\vee((A_\alpha:A)A)) \\ &= (B:A)(A(\vee(A_\alpha:A))) \\ &= ((B:A)A)(\vee(A_\alpha:A)) \\ &= B(\vee(A_\alpha:A)). \end{aligned}$$

Hence

$$I = B:B = (B(\vee(A_\alpha:A))):B$$

$$= (0{:}B) \vee (\vee(A_\alpha{:}A)).$$

Since **L** is totally quasi-local, the above join can equal I only if some term is equal to I. But $0{:}B \neq I$ since $B \neq 0$, thus $A_{\alpha_0}{:}A = I$ for some $A_{\alpha_0} \in \{A_\alpha\}$. Hence $A \leqq A_{\alpha_0}$, so $A = A_{\alpha_0}$. Thus A is completely join-irreducible and we have $4 \Rightarrow 5$.

3. Localization of q-lattices. Here we shall show, by elaborating upon a method introduced by P. McCarthy [**7**], that, given a prime element of a q-lattice **L**, we can construct a totally quasi-local q-lattice whose structure is related in a useful way to that of **L**. This localization approach differs from that of Dilworth [**3**] for Noether lattices and Anderson [**1**] for r-lattices, both of which produce lattices consisting of equivalence classes, in that the elements of our localization lattice are in fact also members of **L**. Actually it can be proved that for the case in which each method mentioned above is applicable (localization of a Noether lattice at a prime) the lattices obtained are all isomorphic. A nice way of showing this is through the use of closure operations. To do so in this paper would entail the introduction of a great deal of unnecessary machinery, but the interested reader may refer to [**8**].

The construction of our localization lattice begins with a mapping which can be defined in a very general setting. Suppose P is a prime of a multiplicative lattice **L**. Let

$$\mathbf{M}(P) = \{X \in \mathbf{L} | X \text{ is principal and } X \nleqq P\}.$$

(Notice $\mathbf{M}(P) \neq \emptyset$ since the greatest element of **L** is always a member. Also since the product of principal elements is principal, it follows easily that $\mathbf{M}(P)$ is closed under finite products.) For $A \in \mathbf{L}$, let

$$\mathbf{N}(P, A) = \{Y \in \mathbf{L} | XY \leqq A \text{ for some } X \in \mathbf{M}(P)\}.$$

Define $\psi_P{:}\mathbf{L} \to \mathbf{L}$ by

$$\psi_P(A) = \vee \mathbf{N}(P, A) \quad \text{for each } A \in \mathbf{L}.$$

We shall need the following properties of ψ_P.

LEMMA 3.1. *Let P be a prime of the multiplicative lattice* **L**. *For all $A, B \in$* **L**

1) $A \leqq \psi_P(A)$

2) $A \leqq B$ *implies* $\psi_P(A) \leqq \psi_P(B)$

3) *if A is principal then $A \nleqq P$ implies $\psi_P(A) = I$*

4) *if A is compact then $A \leqq \psi_P(B)$ implies $A \in \mathbf{N}(P, B)$.*

Proof. Properties 1 through 3 are straightforward and we omit their proofs. To show 4, assume A is compact. Then

$$A \leqq \psi_P(B) = \vee \mathbf{N}(P, B)$$

implies we can find a finite set $\{Y_1, \ldots, Y_n\} \subseteq \mathbf{N}(P, B)$ such that

$$A \leqq Y_1 \vee \ldots \vee Y_n.$$

For each indice i, $Y_i \in \mathbf{N}(P, B)$ implies there exists $X_i \in \mathbf{M}(P)$ such that $X_iY_i \leqq B$. Then the product of the X_i, call it X, belongs to $\mathbf{M}(P)$ and

$$AX \leqq (Y_1 \vee \ldots \vee Y_n)X = Y_1X \vee \ldots \vee Y_nX$$
$$\leqq Y_1X_1 \vee \ldots \vee Y_nX_n \leqq B.$$

For the more specialized setting where $\mathbf{L}$ is a q-lattice, the following properties of ψ_P are useful.

LEMMA 3.2. *Let P be a prime of the q-lattice $\mathbf{L}$. Then for all $A, B \in \mathbf{L}$ and all arbitrary subsets $\{A_\alpha | \alpha \in \Lambda\} \subset \mathbf{L}$,*

1) $A \nleqq P \Rightarrow \psi_P(A) = I$
2) $\psi_P(\psi_P(A)) = \psi_P(A)$
3) $\psi_P(\wedge \psi_P(A_\alpha)) = \wedge \psi_P(A_\alpha)$
4) $\psi_P(\vee \psi_P(A_\alpha)) = \psi_P(\vee A_\alpha)$
5) $A\psi_P(B) \leqq \psi_P(AB)$
6) $\psi_P(A)\psi_P(B) \leqq \psi_P(AB)$
7) $\psi_P(AB) = \psi_P(\psi_P(A)\psi_P(B))$
8) $A = \psi_P(A)$ *implies* $\psi_P(A:B) = A:B$
9) B *is compact implies* $\psi_P(A):B \leqq \psi_P(A:B)$
10) B *is compact implies* $\psi_P(A:B) = \psi_P(A):\psi_P(B)$.

For the remainder of this section $\mathbf{L}$ will denote a q-lattice and P will be a fixed prime of $\mathbf{L}$.

Let

$$\mathbf{L}_P = \{X \in \mathbf{L} | X = \psi_P(X)\}.$$

Clearly $\mathbf{L}_P$ together with the ordering inherited from $\mathbf{L}$ is a poset but we can easily see that it is actually a complete lattice. To show this, it is sufficient to show every subset of $\mathbf{L}_P$ has a greatest lower bound in $\mathbf{L}_P$; so let $\{A_\alpha | \alpha \in \Lambda\}$ be an arbitrary subset of $\mathbf{L}_P$. Then

$$\wedge A_\alpha = \wedge \psi_P(A_\alpha) = \psi_P(\wedge \psi_P(A_\alpha)) = \psi_P(\wedge A_\alpha)$$

where the first and third equalities hold since each A_α belongs to $\mathbf{L}_P$ while the second is just part 3 of Lemma 3.2. Thus $\wedge A_\alpha \in \mathbf{L}_P$ and obviously $\wedge A_\alpha$ is the greatest lower bound of $\{A_\alpha\}$ in $\mathbf{L}_P$. It is important to note that, in general, $\mathbf{L}_P$ does not form a sublattice of $\mathbf{L}$ even though the preceding shows the meet operation is the same for both lattices. The problem is that, even if $\{A_\alpha\} \subseteq \mathbf{L}_P$, $\vee A_\alpha$ does not necessarily belong to $\mathbf{L}_P$. However, $\psi_P(\vee A_\alpha)$ does belong to $\mathbf{L}_P$ and is easily seen to be the least upper bound of $\{A_\alpha\}$ in $\mathbf{L}_P$. Hence, for $\{A_\alpha\} \subseteq \mathbf{L}_P$, we define

$$\bigvee_P A_\alpha = \psi_P(\bigvee A_\alpha)$$

and to be consistent in notation, let

$$\bigwedge_P A_\alpha = \bigwedge A_\alpha.$$

Furthermore, for all $A, B \in \mathbf{L}_P$, set

$$A \cdot_P B = \psi_P(AB).$$

LEMMA 3.3. $(\mathbf{L}_P; \wedge_P, \vee_P, \cdot_P)$ *is a multiplicative lattice.*

Proof. We first note that $\cdot_P$ is obviously commutative; that I, the largest element of $\mathbf{L}$, belongs to $\mathbf{L}_P$; and $I = \psi_P(I)$ serves as the multiplicative identity. For $A, B, C \in \mathbf{L}_P$,

$$\begin{aligned}(A \cdot_P B) \cdot_P C &= \psi_P(\psi_P(AB)C) \\ &= \psi_P(\psi_P(AB)\psi_P(C)) \\ &= \psi_P((AB)C) \\ &= \psi_P(A(BC)) \\ &= \psi_P(\psi_P(A)\psi_P(BC)) \\ &= \psi_P(A\psi_P(BC)) \\ &= A \cdot_P (B \cdot_P C).\end{aligned}$$

Hence it only remains to be shown that $\cdot_P$ is $\vee_P$-distributive. Let $\{A_\alpha\} \subseteq \mathbf{L}_P$ and $A \in \mathbf{L}_P$. Then

$$\begin{aligned}A \cdot_P (\bigvee_P A_\alpha) &= \psi_P(A(\bigvee_P A_\alpha)) \\ &= \psi_P(\psi_P(A)\psi_P(\bigvee A_\alpha)) \\ &= \psi_P(A(\bigvee A_\alpha)) \\ &= \psi_P(\bigvee (AA_\alpha)) \\ &= \psi_P(\bigvee \psi_P(AA_\alpha)) \\ &= \psi_P(\bigvee (A \cdot_P A_\alpha)) \\ &= \bigvee_P (A \cdot_P A_\alpha)\end{aligned}$$

which completes the proof.

If $:_P$ denotes residuation in $(\mathbf{L}_P; \wedge_P, \vee_P, \cdot_P)$ then we have the following result.

LEMMA 3.4. *For all* $A, B \in \mathbf{L}_P$, $A :_P B = A : B$.

Proof. Since $A \in \mathbf{L}_P$, $A = \psi_P(A)$. Hence by part 8 of Lemma 3.2,

$$\psi_P(A:B) = A:B$$

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so at least $A{:}B \in \mathbf{L}_P$. Notice

$$(A{:}B)\cdot_P B = \psi_P((A{:}B)B) \leqq \psi_P(A) = A.$$

Hence $A{:}B \leqq A{:}_P B$.

Also, in $\mathbf{L}$ we have

$$\begin{aligned}(A{:}_P B)B &= (A{:}_P B)\psi_P(B)\\ &\leqq \psi_P((A{:}_P B)B)\\ &= (A{:}_P B)\cdot_P B\\ &\leqq A.\end{aligned}$$

Thus $A{:}_P B \leqq A{:}B$.

We can summarize many of the results about ψ_P shown above by saying ψ_P is a multiplicative lattice homomorphism from $\mathbf{L}$ onto $\mathbf{L}_P$ that preserves residuation by compact elements of $\mathbf{L}$; that is for all $A, B \in \mathbf{L}_P$

1) $\psi_P(A \vee B) = \psi_P(A) \vee_P \psi_P(B)$

2) $\psi_P(A \wedge B) = \psi_P(A) \wedge_P \psi_P(B)$

3) $\psi_P(AB) = \psi_P(A) \cdot_P \psi_P(B)$

4) B is compact in $\mathbf{L}$ implies $\psi_P(A{:}B) = \psi_P(A){:}_P \psi_P(B)$.

Using these properties, the proof of the following lemma is straightforward.

LEMMA 3.5. *Let B be a compact element of $\mathbf{L}_P$. Then*

1) *B is weak meet principal in $\mathbf{L}$ implies $\psi_P(B)$ is weak meet principal in $\mathbf{L}_P$.*

2) *B is weak join principal in $\mathbf{L}$ implies $\psi_P(B)$ is weak join principal in $\mathbf{L}_P$.*

3) *B is meet principal in $\mathbf{L}$ implies $\psi_P(B)$ is meet principal in $\mathbf{L}_P$.*

4) *B is join principal in $\mathbf{L}$ implies $\psi_P(B)$ is join principal in $\mathbf{L}_P$.*

Since, as we saw in Lemma 2.5, weak principal elements of $\mathbf{L}$ are compact, the following is an immediate consequence.

COROLLARY 3.6. *Let $B \in \mathbf{L}$. Then*

1) *B is weak principal in $\mathbf{L}$ implies $\psi_P(B)$ is weak principal in $\mathbf{L}_P$.*

2) *B is principal in $\mathbf{L}$ implies $\psi_P(B)$ is principal in $\mathbf{L}_P$.*

LEMMA 3.7. *Let A be a compact element of $\mathbf{L}$. Then $\psi_P(A)$ is compact in $\mathbf{L}_P$.*

Proof. Suppose $\psi_P(A) \leqq \vee_P A_\alpha$ for some $\{A_\alpha\} \subseteq \mathbf{L}_P$. Then

$$A \leqq \psi_P(A) \leqq \vee_P A_\alpha = \psi_P(\vee A_\alpha).$$

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So by property 4 of Lemma 3.1, there exists $X \in \mathbf{M}(P)$ such that $XA \leqq \vee A_\alpha$. But XA is a product of compact elements and hence itself compact. Thus we can find a finite subset

$$\{B_1, \ldots, B_n\} \subseteq \{A_\alpha\}$$

such that

$$XA \leqq B_1 \vee \ldots \vee B_n.$$

Let Y be principal in $\mathbf{L}$ such that $Y \leqq \psi_P(A)$. Again using 4 of Lemma 3.1, there exists $Z \in \mathbf{M}(P)$ such that $ZY \leqq A$, hence $ZX \in \mathbf{M}(P)$ and

$$ZXY \leqq XA \leqq B_1 \vee \ldots \vee B_n.$$

Thus $Y \in \mathbf{N}(P, B_1 \vee \ldots \vee B_n)$, so

$$Y \leqq \psi_P(B_1 \vee \ldots \vee B_n) = B_1 \vee_P \ldots \vee_P B_n.$$

Since $\mathbf{L}$ is principally generated,

$$\psi_P(A) \leqq B_1 \vee_P \ldots \vee_P B_n.$$

We can now conclude that $\mathbf{L}_P$ is the localization lattice promised at the beginning of this section.

THEOREM 3.8. *$(L_P; \wedge_P, \vee_P, \cdot_P)$ is a totally quasi-local q-lattice with maximal element $\psi_P(P) = P$.*

Proof. It follows easily from Corollary 3.7 that $\mathbf{L}_P$ is principally generated.

Let E be a principal element of $\mathbf{L}_P$. Since $\mathbf{L}$ is principally generated, there exists $\{A_\alpha\} \subseteq \mathbf{L}$ such that each A_α is principal in $\mathbf{L}$ and $E = \vee A_\alpha$. Since $\mathbf{L}$ is a q-lattice, each A_α is compact in $\mathbf{L}$. Then

$$E = \psi_P(E) = \psi_P(\vee A_\alpha) = \vee_P \psi_P(A_\alpha).$$

By Lemma 3.7, each $\psi_P(A_\alpha)$ is compact in $\mathbf{L}_P$; and notice that by the same lemma, $I = \psi_P(I)$ is also compact in $\mathbf{L}_P$, so we can apply Lemma 2.2 to obtain

$$E = \psi(B_1) \vee_P \ldots \vee_P (B_n)$$

where each $B_i \in \{A_\alpha\}$. Thus E is the join of a finite number of compact elements of $\mathbf{L}_P$ and hence E is compact in $\mathbf{L}_P$. Therefore $\mathbf{L}_P$ is a q-lattice.

Suppose Y is a principal element of $\mathbf{L}$ such that $Y \leqq \psi_P(P)$. Then there exists $X \in \mathbf{M}(P)$ such that $XY \leqq P$. But $X \in \mathbf{M}(P)$ implies $X \nleqq P$ and so since P is prime, we must have $Y \leqq P$. It follows that

$$\psi_P(P) \leqq P \leqq \psi_P(P).$$

If $A \in \mathbf{L}_P$ is such that $A \nleqq \psi_P(P)$, then

$$A = \psi_P(A) \nleqq \psi_P(P) = P.$$

Hence, by property 1 of Lemma 3.2,

$$A = \psi_P(A) = I,$$

which shows that $\mathbf{L}_P$ is totally quasi-local.

4. Globalization and applications. In this section we show that identities which hold in every localization of a q-lattice must also hold in the original lattice. In fact, in multiplicative lattices, as can easily be shown, maximal elements are always prime and Theorem 4.1 demonstrates that in order to establish an identity in a q-lattice $\mathbf{L}$, it is sufficient to show the identity holds in each localization of $\mathbf{L}$ at a maximal element. We conclude the section by applying Theorem 4.1 to obtain some conditions equivalent to an element being principal in a general q-lattice.

THEOREM 4.1. *Let A, B be members of the q-lattice* $\mathbf{L}$. *Then $A = B$ if and only if $\psi_M(A) = \psi_M(B)$ for every maximal element M of* $\mathbf{L}$.

Proof. Suppose $\psi_M(A) = \psi_M(B)$ for every maximal element M of $\mathbf{L}$ and $A \neq B$. Without loss of generality, we may assume $A \nleqq B$. Since $\mathbf{L}$ is principally generated, there exists a principal element E of $\mathbf{L}$ such that $E \leqq A$ but $E \nleqq B$ which, in turn implies $B:E \neq I$. Hence by Lemma 2.1, there is a maximal element $M \in \mathbf{L}$ such that $B:E \leqq M < I$, so that

$$\psi_M(B:E) \neq \psi_M(I).$$

Also since $E \leqq A$, $A:E = I$. Thus

$$\psi_M(A:E) = \psi_M(I).$$

However, since E is compact,

$$\psi_M(B:E) = \psi_M(B):\psi_M(E) = \psi_M(A):\psi_M(E) = \psi_M(A:E)$$

which contradicts $\psi_M(B:E) \neq \psi_M(I)$, so $A = B$.

The reverse implication is trival.

THEOREM 4.2. *Let* $\mathbf{L}$ *be a q-lattice and $A \in$* $\mathbf{L}$. *Then the following are equivalent.*

1) *A is principal*

2) *A is weak principal*

3) *A is compact and A is meet principal*

4) *A is compact and A is weak meet principal*

5) *A is compact and $\psi_M(A)$ is weak meet principal in* $\mathbf{L}_M$ *for every maximal element of M of* $\mathbf{L}$

6) *A is compact and $\psi_M(A)$ is meet principal in* $\mathbf{L}_M$ *for every maximal element M of* $\mathbf{L}$

7) *A is compact and $\psi_M(A)$ is weak principal in* $\mathbf{L}_M$ *for every maximal element M of* $\mathbf{L}$

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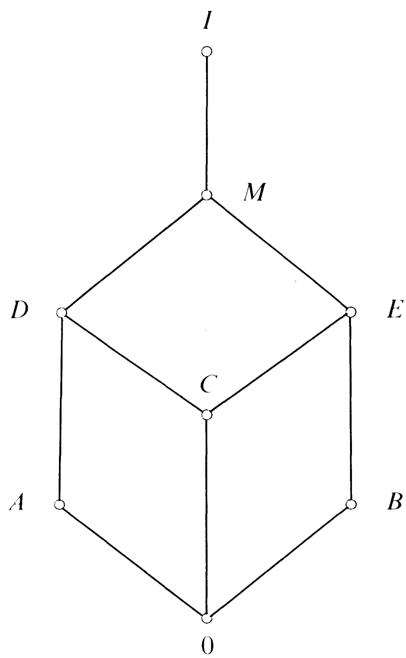
8) *A is compact and* $\psi_M(A)$ *is completely join irreducible in* $\mathbf{L}_M$ *for every maximal element M of* **L**

9) *A is compact and* $\psi_M(A)$ *is principal in* **L** *for every maximal element M of* **L**.

Proof. Clearly $1 \Rightarrow 2$ and $1 \Rightarrow 3 \Rightarrow 4$. Using Lemma 2.1 we see that $2 \Rightarrow 4$. Also $4 \Rightarrow 5$ by Corollary 3.6, and 5 through 9 are equivalent by Lemma 2.5. Finally, Theorem 4.1 and the four properties immediately preceeding Lemma 3.5 allow us to conclude that $9 \Rightarrow 1$.

5. Example. We conclude this paper with an example of a q-lattice which is not modular, and hence neither a Noether lattice nor an r-lattice. Consider the lattice **L** below together with the trivial multiplication ($XI = X$ for all $X \in \mathbf{L}$; $XY = 0$ for all $X, Y \in \mathbf{L}$ different from I):



The reader may easily verify that all elements of **L** are join principal and that the elements $I, A, B, C, 0$ are meet principal. Since **L** is finite, all elements are compact and clearly **L** is generated by $\{I, A, B, C, 0\}$. Thus **L** is a totally quasi-local q-lattice which is not modular.

REFERENCES

1. D. D. Anderson, *Abstract commutative ideal theory without chain condition*, Algebra Universalis 6 (1976), 131-145.

https://doi.org/10.4153/CJM-1986-027-0 Published online by Cambridge University Press

2. K. P. Bogart, *Distributive local Noether lattices*, Michigan Math. J. *16* (1969), 215-223.
3. R. P. Dilworth, *Abstract commutative ideal theory*, Pacific J. Math. *12* (1962), 481-498.
4. G. Grätzer, *Lattice theory* (W. H. Freeman and Company, San Francisco, 1971).
5. E. W. Johnson and J. P. Lediaev, *Join principal elements in Noether lattices*, Proc. Amer. Math. Soc. *36* (1972), 73-78.
6. J. A. Johnson, *Principal elements in multiplicative lattices*, Boll. Un. Mat. Ital. (*6*), 1-B (1982), 673-681.
7. P. J. McCarthy, *Arithmetical rings and multiplicative lattices*, Ann. Mat. Pura Appl. *82* (1969), 267-274.
8. D. C. Perez, Dissertation, University of Houston (1979).

University of Houston-University Park,
Houston, Texas

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