

MAXIMAL OPERATORS AND HILBERT TRANSFORMS ALONG FLAT CURVES NEAR L^1

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Abstract

For a class of convex curves in \mathbb{R}^d we prove that the corresponding maximal operator and Hilbert transform are of weak type $L \log L$. The point of interest here is that this class admits curves which are infinitely flat at the origin. We also prove an analogous weak type result for a class of nonconvex hypersurfaces.

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1. Introduction

Given natural numbers m and d with $d \geq m + 1$, let $\Gamma : \mathbb{R}^m \rightarrow \mathbb{R}^d$ be a smooth m parameter surface in \mathbb{R}^d . Let the associated maximal operator, M_Γ , and Hilbert transform, H_Γ , be given by

$$M_\Gamma f(x) := \sup_{h>0} \frac{1}{h^m} \left| \int_{|t| \in (0,h)} f(x - \Gamma(t)) dt \right|,$$
$$H_\Gamma f(x) := \text{p.v.} \int_{\mathbb{R}^m} f(x - \Gamma(t)) K(t) dt,$$

for appropriate functions f on \mathbb{R}^d . Here, $K : \mathbb{R}^m \rightarrow \mathbb{R}$ is a Calderón–Zygmund kernel; that is, K is C^∞ away from the origin, homogeneous of degree $-d$ and $\int_{|t| \in [a,b]} K(t) dt = 0$ for each $0 < a < b$. In this paper we are concerned with the mapping properties of M_Γ and H_Γ on Orlicz spaces near L^1 and, in particular, with estimates of the form

$$|\{x \in \mathbb{R}^d : |Tf(x)| > \alpha\}| \leq \int_{\mathbb{R}^d} \Phi\left(\frac{C|f(x)|}{\alpha}\right) dx \quad \forall \alpha > 0, \quad (1.1)$$

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where $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a convex function and $T \in \{M_\Gamma, H_\Gamma\}$. As in [13], we shall say that T is of weak type $\Phi(L)$ if (1.1) holds for all simple functions f on \mathbb{R}^d .

Even for model cases of Γ possessing nonvanishing Gaussian curvature, such as $m = 1$, $d = 2$ and $\Gamma(t) = (t, t^2)$, it is unknown whether M_Γ and H_Γ are of weak type L . Currently, the most progress on this question for the parabola is a weak type $L \log \log L$ estimate due to Seeger, Tao and Wright [13]; that is, with the usual abuse of notation, the estimate in (1.1) holds for both operators with $\Phi(t) = t \log \log(100 + t)$. For closely related results, we note that both operators map the parabolic Hardy space to $L^{1,\infty}$ (Christ [7]) and map the smaller product-type Hardy space to the smaller Lorentz space $L^{1,2}$ (Seeger and Tao [12]). Christ [8] has also shown that both operators are not of weak type L if Γ is a ‘piecewise linear parabola’ (see also [3] for further results in this direction).

The main theorem in [13] which we state below covers a large class of hypersurfaces in \mathbb{R}^d for $d \geq 2$, of which the parabolic plane curve is a special case.

THEOREM 1.1. [13] *Let Σ_0 be a smooth compact hypersurface of \mathbb{R}^d whose Gaussian curvature does not vanish to infinite order and let ν be a smooth and compactly supported density on Σ_0 . Then the operator $f \mapsto \sup_{k \in \mathbb{Z}} |\nu_k * f|$ is of weak type $L \log \log L$, where*

$$\langle \nu_k, \psi \rangle := \langle \nu, \psi(\delta_{2^k} \cdot) \rangle, \quad \delta_t := \exp((\log t)P),$$

and P is a real d by d matrix whose eigenvalues each have positive real part. Moreover, if the cancellation condition $\widehat{\nu}(0) = 0$ holds then $f \mapsto \sum_{k \in \mathbb{Z}} \nu_k * f$ extends to an operator of weak type $L \log \log L$.

It is important for us to observe that weaker $L \log L$ estimates for the operators in Theorem 1.1 are true under weaker hypotheses. For the maximal operator, this is the case if ν is a compactly supported measure whose Fourier transform satisfies a decay estimate

$$|\widehat{\nu}(\xi)| \leq C(1 + |\xi|)^{-\tau} \quad \text{for some } \tau > 0 \quad (1.2)$$

and, for the singular integral operator, if additionally we have the cancellation condition $\widehat{\nu}(0) = 0$; see [13, Corollary 3.1]. The nonvanishing curvature hypothesis in Theorem 1.1 guarantees an estimate of the form (1.2) via van der Corput’s lemma. This $L \log L$ result essentially reproved earlier work of Christ and Stein [9] who, for example, considered the class of homogeneous curves in \mathbb{R}^d given by $\Gamma(t) = (t^{\alpha_1}, \dots, t^{\alpha_d})$ for positive integers α_j . We emphasize, however, that even for model cases when the codimension of Γ is high, say $m = 1$, $d = 3$ and $\Gamma(t) = (t, t^2, t^3)$, there has been no improvement beyond $L \log L$.

Our first result in this note is an $L \log L$ estimate for a class of convex curves in \mathbb{R}^d for $d \geq 2$ considered by Carbery *et al.* in [6] (see also [11] for very closely related work). Before describing this class, we highlight that certain curves which are infinitely flat at the origin are admitted into this class and therefore fall outside the scope of previous work.

Suppose that $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_d(t))$, where each γ_j belongs to $C^d(0, \infty)$. The next definition makes precise exactly what we mean by convex.

DEFINITION 1.2. The curve $(t, \gamma_2(t), \dots, \gamma_d(t))$ is said to be *convex* if, for all $j = 1, \dots, d$, $D_j(t) > 0$ for all $t > 0$, where

$$D_j(t) := \det \begin{pmatrix} 1 & \gamma_2'(t) & \cdots & \gamma_j'(t) \\ 0 & \gamma_2''(t) & \cdots & \gamma_j''(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j)}(t) & \cdots & \gamma_j^{(j)}(t) \end{pmatrix}.$$

Carbery *et al.* [6] showed that M_Γ and H_Γ are bounded on L^p for all $p \in (1, \infty)$ if $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_d(t))$ is convex, odd, $\Gamma(0) = 0$ and a certain curvature condition holds, namely,

$$t h_j'(t) \geq C_d h_j(t) \quad \forall t > 0 \text{ and } j = 1, \dots, d, \tag{1.3}$$

where $h_j(t) := N_j(t)/D_{j-1}(t)$, $D_0(t) := 1$ and

$$N_j(t) := \det \begin{pmatrix} t & \gamma_2(t) & \cdots & \gamma_j(t) \\ 1 & \gamma_2'(t) & \cdots & \gamma_j'(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \gamma_2^{(j-1)}(t) & \cdots & \gamma_j^{(j-1)}(t) \end{pmatrix}.$$

This result is an extension of earlier work in [4] for plane curves. Our first main result is as follows.

THEOREM 1.3. *Suppose that $\Gamma(t) = (t, \gamma_2(t), \dots, \gamma_d(t))$ is convex, odd, $\Gamma(0) = 0$ and (1.3) holds. Then M_Γ and H_Γ are of weak type $L \log L$.*

Our second result concerns the class of nonconvex plane curves and associated hypersurfaces considered recently in [1].

THEOREM 1.4. *For $d \geq 2$, let $\Gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^d$ be a hypersurface of the form $\Gamma(y) = (y, P(\gamma(|y|)))$ where P is a real polynomial of degree no less than 2 and γ satisfies the following conditions:*

$$\gamma \in C^2(0, \infty), \text{ convex on } [0, \infty), \gamma(0) = 0 \text{ and } \gamma'(0) \geq 0. \tag{1.4}$$

If γ is extended to be either even or odd, then the operators M_Γ and H_Γ are of weak type $L \log L$ if either (1) $d \geq 3$ or (2) $d = 2$ and $P'(0) = 0$.

REMARKS. (1) In [1] it was shown that if $d = 2$ and $P'(0)$ is nonzero then M_Γ and H_Γ are bounded on L^p for all $p \in (1, \infty)$ under the additional hypothesis that γ' is doubling; that is, there exists a constant $D \in (1, \infty)$ such that $\gamma'(Dt) \geq 2\gamma'(t)$ for

each $t > 0$. See [3] for certain negative results near L^1 when $P(s) = s$ and for a certain class of γ for which γ' is doubling.

(2) The reason for the more relaxed hypothesis when $d \geq 3$ is that the curvature of \mathbb{S}^{d-2} plays a significant role in the proof.

NOTATION. For nonnegative numbers A and B , we write $A \lesssim B$ and $B \gtrsim A$ if $A \leq CB$, where the constant C depends on at most d and Γ . If $A \lesssim B \lesssim A$ then we may write $A \sim B$.

For $y \in \mathbb{R}^d$ and $r > 0$, let $B(y, r)$ denote the open Euclidean ball with centre y and radius r given by

$$B(y, r) := \{x \in \mathbb{R}^d : |x - y| < r\}.$$

2. Proof of Theorem 1.3

We begin with the observation that if

$$\mathcal{M}_\Gamma f(x) := \sup_{k \in \mathbb{Z}} \frac{1}{\lambda^k} \left| \int_{\lambda^k}^{\lambda^{k+1}} f(x - \Gamma(t)) dt \right|$$

for some fixed $\lambda \in (1, \infty)$ depending on d and Γ to be determined later in the proof, then $M_\Gamma f \sim \mathcal{M}_\Gamma f$ for nonnegative functions f ; thus, for the maximal operator, it suffices to consider \mathcal{M}_Γ . The main idea is to follow the arguments in [13] which led to the weaker $L \log L$ version of Theorem 1.1 under the weaker Fourier transform decay hypothesis, as described in the Introduction here. Further key ingredients to the proof are an appropriate Calderón–Zygmund theory and suitable control on the dilations in hand. In our flat setting determined by the curvature condition (1.3), we rely on [4] and [6] to provide these ingredients.

2.1. Dilations and decay estimates All of the work on the choice of dilations and proving the decay estimates that follow was done in [6] and thus we simply state their results; for motivation and proofs we refer the reader to [6]. The dilation matrices $\{A(t) \mid t > 0\}$ are given by

$$A(t) := \begin{pmatrix} t & R_1 t & \cdots & R_{d-1} R_{d-2} \cdots R_1 t \\ \gamma_2(t) & R_1 \gamma_2(t) & \cdots & R_{d-1} R_{d-2} \cdots R_1 \gamma_2(t) \\ \vdots & \vdots & & \vdots \\ \gamma_d(t) & R_1 \gamma_d(t) & \cdots & R_{d-1} R_{d-2} \cdots R_1 \gamma_d(t) \end{pmatrix},$$

where the differential operators R_1, \dots, R_{d-1} are given by

$$R_j f := \left(\frac{f}{h_j} \right)' \frac{h_j^2}{h_j'} \quad \text{for } j = 1, \dots, d - 1.$$

We remark that the convexity hypothesis on Γ ensures that $A(t)$ is well defined because all h_j and h_j' are positive; see [11]. By [6, Lemma 5.3], $A(t)$ is lower

triangular with $A(t)_{1,1} = t$ and $A(t)_{j,j} = h_j(t)$ for $j = 2, \dots, d$; hence $A(t)$ is invertible. Moreover, the curvature hypothesis (1.3) implies that the following Rivière condition holds (in fact, the full strength of the curvature hypothesis (1.3) is not needed to prove Proposition 2.1, and the assumption that each h_j is doubling suffices; see [6]).

PROPOSITION 2.1 [6]. *There exist positive constants C, ε depending on Γ and d such that $\|A(s)^{-1}A(t)\| \leq C (t/s)^\varepsilon$ for all $s \geq t > 0$.*

For each $k \in \mathbb{Z}$, we now define the normalized curves Γ_k by

$$\Gamma_k(t) := A(\lambda^k)^{-1}\Gamma(\lambda^k t) \quad \text{for each } |t| \in [1, \lambda].$$

Also, let the measures $\mu^{(k)}, \mu_k^{(k)}, H^{(k)}$ and $H_k^{(k)}$ be given by

$$\begin{aligned} \langle \mu^{(k)}, \psi \rangle &:= \int_1^\lambda \psi(\Gamma_k(t)) dt, & \langle \mu_k^{(k)}, \psi \rangle &:= \langle \mu^{(k)}, \psi(A(\lambda^k) \cdot) \rangle, \\ \langle H^{(k)}, \psi \rangle &:= \int_{|t| \in [1, \lambda]} \psi(\Gamma_k(t)) \frac{dt}{t}, & \langle H_k^{(k)}, \psi \rangle &:= \langle H^{(k)}, \psi(A(\lambda^k) \cdot) \rangle, \end{aligned}$$

so that $\mathcal{M}_\Gamma f = \sup_{k \in \mathbb{Z}} |\mu_k^{(k)} * f|$ and $H_\Gamma f = \sum_{k \in \mathbb{Z}} H_k^{(k)} * f$. The notation $\mu_k^{(k)}$ and $H_k^{(k)}$ may seem heavy-handed at first. The intention is to maintain the notation from [13] in the sense that $\mu_k^{(k)}$ and $H_k^{(k)}$ are $A(\lambda^k)$ -dilates of the measures $\mu^{(k)}$ and $H^{(k)}$. In general, $\mu^{(k)}$ and $H^{(k)}$ will not be fixed as k varies yet have the property that their Fourier transforms satisfy decay estimates independent of k , and in this sense are ‘essentially’ fixed. These crucial decay estimates are the content of the subsequent lemma. This was proved in [6, Section 5] via a variant of van der Corput’s lemma (see [6, Proposition 3.1] for this variant).

LEMMA 2.2 [6]. *There exists $\tau \in (0, 1)$ depending on d such that*

$$|\widehat{\mu^{(k)}}(\xi)| + |\widehat{H^{(k)}}(\xi)| \lesssim (1 + |\xi|)^{-\tau}$$

for all $k \in \mathbb{Z}$ and all $\xi \in \mathbb{R}^d$.

2.2. Calderón–Zygmund theory In order to use the Calderón–Zygmund theory developed in [4], we shall define balls $\{B_k\}_{k \in \mathbb{Z}}$ satisfying the following conditions:

- (B1) $\bigcup_{k \in \mathbb{Z}} B_k = \mathbb{R}^d$;
- (B2) $\bigcap_{k \in \mathbb{Z}} B_k = \{0\}$;
- (B3) each B_k is open, balanced, convex and bounded;
- (B4) $B_k \subset B_{k+1}$ for each $k \in \mathbb{Z}$;
- (B5) for each $k, |B_{k+1}| \sim |B_k|$.

It follows from Proposition 2.1 that $\|A(\lambda^{k+1})^{-1}A(\lambda^k)\| < 1$ uniformly in k if we choose $\lambda := 4^{\lceil 1 + (\log_2 C) / (2\varepsilon) \rceil}$, where C and ε are those constants appearing in

Proposition 2.1. Let $\widetilde{B}_k := A(\lambda^k)B(0, 1)$; one can check that (B1)–(B4) hold for the \widetilde{B}_k . We then define the collection of B_k by

$$\{B_k\}_{k \in \mathbb{Z}} := \bigcup_{k \in \mathbb{Z}} \bigcup_{l=1}^{n(k)-1} \{E_k^l\} \tag{2.1}$$

where, for each $k \in \mathbb{Z}$, $\{E_k^1, \dots, E_k^{n(k)}\}$ is a finite set of open, balanced, convex and bounded sets such that

$$\widetilde{B}_k = E_k^1 \subset E_k^2 \subset \dots \subset E_k^{n(k)} = \widetilde{B}_{k+1}$$

and $|E_k^l| \leq 2|E_k^{l-1}|$ for $l = 1, \dots, n(k) - 1$.

For each $k \in \mathbb{Z}$, observe that property (B3) allows us to define a norm $\|\cdot\|_k$ such that $B_k = \{x \in \mathbb{R}^d : \|x\|_k < 1\}$.

NOTATION. For each $y \in \mathbb{R}^d$, $k \in \mathbb{Z}$ and $r > 0$, let $B(y, k, r)$ be the open ball with centre y and radius r with respect to $\|\cdot\|_k$ given by

$$B(y, k, r) := \{x \in \mathbb{R}^d : \|x - y\|_k < r\}. \tag{2.2}$$

For each $k \in \mathbb{Z}$, $x \in \mathbb{R}^d$ and nonempty subset S of \mathbb{R}^d , define

$$\text{dist}_k(x, S) := \inf\{\|x - s\|_k : s \in S\}.$$

We now state the Whitney-type decomposition relative to the balls in (2.2).

PROPOSITION 2.3 [4]. *There exists a constant A depending on Γ and d such that the following conditions hold.*

(1) *If Ω is any nonempty proper open subset of \mathbb{R}^d , then $\Omega = \bigcup_{B \in \mathfrak{W}} B$, where*

$$\mathfrak{W} := \{B(x, k, 1) \mid x \in \Omega, k \in \mathbb{Z}, 5 < \text{dist}_k(x, \partial\Omega) < A\}.$$

(2) *If in addition $|\Omega|$ is finite, then we can find a sequence of disjoint balls $Q_i := B(x_i, k_i, 1) \in \mathfrak{W}$ such that $\Omega = \bigcup_i B(x_i, k_i, 3)$.*

If $\mathfrak{B} := \{B(y, k, A) \mid y \in \mathbb{R}^d, k \in \mathbb{Z}\}$, define the associated Hardy–Littlewood-type maximal function M_{HL} by

$$M_{HL}f(x) := \sup_{\substack{B \in \mathfrak{B} \\ x \in B}} \frac{1}{|B|} \int_B |f(y)| dy,$$

where $A > 0$ appears in Proposition 2.3.

2.3. Conclusion of the proof of Theorem 1.3 Fix $\alpha > 0$ and set

$$\Omega := \{x \in \mathbb{R}^d \mid M_{HL}f(x) > \alpha\}.$$

By [4, Proposition 2.2], M_{HL} is of weak type L and therefore $|\Omega| \lesssim \|f\|_1/\alpha$. Next, apply Proposition 2.3 to obtain sequences $\{x_i\} \subseteq \Omega$, $\{j_i\} \subseteq \mathbb{Z}$ and disjoint Whitney cubes $\{Q_i\}$ such that:

- (W1) $\Omega = \bigcup_i Q_i$;
- (W2) $B(x_i, j_i, 1) \subseteq Q_i \subseteq B(x_i, j_i, 3)$;
- (W3) $5 < \inf\{\|x_i - y\|_{j_i} : y \in \partial\Omega\} < A$.

Also define $\Omega^* := \bigcup_i B(x_i, j_i, C + 10)$, where C is the constant appearing in the statement of Proposition 2.1. Observe that (W1) implies that $|\Omega^*| \sim |\Omega|$; in particular, $|\Omega^*| \lesssim \|f\|_1/\alpha$.

By an analogue of the Lebesgue differentiation theorem, $|f(x)| \leq \alpha$ for $x \notin \Omega$. Our first decomposition is $f = g + \sum_i f_{Q_i}$, where

$$f_{Q_i}(x) := \begin{cases} f(x) & \text{if } x \in Q_i \text{ and } |f(x)| > \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

This decomposition is akin to that of classical Calderón–Zygmund theory. Next, decompose f_{Q_i} further by letting

$$f_{Q_i}^n(x) := \begin{cases} f_{Q_i}(x) & \text{if } \lambda^{(n-1)\tau}\alpha < |f_{Q_i}(x)| \leq \lambda^{n\tau}\alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where τ is the decay exponent from Lemma 2.2. Notice that $f_{Q_i} = \sum_{n \geq 1} f_{Q_i}^n$ and

$$\sum_{n \geq 1} \frac{1}{|Q_i|} \int |f_{Q_i}^n(x)| \, dx \lesssim \alpha. \tag{2.3}$$

Now let $g_{Q_i}^n := \chi_{Q_i}|Q_i|^{-1} \int_{Q_i} f_{Q_i}^n$ and $b_{Q_i}^n := f_{Q_i}^n - g_{Q_i}^n$. Furthermore, define $g^n := \sum_i g_{Q_i}^n$, $b^n := \sum_i b_{Q_i}^n$ and $f^n := \sum_i f_{Q_i}^n$. Observe that (2.3) implies that

$$\sum_{n \geq 1} |g_{Q_i}^n(x)| \leq \chi_{Q_i}(x) \sum_{n \geq 1} \frac{1}{|Q_i|} \int_{Q_i} |f_{Q_i}^n(y)| \, dy \lesssim \alpha \chi_{Q_i}(x). \tag{2.4}$$

Moreover, by (2.3) and (2.4),

$$\sum_{n \geq 1} \|b_{Q_i}^n\|_1 \lesssim \alpha |Q_i|. \tag{2.5}$$

We first consider \mathcal{M}_Γ , and the next step is to decompose the measures $\mu^{(k)}$, first by the following localization. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ have compact support in $B(0, 1/2)$ with $\int \phi(x) \, dx = 1$, and $\int x_k \phi(x) \, dx = 0$ for all $k = 1, \dots, d$. Also define

$$\mu^{(k),0} := \mu^{(k)} \quad \text{and} \quad \mu^{(k),n} := \phi_n * \mu^{(k)},$$

where $\phi_n(x) := \lambda^{nd} \phi(\lambda^n x)$ for each $n \geq 1$. For each $k \in \mathbb{Z}$ and $n \geq 1$ define the dilates

$$\mu_k^{(k),n} := \det A(\lambda^k)^{-1} \mu^{(k),n} (A(\lambda^k)^{-1} \cdot).$$

We are now in a position to decompose the operator \mathcal{M}_Γ as

$$\mathcal{M}_\Gamma f = \sup_{k \in \mathbb{Z}} |\mu_k^{(k)} * f| \leq \mathcal{M}_{I,1} + \mathcal{M}_{I,2} + \mathcal{M}_{I,3} + \mathcal{M}_{I,4} + \mathcal{M}_{II},$$

where

$$\begin{aligned} \mathcal{M}_{I,1} &:= \sup_{k \in \mathbb{Z}} |\mu_k^{(k)} * g|, \\ \mathcal{M}_{I,2} &:= \sup_{k \in \mathbb{Z}} \left| \mu_k^{(k)} * \sum_{n \geq 1} g^n \right|, \\ \mathcal{M}_{I,3} &:= \sum_{n \geq 1} \sup_{k \in \mathbb{Z}} |(\mu_k^{(k)} - \mu_k^{(k),n}) * f^n|, \\ \mathcal{M}_{I,4} &:= \sum_{m \geq 0} \sup_{k \in \mathbb{Z}} \left| (\mu_k^{(k),m+1} - \mu_k^{(k),m}) * \sum_{n > m} g^n \right|, \\ \mathcal{M}_{II} &:= \sum_{n \geq 1} \sup_{k \in \mathbb{Z}} |\mu_k^{(k),n} * b^n|. \end{aligned}$$

In order to handle each $\mathcal{M}_{I,i}$ term we shall show that

$$\sum_{i=1}^4 \|\mathcal{M}_{I,i}\|_2^2 \lesssim \alpha \|f\|_1, \tag{2.6}$$

Chebyshev’s inequality immediately implying that the contribution from these terms is suitably under control. Before proving (2.6), we outline how we control the more difficult term \mathcal{M}_{II} using L^1 arguments. Recalling the definition of our balls B_j from (2.1), for each i let l_i be the integer satisfying

$$\widetilde{B}_{l_i-1} \subseteq B_{j_i} \subset \widetilde{B}_{l_i}.$$

For each i and $n \geq 1$, set

$$S_{n,i} := \{k \in \mathbb{Z} \mid l_i - 2 \leq k \leq l_i + \varepsilon^{-1}n\},$$

where ε appears in Proposition 2.1. Then $\mathcal{M}_{II} \leq \mathcal{M}_{II,1} + \mathcal{M}_{II,2}$, where

$$\mathcal{M}_{II,1} := \sum_{n \geq 1} \sum_i \sum_{k \notin S_{n,i}} |\mu_k^{(k),n} * b_{Q_i}^n| \quad \text{and} \quad \mathcal{M}_{II,2} := \sum_{n \geq 1} \sum_i \sum_{k \in S_{n,i}} |\mu_k^{(k),n} * b_{Q_i}^n|.$$

We claim that

$$\|\mathcal{M}_{II,1}\|_{L^1(\mathbb{R}^d \setminus \Omega^*)} \lesssim \|f\|_1 \tag{2.7}$$

which, granted, implies that the contribution from $\mathcal{M}_{II,1}$ is under control by Chebyshev’s inequality. For $\mathcal{M}_{II,2}$, notice that $\|\mu_k^{(k),n}\|_1 \lesssim 1$ because of $\|\mu^{(k)}\| \lesssim 1$ and L^1 dilation invariance. Thus, by Chebyshev’s inequality and the fact that $|S_{n,i}| \lesssim n$ for each i ,

$$\begin{aligned} |\{x \in \mathbb{R}^d \mid \mathcal{M}_{II,2}(x) > \alpha/6\}| &\lesssim \frac{1}{\alpha} \sum_{n \geq 1} \sum_i n \|b_{Q_i}^n\|_1 \\ &\lesssim \int \frac{|f(x)|}{\alpha} \log \left(\frac{|f(x)|}{\alpha} + 100 \right) dx. \end{aligned}$$

The rest of the proof of Theorem 1.3 is then dedicated to (2.6) and (2.7).

From [6] we know that \mathcal{M}_Γ is a bounded operator on L^2 and therefore

$$\|\mathcal{M}_{I,1}\|_2^2 \lesssim \|g\|_2^2 \lesssim \alpha \|f\|_1.$$

Moreover, by (2.4),

$$\left\| \sum_{n \geq 1} g^n \right\|_2^2 \lesssim \alpha \sum_{n \geq 1} \sum_i \int |g_{Q_i}^n(x)| dx = \alpha \sum_{n \geq 1} \sum_i \int |f_{Q_i}^n(x)| dx \leq \alpha \|f\|_1.$$

Thus, $\|\mathcal{M}_{I,2}\|_2^2 \lesssim \alpha \|f\|_1$. To handle $\mathcal{M}_{I,3}$ and $\mathcal{M}_{I,4}$ we use the following estimates concerning our localized measures.

LEMMA 2.4. *For each $m \geq 0$,*

$$\left\| \sup_{k \in \mathbb{Z}} |(\mu_k^{(k),m+1} - \mu_k^{(k),m}) * f| \right\|_2 \lesssim \lambda^{-m\tau} \|f\|_2.$$

PROOF. By dominating the supremum over $k \in \mathbb{Z}$ with an $l^2(\mathbb{Z})$ -sum and Plancherel’s theorem, it suffices to show that

$$\sum_{k \in \mathbb{Z}} |\widehat{\mu^{(k),m+1}}(A(\lambda^k)*\xi) - \widehat{\mu^{(k),m}}(A(\lambda^k)*\xi)|^2 \lesssim \lambda^{-2m\tau} \tag{2.8}$$

for each $\xi \neq 0$. We claim that

$$|\widehat{\mu^{(k),m+1}}(\xi) - \widehat{\mu^{(k),m}}(\xi)| \lesssim \lambda^{-m\tau} \min(\lambda^{-m}|\xi|, (\lambda^{-m}|\xi|)^{-1}). \tag{2.9}$$

That (2.9) implies (2.8) easily follows because Proposition 2.1 allows us to estimate the left-hand side of (2.8) by a convergent geometric series. Estimate (2.9) follows from the uniform decay estimate on the Fourier transform of $\mu^{(k)}$ using Lemma 2.2 and properties of ϕ .

Lemma 2.4 implies that

$$\begin{aligned} \|\mathcal{M}_{I,3}\|_2 &\leq \sum_{n \geq 1} \sum_{m \geq n} \left\| \sup_{k \in \mathbb{Z}} |(\mu_k^{(k),m+1} - \mu_k^{(k),m}) * f^n| \right\|_2 \lesssim \sum_{n \geq 1} \lambda^{-n\tau} \|f^n\|_2 \\ &\lesssim \alpha^{1/2} \|f\|_1^{1/2} \sum_{n \geq 1} \lambda^{-n\tau/2} \lesssim \alpha^{1/2} \|f\|_1^{1/2} \end{aligned}$$

and, moreover,

$$\|\mathcal{M}_{I,4}\|_2 \lesssim \sum_{m \geq 0} \lambda^{-m\tau} \left\| \sum_{n > m} g^n \right\|_2 \lesssim \alpha \|f\|_1.$$

This concludes the proof of (2.6).

To see (2.7), we first claim that for fixed i , those k such that $k \leq l_i - 2$ do not contribute to $\|\mathcal{M}_{II,1}\|_{L^1(\mathbb{R}^d \setminus \Omega^*)}$. To see this, write

$$A(\lambda^k) \text{supp } \mu^{(k)} = \{A(\lambda^{k+1})\Gamma_{k+1}(\lambda^{-1}t) \mid t \in [1, \lambda]\}.$$

By Proposition 2.1 and the fact that $\Gamma_{k+1}(s) = A(\lambda^{k+1})^{-1}A(\lambda^{k+1}s)e_1$, where e_1 is the first canonical basis vector of \mathbb{R}^d , it follows that

$$A(\lambda^k) \text{supp } \mu^{(k)} \subseteq C\widetilde{B_{k+1}} \subseteq C\widetilde{B_{l_i-1}} \subseteq CB_{j_i}.$$

Here, C is the constant appearing in Proposition 2.1. Since each ϕ_n is supported in $B(0, 1/2)$, it follows that $A(\lambda^k) \text{supp } \phi_n \subseteq B_{j_i}$ and hence, $\text{supp}(\mu_k^{(k),n} * b_{Q_i}^n) \subseteq \Omega^*$. So it suffices to prove that

$$\sum_i \sum_{n \geq 1} \sum_{k \geq l_i + \varepsilon^{-1}n} \|\mu_k^{(k),n} * b_{Q_i}^n\|_1 \lesssim \|f\|_1.$$

For fixed $x \in \mathbb{R}^d$ we use the cancellation in $b_{Q_i}^n$ and Taylor’s theorem to get

$$\begin{aligned} & \det A(\lambda^k)(\mu_k^{(k),n} * b_{Q_i}^n(x)) \\ &= \int_{Q_i} b_{Q_i}^n(y) [\mu^{(k),n}(A(\lambda^k)^{-1}(x - y)) - \mu^{(k),n}(A(\lambda^k)(x - x_i))] dy \\ &= \int_0^1 \int_{Q_i} b_{Q_i}^n(y) \langle A(\lambda^k)^{-1}(x_i - y), \nabla \mu^{(k),n}(A(\lambda^k)^{-1}(x - x_i + t(x_i - y))) \rangle dy dt. \end{aligned}$$

For $y \in Q_i$,

$$A(\lambda^k)^{-1}(x_i - y) \in A(\lambda^k)^{-1}A(\lambda^{l_i})B(0, 3).$$

Since $k \geq l_i$, it follows by Proposition 2.1 that $|A(\lambda^k)^{-1}(x_i - y)| \lesssim \lambda^{(l_i-k)\varepsilon}$. Also, $\|\nabla \mu^{(k),n}\|_1 \lesssim \lambda^n$ because $\|\mu^{(k)}\| \lesssim 1$. Therefore,

$$\int |\mu_k^{(k),n} * b_{Q_i}^n(x)| dx \lesssim \lambda^{(l_i-k)\varepsilon+n} \|b_{Q_i}^n\|_1,$$

and by (2.5) this implies that

$$\sum_i \sum_{n \geq 1} \sum_{k \geq l_i + \varepsilon^{-1}n} \|\mu_k^{(k),n} * b_{Q_i}^n\|_1 \lesssim \sum_i \sum_{n \geq 1} \|b_{Q_i}^n\|_1 \lesssim \alpha |\Omega| \lesssim \|f\|_1$$

as required. This concludes the proof of Theorem 1.3 for M_Γ . As in [13], a very similar argument works for H_Γ . Indeed, no further ideas beyond those in [13] and those contained in our proof of Theorem 1.3 for the maximal operator are needed. Thus we refer the reader to [13] (or to [2]) for a fleshed-out proof for H_Γ .

3. Proof of Theorem 1.4

Without loss of generality $P(0) = 0$, so $P(s) = \sum_{j=1}^n p_j s^j$ for some real numbers p_j . As in [1], we use a decomposition of the form

$$(0, \infty) = \bigcup_{j \in \mathfrak{J}} \gamma|_{(0, \infty)}^{-1}(G_j) \cup \bigcup_{l \in \mathfrak{L}} \gamma|_{(0, \infty)}^{-1}(D_l),$$

which is based on properties of P . Here, \mathfrak{J} are those $j = 1, \dots, n$ such that the interval $G_j := (\kappa|z_j|, \kappa^{-1}|z_{j+1}|)$ is nonempty, where κ is some constant depending only on n and the z_j are the roots of P ordered as

$$0 = |z_1| \leq |z_2| \leq \dots \leq |z_n|$$

with $z_{n+1} := \infty$. We may choose the D_l to be disjoint and such that if $D_l = (\alpha_l, \beta_l)$ then $\alpha_l \sim \beta_l$. Moreover, $\gamma^{-1}(\alpha_l) \sim \gamma^{-1}(\beta_l)$ since γ is convex and passes through the origin (see, for example, [1]). Therefore, if

$$\mathcal{M}_{\Gamma, I} f(x) := \sup_{k \in \mathbb{Z}} 2^{-(d-1)k} \left| \int_{|t| \in [2^k, 2^{k+1}] \cap \gamma|_{(0, \infty)}^{-1}(I)} f(x - \Gamma(t)) dt \right| \quad \text{for } I \subset (0, \infty)$$

then each $\mathcal{M}_{\Gamma, D_l}$ is bounded on L^1 by Minkowski's inequality. Since the cardinalities of \mathfrak{J} and \mathfrak{L} cannot exceed n , we only need consider each $\mathcal{M}_{\Gamma, G_j}$. The point is that P is suitably under control on each G_j in the following sense.

LEMMA 3.1. *There exists a constant $C_n > 1$ such that, for any $\kappa \geq C_n$:*

- (1) $|P(s)| \sim |p_j|s^j$ for all $j \in \mathfrak{J}$ and $s \in G_j$;
- (2) $P'(s)/P(s) > 0$ for all $j \in \mathfrak{J}$ and $s \in G_j$;
- (3) $sP'(s)/P(s) \sim 1$ for all $j \in \mathfrak{J}$ and $s \in G_j$;
- (4) $P''(s)/P(s) > 0$ and $s^2P''(s)/P(s) \sim 1$ for all $j \in \mathfrak{J} \setminus \{1\}$ and $s \in G_j$.

Lemma 3.1 appeared in this form in [2] but the ideas originated in [5]. We fix $j \in \mathfrak{J}$ and proceed as in our proof of Theorem 1.3. The appropriate d by d dilation matrices $\{A(t) \mid t > 0\}$ are given by

$$A(t)_{k,l} := \begin{cases} t & \text{for } k = l \text{ and } k = 1, \dots, d - 1 \\ |p_j|\gamma(t)^j & \text{for } k = l = d \\ 0 & \text{for } k \neq l. \end{cases}$$

The convexity of γ implies that the Rivière condition in Proposition 2.1 holds. We again use the Calderón–Zygmund theory developed in [4] with $\widetilde{B}_k := A(2^k)B(0, 1)$ and an appropriately chosen supersequence of B_k to ensure that condition (B5) holds. In fact, our only job is to verify that an appropriate decay estimate holds. We normalize Γ in the analogous way; that is, for each $k \in \mathbb{Z}$, Γ_k is given by

$$\Gamma_k(t) := A(2^k)^{-1}\Gamma(2^k t) \quad \text{for } t \in \mathbb{R}^{d-1}.$$

Therefore, $\Gamma_k(t) = (t, \gamma_k(|t|))$ where

$$\gamma_k := \frac{P(\gamma(2^k \cdot))}{|p_j|\gamma(2^k)^j}.$$

We write $\mathcal{M}_{\Gamma, G_j} f = \sup_{k \in \mathbb{Z}} |\mu_k^{(k)} * f|$ where

$$\langle \mu^{(k)}, \psi \rangle := \int_{|t| \in I_k} \psi(\Gamma_k(t)) dt, \quad \langle \mu_k^{(k)}, \psi \rangle := \langle \mu^{(k)}, \psi(A(2^k) \cdot) \rangle$$

and $I_k := [1, 2] \cap 2^{-k}\gamma|_{(0, \infty)}^{-1}(G_j)$.

LEMMA 3.2. *For each $k \in \mathbb{Z}$ and each $\xi \in \mathbb{R}^d$,*

$$|\widehat{\mu^{(k)}}(\xi)| \lesssim (1 + |\xi|)^{-1/2}.$$

PROOF. Under hypothesis (1) of Theorem 1.4 we capitalize on the decay of the Fourier transform of surface measure on \mathbb{S}^{d-2} . The details of this argument can be found in [1, Lemma 3.1] with the main impetus of ideas originating in [10]. Assuming that (2) of Theorem 1.4 holds, we claim that

$$|\gamma_k''(t)| \gtrsim |\gamma_k'(t)| \gtrsim |\gamma_k(t)| \gtrsim 1 \quad \forall t \in I_k.$$

Given the claim, the proof of Lemma 3.2 follows from standard arguments using van der Corput’s lemma.

To prove the claim, it is immediate that Lemma 3.1 and (1.4) give $|\gamma_k(t)| \gtrsim 1$ for all $t \in I_k$. Moreover, (1.4) in Theorem 1.4 implies that $t\gamma'(t) \geq \gamma(t)$ for all $t > 0$. Thus, by Lemma 3.1,

$$\frac{|\gamma_k'(t)|}{|\gamma_k(t)|} = 2^k \frac{|P'(\gamma(2^k t))|}{|P(\gamma(2^k t))|} \gamma'(2^k t) \gtrsim 2^k \frac{\gamma'(2^k t)}{\gamma(2^k t)} \gtrsim 1,$$

which implies that $|\gamma_k'(t)| \gtrsim 1$. For the remaining assertion, observe that

$$\frac{P''(\gamma(2^k t))\gamma'(2^k t)}{P'(\gamma(2^k t))} \quad \text{and} \quad \frac{\gamma''(2^k t)}{\gamma'(2^k t)}$$

are both positive on I_k by Lemma 3.1. Therefore,

$$\frac{|\gamma_k''(t)|}{|\gamma_k'(t)|} = 2^k \left| \frac{P''(\gamma(2^k t))\gamma'(2^k t)}{P'(\gamma(2^k t))} + \frac{\gamma''(2^k t)}{\gamma'(2^k t)} \right| \geq 2^k \frac{P''(\gamma(2^k t))\gamma'(2^k t)}{P'(\gamma(2^k t))} \gtrsim 1,$$

where the last bound follows from another application of Lemma 3.1. This completes the proof of Lemma 3.2.

One can easily verify that the corresponding decay estimates also hold for the normalized measures associated with H_Γ . Thus, Theorem 1.4 now follows from the argument used to prove Theorem 1.3.

REMARK. It follows from the proof of Theorem 1.4 that the conclusion holds with a finite constant C which one can take to be independent of the coefficients of P .

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