

THE EXISTENCE OF PERIODIC AND SUBHARMONIC SOLUTIONS TO SUBQUADRATIC DISCRETE HAMILTONIAN SYSTEMS

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Abstract

In this paper, by using critical point theory, we establish some results for the existence of periodic and subharmonic solutions to subquadratic discrete Hamiltonian systems.

1. Introduction

Let \mathbb{N} , \mathbb{Z} , \mathbb{R} be the set of all natural numbers, integers and real numbers respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a + 1, \dots\}$ and $\mathbb{Z}(a, b) = \{a, a + 1, \dots, b\}$ when $a \leq b$.

Consider the discrete Hamiltonian system

$$\begin{cases} \Delta x_1(n) = -H_{x_2}(n, x_1(n+1), x_2(n)), \\ \Delta x_2(n) = H_{x_1}(n, x_1(n+1), x_2(n)), \end{cases} \quad (1.1)$$

where $x_1, x_2 \in \mathbb{R}^d$, $H(n, x_1, x_2) \in C^1(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ for each $n \in \mathbb{Z}$ and $\Delta x_i(n) = x_i(n+1) - x_i(n)$, $i = 1, 2$.

Suppose that $H(n, x_1, x_2)$ is m -periodic in the first variable n where m is a positive integer. We are interested in the existence of m -periodic solutions of (1.1). For any positive integer p , a pm -periodic solution of (1.1), as usual, is also called a subharmonic solution of (1.1).

We may think of system (1.1) as being a discrete analogue of the following Hamiltonian system:

$$\begin{cases} dx/dt = -H_y(t, x(t), y(t)), \\ dy/dt = H_x(t, x(t), y(t)), \end{cases} \quad (1.2)$$

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which has been studied extensively by many scholars. In particular, by using critical point theory, P. H. Rabinowitz, V. Benci, J. Mahwin and K. C. Chang *etc.* have obtained some significant results for the existence of periodic and subharmonic solutions of (1.2). We refer the reader to [4, 6, 12–15] and references therein for further details.

Since the behaviour of a discrete system is sometimes sharply different to the behaviour of the corresponding continuous system (see, for example, [1, 16, 17]), many scholars have investigated discrete Hamiltonian systems for the disconjugacy, boundary value problems, oscillations and asymptotic behaviour of (1.1) [3, 5, 7, 8, 11]. Only very few papers deal with the existence of periodic solutions even to general difference equations, see for example [2, 9, 16, 17]. As is well known, critical point theory is an important tool for dealing with the existence of periodic solutions of differential equations. It is natural for us to think that critical point theory may be applied to prove the existence of periodic solutions of system (1.1).

In this paper, we will consider the existence of periodic and subharmonic solutions of system (1.1) by using critical point theory.

Let $z = (x_1^T, x_2^T)^T$ where x^T denotes the transpose of x , $H(n, z) = H(n, x_1, x_2)$ and $\nabla_z H(n, z)$ represents the gradient of $H(n, z)$ in z . For any $k \in \mathbb{N}$, $|\cdot|$ will denote the Euclidean norm in \mathbb{R}^k defined by

$$|x| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2}, \quad \text{for any } x = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^k.$$

The main results of this paper are as follows.

THEOREM 1.1. *Suppose that $H(n, z)$ satisfies*

(H₁) $H(n, z) \in C(\mathbb{R}^{2d}, \mathbb{R})$ for each $n \in \mathbb{Z}$ and there exists a positive integer m such that for all $(n, z) \in \mathbb{Z} \times \mathbb{R}^{2d}$, $H(n + m, z) = H(n, z)$;

(H₂) there is a constant $M_0 > 0$ such that for all $(n, z) \in \mathbb{Z} \times \mathbb{R}^{2d}$, $|H_z(n, z)| \leq M_0$;

(H₃) $H(n, z) \rightarrow +\infty$ uniformly for $n \in \mathbb{Z}$ as $|z| \rightarrow \infty$.

Then for any given positive integer p , system (1.1) possesses at least one pm -periodic solution.

COROLLARY 1.2. *If $H(n, z)$ satisfies (H₁)–(H₂), and (H₃) is replaced by*

(H₄) $H(n, z) \rightarrow -\infty$ uniformly for $n \in \mathbb{Z}$ as $|z| \rightarrow \infty$,

then the conclusion of Theorem 1.1 holds.

THEOREM 1.3. *Suppose $H(n, z)$ satisfies (H₁) and*

(H₅) there exist constants $R_1 > 0$ and $\alpha \in (1, 2)$ such that for any $(n, z) \in \mathbb{Z} \times \mathbb{R}^{2d}$, $|z| \geq R_1$,

$$0 < z \cdot H_z(n, z) \leq \alpha H(n, z); \tag{1.3}$$

(H₆) *there exist constants $a_1 > 0$, $a_2 \geq 0$ and $\gamma \in (1, \alpha]$ such that*

$$H(n, z) \geq a_1|z|^\gamma - a_2, \quad \forall(n, z) \in \mathbb{Z} \times \mathbb{R}^{2d}. \tag{1.4}$$

Then for any positive integer $p > 0$, system (1.1) possesses at least one pm -periodic solution.

REMARK 1.1. By integrating inequality (1.3), we have that

$$H(n, z) \leq a_3|z|^\alpha + a_4 \tag{1.5}$$

holds for some positive constants a_3 and a_4 , which implies that

$$\lim_{|z| \rightarrow \infty} \frac{H(n, z)}{z^2} = 0.$$

In this case, $F(n, z)$ is called subquadratic at infinity. When $H(n, z)$ grows superquadratic both at zero and at infinity, we refer to [10] for the existence of periodic solutions of system (1.1).

REMARK 1.2. The pm -periodic solutions obtained in Theorem 1.1, Corollary 1.2 and Theorem 1.3 may be constant. On some occasions, we may be more interested in nonconstant periodic solutions. The next theorem establishes such a result.

THEOREM 1.4. *Suppose $H(n, z)$ satisfies (H₁) and*

(H₇) $H(n, 0) = 0$, for all $n \in \mathbb{Z}$;

(H₈) *there exists a constant $\alpha \in (1, 2)$ such that*

$$0 < z \cdot H_z(n, z) \leq \alpha H(n, z), \quad \forall(n, z) \in \mathbb{Z} \times \mathbb{R}^{2d} \text{ and } |z| \neq 0; \tag{1.6}$$

(H₉) *there exist constants $a_5 > 0$ and $\gamma \in (1, \alpha]$ such that*

$$H(n, z) \geq a_5|z|^\gamma, \quad \forall(n, z) \in \mathbb{Z} \times \mathbb{R}^{2d}. \tag{1.7}$$

Then for any positive integer $p > 0$, system (1.1) possesses at least one nonconstant pm -periodic solution.

The remaining part of this paper is organised as follows. In the next section, we will find a functional F such that the critical point of F corresponds to the periodic solution of system (1.1). In Section 3, by using critical point theory, we will prove that F possesses at least one critical point and an illustrative example is also given.

2. Some basic lemmas

Let S be the set of sequences $x = \{x(n)\}_{n \in \mathbb{Z}}$, that is,

$$S = \{ \{x(n)\} \mid x(n) = (x_1^T(n), x_2^T(n))^T \in \mathbb{R}^{2d}, x_j(n) \in \mathbb{R}^d, j = 1, 2, n \in \mathbb{Z} \}.$$

For any $x, y \in S, a, b \in \mathbb{R}, ax + by$ is defined by

$$ax + by \triangleq \{ax(n) + by(n)\}.$$

Then S is a vector space.

For any given positive integer p, m, E_{pm} is defined as a subspace of S by

$$E_{pm} = \{x = \{x(n)\} \in S \mid x(n + pm) = x(n), n \in \mathbb{Z}\}.$$

We note that E_{pm} can be equipped with the inner product $\langle \cdot, \cdot \rangle_{E_{pm}}$ and norm $\| \cdot \|_{E_{pm}}$ as follows:

$$\langle x, y \rangle_{E_{pm}} = \sum_{j=1}^{pm} (x(j), y(j)), \quad \forall x, y \in E_{pm} \tag{2.1}$$

and

$$\|x\|_{E_{pm}} = \left(\sum_{j=1}^{pm} |x(j)|^2 \right)^{1/2}, \quad \forall x \in E_{pm}, \tag{2.2}$$

where (\cdot, \cdot) denotes the usual inner product and $|\cdot|$ denotes the Euclidean norm in \mathbb{R}^{2d} .

We define the functional F on E_{pm} as follows:

$$F(x) = \frac{1}{2} \sum_{n=1}^{pm} (J \Delta Lx(n), x(n)) + \sum_{n=1}^{pm} H(n - 1, x_1(n), x_2(n - 1)), \tag{2.3}$$

for all $x \in E_{pm}$, where $Lx(n) = \begin{pmatrix} x_1(n) \\ x_2(n-1) \end{pmatrix}, J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ is the standard $2d \times 2d$ symplectic matrix and I is the $d \times d$ identity matrix.

By a simple computation, we can get

$$\begin{aligned} F(x) &= \sum_{n=1}^{pm} (\Delta x_1(n), x_2(n)) + \sum_{n=1}^{pm} H(n, x_1(n + 1), x_2(n)) \\ &= - \sum_{n=1}^{pm} (x_1(n), \Delta x_2(n - 1)) + \sum_{n=1}^{pm} H(n - 1, x_1(n), x_2(n - 1)). \end{aligned} \tag{2.4}$$

Therefore $F'(x) = 0$ if and only if for any $n \in Z(1, pm),$

$$\begin{cases} \Delta x_1(n) = -H'_{x_2}(n, x_1(n + 1), x_2(n)), \\ \Delta x_2(n) = H'_{x_1}(n, x_1(n + 1), x_2(n)), \end{cases}$$

where $x \in E_{pm}$.

It is clear now that the problem of finding a pm -periodic solution of (1.1) has been reduced to that of seeking the critical point of the functional F defined on E_{pm} .

Now we will discuss the existence of a critical point of F . To this end, for any $x = \{x(n)\} \in E_{pm}$, where $x(n) = (x_1^T(n), x_2^T(n))^T \in \mathbb{R}^{2d}$, we assume $x_j(n) = (x_j^1(n), x_j^2(n), \dots, x_j^d(n))^T$ for $j = 1, 2$ and let

$$x^i = (x_1^i(1), x_1^i(2), \dots, x_1^i(pm), x_2^i(1), x_2^i(2), \dots, x_2^i(pm))^T \in \mathbb{R}^{2pm}.$$

Define a map ϕ from E_{pm} to \mathbb{R}^{2dpm} as

$$\phi(x) = ((x^1)^T, (x^2)^T, \dots, (x^d)^T)^T. \tag{2.5}$$

Then ϕ is a linear homeomorphism and $\|x\|_{E_{pm}} = |\phi(x)|$.

By (2.4), we have

$$\begin{aligned} F(x) &= \sum_{i=1}^d \sum_{n=1}^{pm} (x_1^i(n+1) - x_1^i(n))x_2^i(n) + \sum_{n=1}^{pm} H(n-1, x_1(n), x_2(n-1)) \\ &= \frac{1}{2}(A\phi(x), \phi(x)) + \sum_{n=1}^{pm} H(n-1, x_1(n), x_2(n-1)), \end{aligned} \tag{2.6}$$

$$A = \begin{pmatrix} B & & 0 \\ & \ddots & \\ 0 & & B \end{pmatrix}_{2dpm \times 2dpm} \quad \text{and} \quad B = \begin{pmatrix} 0 & Q^T \\ Q & 0 \end{pmatrix}_{2pm \times 2pm},$$

where

$$Q = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 & 1 \\ 1 & 0 & 0 & \dots & 0 & -1 \end{pmatrix}_{pm \times pm}.$$

It is easy to see that λ is an eigenvalue of B if and only if λ^2 is an eigenvalue of $Q^T Q$ since

$$Q^T Q = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ -1 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}_{pm \times pm}$$

is positive semi-definite and $Q^T Q$ has the eigenvalue 0 and $pm - 1$ other positive eigenvalues denoted by $0, \lambda_1^2, \lambda_2^2, \dots, \lambda_{pm-1}^2$, where $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{pm-1}$. Then the eigenvalues of A can be denoted by $0, \lambda_{\pm 1}, \lambda_{\pm 2}, \dots, \lambda_{\pm pm-1}$ with multiplicity $2d$ of 0, where $\lambda_{-j} = -\lambda_j$. Let A^0, A^+, A^- denote the eigenspace associated with 0, all positive eigenvalues and all negative eigenvalues respectively, and

$$E_{pm}^0 = \phi^{-1}(A^0), \quad E_{pm}^+ = \phi^{-1}(A^+) \quad \text{and} \quad E_{pm}^- = \phi^{-1}(A^-).$$

Then $E_{pm}^0 = \{x \in E_{pm} \mid x = \{v\}, v \in \mathbb{R}^{2d}\}$ and $E_{pm} = E_{pm}^0 \oplus E_{pm}^+ \oplus E_{pm}^-$.

For any $x = (x_1, x_2, \dots, x_{2pm})^T \in \mathbb{R}^{2dpm}$, and $r > 1$, define

$$|x|_r = \left(\sum_{i=1}^{2dpm} x_i^r \right)^{1/r}.$$

Then $|\cdot|_r$ is a norm in \mathbb{R}^{2dpm} . We can equip E_{pm} with the norm $\|x\|_r$ deduced by

$$\|x\|_r = |\phi(x)|_r, \quad \forall x \in E_{pm}. \tag{2.7}$$

Clearly, $\|x\|_2 = \|x\|_{E_{pm}}$. Since E_{pm} is equivalent to the finite-dimensional Hilbert space \mathbb{R}^{2dpm} , $\|\cdot\|_2$ and $\|\cdot\|_r$ are equivalent, that is, there exist constants C_{1r}, C_{2r} such that $C_{2r} \geq C_{1r} > 0$, and

$$C_{1r} \|x\|_r \leq \|x\|_2 \leq C_{2r} \|x\|_r, \quad \forall x \in E_{pm}. \tag{2.8}$$

On the other hand, we have

$$\lambda_1 \|u\|_2^2 \leq \langle A\phi(u), \phi(u) \rangle \leq \lambda_{pm-1} \|u\|_2^2, \quad \text{for } u \in E_{pm}^+, \tag{2.9}$$

$$-\lambda_{pm-1} \|v\|_2^2 \leq \langle A\phi(v), \phi(v) \rangle \leq -\lambda_1 \|v\|_2^2, \quad \text{for } v \in E_{pm}^-, \tag{2.10}$$

$$\langle A\phi(w), \phi(w) \rangle = 0, \quad \text{for } w \in E_{pm}^0. \tag{2.11}$$

Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$, that is, I is a continuously Fréchet differentiable functional defined on X . The functional I is said to satisfy the Palais-Smale condition (P-S condition) if any sequence $\{u_n\} \subset X$ for which $\{I(u_n)\}$ is bounded and $I'(u_n) \rightarrow 0 (n \rightarrow \infty)$ possesses a convergent subsequence in X .

Let B_r denote the open ball in X about 0 of radius r and let ∂B_r denote its boundary.

LEMMA 2.1 (Saddlepoint theorem (see [15])). *Let X be a real Banach space, $X = X_1 \oplus X_2$, where $X_1 \neq \{0\}$ and is finite dimensional. Suppose $I \in C^1(X, \mathbb{R})$ satisfies the P-S condition and*

(I₁) *there exist constants $\sigma, \rho > 0$ such that $I|_{\partial B_\rho \cap X_1} \leq \sigma$, and*

(I₂) *there exist $e \in B_\rho \cap X_1$ and a constant $\omega > \sigma$ such that $I|_{e+X_2} \geq \omega$.*

Then I possesses a critical value $c \geq \omega$ and $c = \inf_{h \in \Gamma} \max_{u \in B_\rho \cap X_1} I(h(u))$, where $\Gamma = \{h \in C(\bar{B}_\rho \cap X_1, X) \mid h|_{\partial B_\rho \cap X_1} = \text{id}\}$.

3. Proofs of the main results

PROOF OF THEOREM 1.1. Let $F(x)$ be defined as in (2.3). Clearly, $F \in C^1(E_{pm}, \mathbb{R})$. In view of (H_2) , there exists a constant $M_1 > 0$ such that

$$|H(n, z)| \leq M_1 + M_0|z|, \quad \forall(n, z) \in \mathbb{Z} \times \mathbb{R}^m. \tag{3.1}$$

We will first show that F satisfies the P-S condition. In fact, suppose that $\{x^{(k)}\}$ is a sequence in E_{pm} such that for any $k \in \mathbb{Z}(1)$, $|F(x^{(k)})| \leq M_2$ for some positive constant M_2 and $F'(x^{(k)}) \rightarrow 0$ as $k \rightarrow \infty$, then for sufficiently large k , $|\langle F'(x^{(k)}), x \rangle| \leq \|x\|_2$.

Let $x^{(k)} = y^{(k)} + w^{(k)}$, where $y^{(k)} = u^{(k)} + v^{(k)} \in E_{pm}^+ \oplus E_{pm}^-$ and $w^{(k)} \in E_{pm}^0$. According to (2.6) and the periodicity of H , we have

$$\langle F'(x^{(k)}), x \rangle = \langle A\phi(x^{(k)}), \phi(x) \rangle + \sum_{n=1}^{pm} H_z(n-1, Lx^{(k)}(n)) \cdot Lx(n), \quad \forall x \in E_{pm}.$$

Then for sufficiently large k ,

$$\begin{aligned} |\langle A\phi(x^{(k)}), \phi(u^{(k)}) \rangle| &\leq \sum_{n=1}^{pm} |H_z(n-1, Lx^{(k)}(n)) \cdot Lu^{(k)}(n)| + \|u^{(k)}\|_2 \\ &\leq M_0 \sum_{n=1}^{pm} |Lu^{(k)}(n)| + \|u^{(k)}\|_2 \leq (M_0\sqrt{pm} + 1)\|u^{(k)}\|_2 \end{aligned}$$

and according to (2.9), we get

$$|\langle A\phi(x^{(k)}), \phi(u^{(k)}) \rangle| = |\langle A\phi(u^{(k)}), \phi(u^{(k)}) \rangle| \geq \lambda_1 \|u^{(k)}\|_2^2.$$

Thus we have $\lambda_1 \|u^{(k)}\|_2^2 \leq (M_0\sqrt{pm} + 1)\|u^{(k)}\|_2$, which implies that $\{u^{(k)}\}$ is bounded.

Similarly, we can show that $\{v^{(k)}\}$ is bounded. So $\{y^{(k)}\}$ is bounded.

Next, we need to prove that $\{w^{(k)}\}$ is bounded. In fact,

$$\begin{aligned} F(x^{(k)}) &= \frac{1}{2} \langle A\phi(x^{(k)}), \phi(x^{(k)}) \rangle + \sum_{n=1}^{pm} H(n-1, Lx^{(k)}(n)) \\ &= \frac{1}{2} \langle A\phi(y^{(k)}), \phi(y^{(k)}) \rangle + \sum_{n=1}^{pm} H(n-1, Lw^{(k)}(n)) \\ &\quad + \sum_{n=1}^{pm} [H(n-1, Lx^{(k)}(n)) - H(n-1, Lw^{(k)}(n))]. \end{aligned}$$

So,

$$\begin{aligned}
 & \left| \sum_{n=1}^{pm} H(n-1, Lw^{(k)}(n)) \right| \\
 & \leq |F(x^{(k)})| + \frac{1}{2} |\langle A\phi(y^{(k)}), \phi(y^{(k)}) \rangle| \\
 & \quad + \sum_{n=1}^{pm} |H(n-1, Lx^{(k)}(n)) - H(n-1, Lw^{(k)}(n))| \\
 & \leq M_2 + \frac{1}{2} \lambda_{pm-1} |\phi(y^{(k)})|^2 + \sum_{n=1}^{pm} |H_z(n-1, L(w^{(k)}(n) + \theta y^{(k)}(n)))| |Ly^{(k)}(n)| \\
 & \leq M_2 + \frac{1}{2} \lambda_{pm-1} \|y^{(k)}\|_2^2 + M_0 \sqrt{pm} \|y^{(k)}\|_2,
 \end{aligned}$$

where $\theta \in (0, 1)$. This implies that $\{ \sum_{n=1}^{pm} H(n-1, Lw^{(k)}(n)) \}$ is bounded.

By Assumption (H_3) , we have that $\{w^{(k)}\}$ is bounded. If otherwise, there is no harm in assuming that $\|w^{(k)}\|_2 \rightarrow \infty$ as $k \rightarrow \infty$. Since there exist $z^{(k)} \in \mathbb{R}^{2d}$, $k \in \mathbb{Z}(1)$, such that $w^{(k)} = \{z^{(k)}\} \in E_{pm}$, then

$$\|w^{(k)}\| = \left(\sum_{n=1}^{pm} |w^{(k)}(n)|^2 \right)^{1/2} = \left(\sum_{n=1}^{pm} |z^{(k)}|^2 \right)^{1/2} = \sqrt{pm} |z^{(k)}| \xrightarrow{k \rightarrow \infty} \infty.$$

Since $H(n-1, Lw^{(k)}(n)) = H(n-1, z^{(k)})$, we have $H(n-1, Lw^{(k)}(n)) \rightarrow +\infty$ as $k \rightarrow \infty$. This contradicts the fact that $\{ \sum_{n=1}^{pm} H(n-1, Lw^{(k)}(n)) \}$ is bounded.

Now we will check that the conditions (I_1) and (I_2) in the Saddlepoint theorem hold. To this end, let $X_1 = E_{pm}^-$ and $X_2 = E_{pm}^+ \oplus E_{pm}^0$. For any $x = u + w \in E_{pm}^+ \oplus E_{pm}^0 = X_2$, there exist $z^* \in \mathbb{R}^{2d}$ such that $w(n) = z^*$, $\forall n \in \mathbb{Z}$. Thus

$$\begin{aligned}
 F(x) &= \frac{1}{2} \langle A\phi(x), \phi(x) \rangle + \sum_{n=1}^{pm} H(n-1, Lx(n)) \\
 &= \frac{1}{2} \langle A\phi(u), \phi(u) \rangle + \sum_{n=1}^{pm} (H(n-1, Lx(n)) - H(n-1, Lw(n))) \\
 & \quad + \sum_{n=1}^{pm} H(n-1, Lw(n)) \\
 &\geq \frac{1}{2} \lambda_1 \|u\|_2^2 + \sum_{n=1}^{pm} H_z(n-1, Lw(n) + \theta Lu(n)) \cdot Lu(n) + \sum_{n=1}^{pm} H(n-1, z^*) \\
 &\geq \frac{1}{2} \lambda_1 \|u\|_2^2 - M_0 \sum_{n=1}^{pm} |Lu(n)| + \sum_{n=1}^{pm} H(n-1, z^*)
 \end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{2}\lambda_1\|u\|_2^2 - M_0\sqrt{pm} \left(\sum_{n=1}^{pm} |Lu(n)|^2 \right)^{1/2} + \sum_{n=1}^{pm} H(n-1, z^*) \\ &= \frac{1}{2}\lambda_1\|u\|_2^2 - M_0\sqrt{pm}\|u\|_2 + \sum_{n=1}^{pm} H(n-1, z^*), \end{aligned}$$

where $\theta \in (0, 1)$. According to (H_3) , it is easy to see that $H(n, z)$ is bounded from below for $n \in \mathbb{Z}$ and $z \in \mathbb{R}^{2d}$. That is, $H(n, z) \geq M_3$, for all $(n, z) \in \mathbb{Z} \times \mathbb{R}^{2d}$ for some constant M_3 . Thus

$$\begin{aligned} F(x) &\geq \frac{1}{2}\lambda_1\|u\|_2^2 - M_0\sqrt{pm}\|u\|_2 + pmM_3 \\ &= \frac{1}{2}\lambda_1 \left(\|u\|_2 - \frac{M_0}{\lambda_1}\sqrt{pm} \right)^2 - \frac{M_0^2}{2\lambda_1}pm + pmM_3 \geq pm \left(M_3 - \frac{M_0^2}{2\lambda_1} \right). \end{aligned}$$

Take $\omega = pm(M_3 - M_0^2/2\lambda_1)$, then we have

$$F(x) \geq \omega, \quad \forall x \in X_2. \tag{3.2}$$

Let $e = 0$, then (I_2) holds.

For any $v \in X_1 = E_{pm}^-$, according to (3.1), we have

$$\begin{aligned} F(v) &= \frac{1}{2}(A\phi(v), \phi(v)) + \sum_{n=1}^{pm} H(n-1, Lv(n)) \\ &\leq -\frac{1}{2}\lambda_1\|v\|_2^2 + \sum_{n=1}^{pm} (M_0|Lv(n)| + M_1) \\ &\leq -\frac{1}{2}\lambda_1\|v\|_2^2 + M_0\sqrt{pm}\|v\|_2 + pmM_1. \end{aligned}$$

This implies that $F(v) \rightarrow -\infty$ as $\|v\|_2 \rightarrow \infty$. Let $\sigma = \omega - 1$, then there exists a sufficiently large $\rho > 0$ such that

$$F(v) \leq \sigma, \quad \forall v \in X_1, \text{ and } \|v\|_2 = \rho. \tag{3.3}$$

Thus (I_1) is satisfied. By the Saddlepoint theorem, the proof of Theorem 1.1 is complete. \square

PROOF OF COROLLARY 1.2. Let $G(x) = -F(x)$, $X_1 = E_{pm}^+$ and $X_2 = E_{pm}^- \oplus E_{pm}^0$. By a similar argument to the proof of Theorem 1.1, we can prove Corollary 1.2. \square

PROOF OF THEOREM 1.3. First, we need to show that F satisfies the P-S condition.

Clearly, $F \in C^1(E_{pm}, \mathbb{R})$. Let $x^{(k)} \in E_{pm}$, $k \in \mathbb{Z}(1)$ be such that $\{F(x^{(k)})\}$ is bounded and $F'(x^{(k)}) \rightarrow 0$ as $k \rightarrow \infty$. Then there exists a constant $M_4 > 0$ and $k_0 \in \mathbb{Z}(1)$ such that $|F(x^{(k)})| \leq M_4$ for $k \in \mathbb{Z}(1)$ and $|\langle F'(x^{(k)}), x \rangle| \leq \|x\|_2$ for $k \in \mathbb{Z}(k_0)$, $x \in E_{pm}$.

Since

$$\langle F'(x^{(k)}), x^{(k)} \rangle = \langle A\phi(x^{(k)}), \phi(x^{(k)}) \rangle + \sum_{n=1}^{pm} H_z(n-1, Lx^{(k)}(n)) \cdot Lx^{(k)}(n),$$

we see that, for $k \in \mathbb{Z}(k_0)$,

$$\begin{aligned} M_4 + \frac{1}{2}\|x^{(k)}\|_2 &\geq F(x^{(k)}) - \frac{1}{2}\langle F'(x^{(k)}), x^{(k)} \rangle \\ &= \sum_{n=1}^{pm} \left[H(n-1, Lx^{(k)}(n)) - \frac{1}{2}Lx^{(k)}(n) \cdot H_z(n-1, Lx^{(k)}(n)) \right]. \end{aligned}$$

For any $k \in \mathbb{Z}(k_0)$, denote

$$S_1^k = \{n \in \mathbb{Z}(1, pm) \mid |x^{(k)}(n)| \geq R_1\} \quad \text{and} \quad S_2^k = \{n \in \mathbb{Z}(1, pm) \mid |x^{(k)}(n)| < R_1\}.$$

Then $S_1^k \cup S_2^k = \mathbb{Z}(1, pm)$ and

$$\begin{aligned} M_4 + \frac{1}{2}\|x^{(k)}\|_2 &\geq \sum_{n=1}^{pm} H(n-1, Lx^{(k)}(n)) - \frac{1}{2} \sum_{n \in S_1^k} Lx^{(k)}(n) \cdot H_z(n-1, Lx^{(k)}(n)) \\ &\quad - \frac{1}{2} \sum_{n \in S_2^k} Lx^{(k)}(n) \cdot H_z(n-1, Lx^{(k)}(n)). \end{aligned}$$

In view of (1.3), we have

$$\begin{aligned} M_4 + \frac{1}{2}\|x^{(k)}\|_2 &\geq \sum_{n=1}^{pm} H(n-1, Lx^{(k)}(n)) - \frac{\alpha}{2} \sum_{n \in S_1^k} H(n-1, Lx^{(k)}(n)) \\ &\quad - \frac{1}{2} \sum_{n \in S_2^k} Lx^{(k)}(n) \cdot H_z(n-1, Lx^{(k)}(n)) \\ &= \left(1 - \frac{\alpha}{2}\right) \sum_{n=1}^{pm} H(n-1, Lx^{(k)}(n)) \\ &\quad + \frac{1}{2} \sum_{n \in S_2^k} [\alpha H(n-1, Lx^{(k)}(n)) - Lx^{(k)}(n) \cdot H_z(n-1, Lx^{(k)}(n))]. \end{aligned}$$

Since $\alpha H(n - 1, z) - z \cdot H_z(n - 1, z)$ is continuous with respect to $z \in \mathbb{R}^{2d}$ for each $n \in \mathbb{Z}$, there exists a constant $M_5 > 0$ such that $|\alpha H(n - 1, z) - z \cdot H_z(n - 1, z)| \leq M_5$, for all $z \in \mathbb{R}^{2d}$ and $|z| \leq R_1, n \in \mathbb{Z}(1, pm)$. Thus

$$M_4 + \frac{1}{2} \|x^{(k)}\|_2 \geq \left(1 - \frac{\alpha}{2}\right) \sum_{n=1}^{pm} H(n - 1, Lx^{(k)}(n)) - \frac{1}{2} pm M_5.$$

By (1.4) and (2.8), we get

$$\begin{aligned} M_4 + \frac{1}{2} \|x^{(k)}\|_2 &\geq \left(1 - \frac{\alpha}{2}\right) a_1 \sum_{n=1}^{pm} |Lx^{(k)}(n)|^\gamma - \left(1 - \frac{\alpha}{2}\right) a_2 pm - \frac{1}{2} pm M_5 \\ &\geq \left(1 - \frac{\alpha}{2}\right) a_1 \left(\frac{1}{C_{2\gamma}}\right)^\gamma \|x^{(k)}\|_2^\gamma - M_6, \end{aligned}$$

where $M_6 = (1 - \alpha/2)a_2 pm + (pm M_5)/2$.

That is,

$$\left(1 - \frac{\alpha}{2}\right) a_1 \left(\frac{1}{C_{2\gamma}}\right)^\gamma \|x^{(k)}\|_2^\gamma - \frac{1}{2} \|x^{(k)}\|_2 \leq M_4 + M_6.$$

Because $\gamma \in (1, 2)$, we see that $\{\|x^{(k)}\|_2\}$ is bounded. Since E_{pm} is finite dimensional, $\{x^{(k)}\}$ has a subsequence which is convergent in E_{pm} . Therefore F satisfies the P-S condition.

Now we prove that F satisfies (I_1) and (I_2) . To this end, let $X_1 = E_{pm}^-$ and $X_2 = E_{pm}^+ \oplus E_{pm}^0$. Then for any $x = u + w \in E_{pm}^+ \oplus E_{pm}^0 = X_2$,

$$\begin{aligned} F(x) &= \frac{1}{2} (A\phi(u), \phi(u)) + \sum_{n=1}^{pm} H(n - 1, Lx(n)) \\ &\geq \frac{1}{2} \lambda_1 \|u\|_2^2 + \sum_{n=1}^{pm} (a_1 |Lx(n)|^\gamma - a_2) \\ &\geq -a_2 pm. \end{aligned}$$

Let $\omega = -a_2 pm, e = 0$ and $\sigma = \omega - 1$. Then F satisfies (I_2) . By (2.10), for $v \in X_1 = E_{pm}^-$, according to (1.5),

$$\begin{aligned} F(x) &= \frac{1}{2} (A\phi(v), \phi(v)) + \sum_{n=1}^{pm} H(n - 1, Lv(n)) \\ &\leq -\frac{1}{2} \lambda_1 \|v\|_2^2 + a_3 \sum_{n=1}^{pm} |Lv(n)|^\alpha + a_4 pm \\ &\leq -\frac{1}{2} \lambda_1 \|v\|_2^2 + a_3 \left(\frac{1}{C_{1\alpha}}\right)^\alpha \|v\|_2^\alpha + a_4 pm. \end{aligned}$$

Since $1 < \alpha < 2$, there exists a sufficiently large constant $\rho > 0$ such that $F(v) \leq \sigma$ for all $v \in X_1$ and $\|v\| = \rho$. Thus (I_1) holds. According to the Saddlepoint theorem, there exists at least one critical point of F . And the proof is complete. \square

PROOF OF THEOREM 1.4. According to the proof of Theorem 1.3, we know that the functional $F \in C^1(E_{pm}, \mathbb{R})$ has a critical value c and (1.1) has a corresponding solution. By (H_7) , we see that the zero solution is a constant solution of system (1.1). If $x(n) \equiv z \in \mathbb{R}^{2d}$ is another constant solution of system (1.1), then $H_{x_1}(n, z) = 0$ and $H_{x_2}(n, z) = 0$. This implies that $H_z(n, z) = 0$ for $n \in \mathbb{Z}$. By (1.6), we have $z = 0$. Thus the zero solution is the only constant solution of system (1.1). Since $F(0) = 0$, to complete the proof it suffices to show that F has a critical value $c > 0$. To this end, assume that $\bar{y}_0 \in \mathbb{R}^{2dp_m}$ is a nonzero eigenvector of A associated with the eigenvalue $-\lambda_1$ and with $|\bar{y}_0| = \delta$ small enough, it holds that

$$\omega = \delta^\gamma \left(a_5 \left(\frac{1}{C_{2\gamma}} \right)^\gamma - \frac{1}{2} \lambda_1 \delta^{2-\gamma} \right) > 0.$$

Let $y_0 = \phi^{-1}(\bar{y}_0) \in E_{pm}^-, X_1 = E_{pm}^-$ and $X_2 = y_0 + E_{pm}^+ \oplus E_{pm}^0$.

For any $x = y_0 + u + w \in X_2$, where $u \in E_{pm}^+, w \in E_{pm}^0$. According to (2.6), we have

$$\begin{aligned} F(x) &= \frac{1}{2} \langle A\phi(x), \phi(x) \rangle + \sum_{n=1}^{pm} H(n-1, Lx(n)) \\ &\geq \frac{1}{2} \langle A\phi(y_0), \phi(y_0) \rangle + \frac{1}{2} \langle A\phi(u), \phi(u) \rangle + a_5 \sum_{n=1}^{pm} |Lx(n)|^\gamma \\ &\geq -\frac{1}{2} \lambda_1 \|y_0\|_2^2 + \frac{1}{2} \lambda_1 \|u\|_2^2 + a_5 \left(\frac{1}{C_{2\gamma}} \right)^\gamma (\|y_0\|_2^2 + \|u\|_2^2 + \|w\|_2^2)^{\gamma/2} \\ &\geq -\frac{1}{2} \lambda_1 \|y_0\|_2^2 + a_5 \left(\frac{1}{C_{2\gamma}} \right)^\gamma \|y_0\|_2^\gamma \\ &= \delta^\gamma \left(a_5 \left(\frac{1}{C_{2\gamma}} \right)^\gamma - \frac{1}{2} \lambda_1 \delta^{2-\gamma} \right) = \omega. \end{aligned}$$

This shows that F satisfies the condition (I_1) .

For any $v \in X_1$, similar to the proof of Theorem 1.3, we have $F(v) \rightarrow -\infty$ as $\|v\|_2 \rightarrow \infty$. So, there exists a sufficiently large constant ρ such that $F(v) \leq \omega - 1$ for $v \in X_1, \|v\| = \rho$, which implies that F satisfies the condition (I_2) . Therefore F possesses a critical value $c \geq \omega > 0$. The proof of Theorem 1.4 is complete. \square

Finally, we give an example to illustrate our conclusions.

EXAMPLE. Consider the system (1.1) with

$$H(n, x_1, x_2) = (a|x_1|^\gamma + b|x_2|^\gamma)\psi(n), \quad \forall n \in \mathbb{Z}, (x_1, x_2)^T \in \mathbb{R}^2, \quad (3.4)$$

where $a, b > 0$, $\gamma \in (1, 2)$ and $\psi(n)$ is a positive m -periodic function.

We now check that all conditions in Theorem 1.4 hold. Clearly, (H_1) and (H_7) hold. Let $z = (x_1, x_2)^T$ and $\bar{a} = \min\{\psi(n) | n \in \mathbb{Z}(1, m)\} > 0$. Then

$$z \cdot H_z(n, z) = (a\gamma|x_1|^\gamma + b\gamma|x_2|^\gamma)\psi(n) = \gamma H(n, z),$$

which implies that (H_8) holds.

At last, we have

$$\begin{aligned} H(n, z) &\geq \bar{a} \min\{a, b\}(|x_1|^\gamma + |x_2|^\gamma) \\ &\geq \bar{a} \min\{a, b\} \left(\frac{1}{C_{2\gamma}}\right)^\gamma (|x_1|^2 + |x_2|^2)^{\gamma/2} \\ &= \bar{a} \min\{a, b\} \left(\frac{1}{C_{2\gamma}}\right)^\gamma |z|^\gamma. \end{aligned}$$

So, (H_9) holds.

According to Theorem 1.4, we see that, for any given positive integer p , the system (1.1) with H as defined in (3.4) possesses at least one nonconstant pm -periodic solution.

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