# ON THE CONTINUITY OF BROWNIAN MOTION WITH A MULTIDIMENSIONAL PARAMETER

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## § 1. Introduction

A stochastic process $X(A, \omega)$ is called Brownian motion with an $N$-dimensional parameter when it satisfies the following conditions:

1) For any positive integer $n$ and any set of points $A_1, A_2, \ldots, A_n$ in an $N$-dimensional Euclidian space $E_N$, the joint variable $\langle X_i = X(A_i)\,;\ i = 1, 2, \ldots, n\rangle$ is subject to an $n$-dimensional Gaussian distribution having the vector $\mathbf{0}$ as its mean vector.

2) $$E(X_i X_j) = \{\text{dis}(O, A_i) + \text{dis}(O, A_j) - \text{dis}(A_i, A_j)\}/2,$$

where $E(X)$, $O$, and $\text{dis}(A, B)$ denote the expectation of $X$, the origin of $E_N$, and the Euclidian distance between $A$ and $B$ respectively.

3) For almost every sample point $\omega$, $X(A, \omega)$ is continuous in $A$ and $X(O, \omega) = 0$. The random variables $X(A) - X(B)$ evidently form Wiener process if $A$ moves on some demi-straight line with the terminal point $B$. In this paper, we study the continuity of Brownian motion process with an $N$-dimensional parameter.

Let us begin with the definitions of the concepts of upper class and lower class with respect to $\{X(A)\,;\ A \in E_N\}$. Let $\psi(t)$ be a non-negative and non-decreasing function defined for large $t$'s.

i) If the set of $A$ satisfying

$$X(A, \omega) > (\text{dis}(O, A))^{1/2}\, \psi(\text{dis}(O, A))$$

is bounded (unbounded) for almost all $\omega$, we say that $\psi(t)$ belongs to the upper (lower) class with respect to $\{X(A)\,;\ A \in E_N\}$ at $\infty$ and denote it by $\psi(t) \in \mathfrak{U}_N^{\infty}$ $(\psi(t) \in \mathfrak{L}_N^{\infty})$.

ii) If the set of $A$ satisfying

$$X(A, \omega) > (\text{dis}(O, A))^{1/2}\, \psi(1/\text{dis}(O, A))$$

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is separated (not separated) from $O$ for almost all $\omega$, we say that $\psi(t)$ belongs to the upper (lower) class with respect to $\{X(A);\ A \in E_N\}$ at $O$ and denote it by $\psi(t) \in \mathfrak{U}_N^\circ$ ($\psi(t) \in \mathfrak{L}_N^\circ$).

According to the theorem of projective invarince, $\psi(t)$ belongs to $\mathfrak{U}_N^\infty$ ($\mathfrak{L}_N^\infty$) if and only if $\psi(t)$ belongs to $\mathfrak{U}_N^\circ$ ($\mathfrak{L}_N^\circ$). Therefore, we have only to discuss the behavior of $X(A)$ in the neighborhood of $O$.

For Wiener process, namely for Brownian motion with 1-dimensional parameter, we have the following criterion of Kolmogorov [1]: a monotone non-decreasing function $\varphi(t)$ belongs to $\mathfrak{U}_1^\circ$ ($\mathfrak{L}_1^\circ$) if and only if

$$\int^\infty \frac{1}{t}\,\psi(t)\,e^{-\frac{1}{2}\psi^2(t)}\,dt < +\infty\,( = +\infty).$$

This criterion shows that the function

$$\psi(t) = \{2\ \log_{(2)} t + 3\ \log_{(3)} t + \cdots + 2\ \log_{(n-1)} t + (2+\delta)\ \log_{(n)} t\}^{1/2}$$

belongs to $\mathfrak{U}_1^\circ$ for $\delta > 0$ and belongs to $\mathfrak{L}_1^\circ$ for $\delta \leqq 0$, where $\log_{(n)} t$ denotes the $n$-time iterated logarithm. We shall extend this result to Brownian motion with an $N$-dimensional parameter using Chung-Erdös' method in § 3.

Secondly, we define similar concepts with regard to the uniform continuity of $X(A)$. Let $\varphi(t)$ be a non-negative, continuous, and non-decreasing function defined in some finite interval $(O, T)$, and $f(A)$ be a function defined on some region in $E_N$.

If there exists a positive number $\varepsilon$ such that dis $(A, B) \leqq \varepsilon$ implies

$$|f(A) - f(B)| \leqq \varphi(\text{dis}\,(A, B)),$$

we say that $f(A)$ satisfies Lipschitz's condition relative to $\varphi(t)$. We put now $\psi(t) = \varphi(1/t)\ t^{1/2}$ and consider the cube $U_N = \{A = (a_1, a_2, \ldots, a_N);\ \max_{1 \leqq i \leqq N} |a_i| \leqq 1\}$. If the process $X(A, \omega)$ with the parameter domain $U_N$ satisfies (does not satisfy) Lipschitz's condition relative to $\psi(t)$ for almost all $\omega$, we say that $\varphi(t)$ belongs to the upper (lower) class with regard to the uniform continuity of $\{X(A);\ A \in U_N\}$, and denote it by $\varphi(t) \in \mathfrak{U}_N^u$ ($\mathfrak{L}_N^u$).

P. Lévy remarked in his book [2] that the concepts of upper class and lower class with regard to the uniform continuity of $X(A)$ are meaningful only for the process with a bounded parameter domain. Accordingly, it is sufficient to define the concepts for $\{X(A);\ A \in U_N\}$.

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For Wiener process, P. Lévy [3] proved that the function

$$\xi(t) = \{2\, c \log t\}^{1/2}$$

belongs to $\mathfrak{U}_1^u$ for $c>1$ and belongs to $\mathfrak{L}_1^u$ for $c<1$. Recently K. L. Chung, P. Erdös, and T. Sirao [4] proved a final form of the criterion which reads: $\varphi(t)$ belongs to $\mathfrak{U}_1^u$ ($\mathfrak{L}_1^u$) if and only if the integral

$$\int^{\infty} \varphi^3(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt$$

is convergent (divergent). In virtue of this criterion, we can easily see that the function

$$\varphi(t) = \{2 \log t + 5 \log_{(2)} t + 2 \log_{(3)} t + \cdots + 2 \log_{(n-1)} t + (2+\delta) \log_{(n)} t\}^{1/2}$$

belongs to $\mathfrak{U}_1^u$ for $\delta>0$ and belongs to $\mathfrak{L}_1^u$ for $\delta \leqq 0$.

Also, for Brownian motion with an $N$-dimensional parameter, P. Lévy [5] proved that the function

$$\eta(t) = \{2\, Nc \log t\}^{1/2}$$

belongs to $\mathfrak{U}_N^u$ for $c>1$ and belongs to $\mathfrak{L}_N^u$ for $c<1$. This result was improved by T. Hida [6] as follows:

$$\zeta(t) = \{2\, N \log t + c \log_{(2)} t\}^{1/2}$$

belongs to $\mathfrak{U}_N^u$ for $c>8N+1$ and belongs to $\mathfrak{L}_N^u$ for $c<1$. In § 2, the author proves a final form of the criterion, a generalization of Chung-Erdös-Sirao's result, for Brownian motion with an $N$-dimensional parameter. We shall here use the same method as in the 1-dimensional case [4] with some device of computation which will be necessary to overcome the difficulty due to high dimensionality.

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## § 2. Uniform continuity of Brownian motion with an N-dimensional parameter

Concerning the uniform continuity of $X(A)$, we have

THEOREM 1. *Let $\varphi(t)$ be a non-negative, continuous, and non-decreasing function defined for large $t$'s. Then $\varphi(t)$ belongs to $\mathfrak{U}_N^u$ or $\mathfrak{L}_N^u$ according as the*

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*integral*

$$\int^{\infty} t^{N-1} \varphi^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt \tag{1}$$

*is convergent or divergent.*

In virtue of this theorem, we obtain easily

COR. 1. *The function*

$$\varphi(t) = \{2N \log t + (4N+1) \log_{(2)} t + 2 \log_{(3)} t + \cdots + 2 \log_{(n-1)} t + (2+\delta) \log_{(n)} t\}^{1/2}$$

*belongs to* $\mathfrak{U}_N^u$ *for* $\delta > 0$ *and belongs to* $\mathfrak{L}_N^u$ *for* $\delta \leqq 0$.

By $\log_{(n)}^+ t$, let us denote $\log_{(n)} t$ so long as it is defined and positive, and 0 elsewhere. Namely,

$$\log_{(n)}^+ t = \begin{cases} \log_{(n)} t & \text{for } a_n < t < +\infty \\ 0 & \text{for } 0 < t \leqq a_n, \end{cases} \tag{2}$$

where $a_n$ is defined by $\log_{(n)} a_{n+1} = 1$ and $a_1 = 1$. Then we obtain

COR. 2. *The function*

$$\varphi_\infty(t) = \{2N \log^+ t + (4N+1) \log_{(2)}^+ t + 2 \sum_{n=3}^{\infty} \log_{(n)}^+ t\}^{1/2}$$

*belongs to* $\mathfrak{L}_N^u$.

*Proof.* By the definition of $\log_{(n)}^+ t$, we have

$$\begin{aligned} \int_{a_2}^{\infty} t^{N-1} \varphi_\infty^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2_\infty(t)} dt &= \sum_{n=2}^{\infty} \int_{a_n}^{a_{n+1}} t^{N-1} \varphi_\infty^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2_\infty(t)} dt \\ &> (2N)^N \sum_{n=2}^{\infty} \int_{a_n}^{a_{n+1}} (t \prod_{k=1}^{n-1} \log_{(k)} t)^{-1} dt \\ &= (2N)^N \sum_{n=2}^{\infty} [\log_{(n)} t]_{a_n}^{a_{n+1}} = +\infty. \end{aligned} \tag{3}$$

So our assertion follows from Theorem 1.

Before going into the proof of Theorem 1, we state

LEMMA 1. *Theorem* 1 *holds, if it holds under the following condition:*

$$(2N \log t - 10N \log_{(2)} t)^{1/2} \leqq \varphi(t) \leqq (2N \log t + 10N \log_{(2)} t)^{1/2}. \tag{4}$$

*Proof.* If we put

$$\hat{\varphi}(t) = \min\{\max(\varphi(t), \varphi_1(t)), \varphi_2(t)\}, \tag{5}$$

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where

$$\varphi_1(t) = \{2\, N \log t - 10\, N \log_{(2)} t\}^{1/2},$$
$$\varphi_2(t) = \{2\, N \log t + 10\, N \log_{(2)} t\}^{1/2},$$

then $\hat{\varphi}(t)$ satisfies the condition (4).

First, let us consider the case in which the integral (1) for $\varphi(t)$ is convergent. If there exists a monotone increasing sequence $\{t_n\}$ such that $\varphi(t_n)$ is less than $\varphi_1(t_n)$, and $t_n$ tends to infinity with $n$, we have

$$(6)\qquad \begin{aligned} \int^{\infty} t^{N-1} \varphi^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt &> \int_{t_1}^{t_n} t^{N-1} \varphi^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt \\ &\geqq \int_{t_1}^{t_n} t^{N-1} \varphi^{4N-1}(t_n)\, e^{-\frac{1}{2}\varphi^2(t_n)}\, dt \\ &\geqq c\, t_n^N \varphi^{4N-1}(t_n)\, e^{-\frac{1}{2}\varphi^2(t_n)} \\ &\geqq c\, t_n^N \varphi_1^{4N-1}(t_n)\, e^{-\frac{1}{2}\varphi_1^2(t_n)} \\ &= c(\log t_n)^{7N-\frac{1}{2}} \end{aligned}$$

because $\varphi(t)$ is monotone non-decreasing, where $c$ is a suitably chosen positive constant. Since $\log t_n$ tends to infinity with $n$, no such $\{t_n\}$ can exist in the present case. Therefore, $\varphi(t) > \varphi_1(t)$ and also $\varphi(t) \geqq \hat{\varphi}(t)$ for sufficiently large $t$'s. Moreover, the integral (1) for $\varphi_2(t)$ is convergent, so the integral (1) for $\hat{\varphi}(t)$ is convergent and $\hat{\varphi}(t)$ belongs to $\mathfrak{U}_N^u$ if Theorem 1 holds under the condition (4). As $\varphi(t) \geqq \hat{\varphi}(t)$ for sufficiently large $t$'s, $\varphi(t)$ belongs to $\mathfrak{U}_N^u$.

Secondly, let us consider the case in which the integral (1) for $\varphi(t)$ is divergent. If there is an increasing sequence $\{t_n\}$ such that $\varphi(t_n) < \varphi_1(t_n)$ and $t_n$ tends to infinity with $n$, we have

$$(7)\qquad \begin{aligned} \int^{\infty} t^{N-1} \hat{\varphi}^{4N-1}(t)\, e^{-\frac{1}{2}\hat{\varphi}^2(t)}\, dt &\geqq c t_n^N \hat{\varphi}(t_n)\, e^{-\frac{1}{2}\hat{\varphi}^2(t_n)} \\ &= c t_n^N \varphi_1(t_n)\, e^{-\frac{1}{2}\varphi_1^2(t_n)} \\ &= c(\log t_n)^{7N-\frac{1}{2}} \end{aligned}$$

because $\hat{\varphi}(t)$ is monotone non-decreasing and $\hat{\varphi}(t_n) = \varphi_1(t_n)$, where $c$ is a suitably chosen positive constant. On the contrary, if $\varphi_1(t)$ is less than $\varphi(t)$ for large $t$'s, then $\varphi(t) \geqq \hat{\varphi}(t)$ for large $t$'s and hence there exists a positive constant $c$ such that

$$(8)\qquad \int^{\infty} t^{N-1} \hat{\varphi}^{4N-1}(t)\, e^{-\frac{1}{2}\hat{\varphi}^2(t)}\, dt \geqq c \int^{\infty} t^{N-1} \varphi^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt = +\infty.$$

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Now (7) and (8) show that the integral for $\hat{\varphi}(t)$ is divergent in the present case. Namely, $\hat{\varphi}(t)$ belongs to $\mathfrak{L}_N^u$, if Theorem 1 holds under the condition (4), i.e. for almost all $\omega$, there exists a sequence $\{(A_n, B_n);\ A_n, B_n \in U_N\}$ in which dis $(A_n, B_n)$ tends to 0 as $n$ increases to infinity and satisfying the condition

$$(9) \qquad |X(A_n, \omega) - X(B_n, \omega)| > (\text{dis}\,(A_n, B_n))^{1/2}\, \hat{\varphi}(1/\text{dis}\,(A_n, B_n)).$$

Moreover, $\varphi_2(t)$ belongs to $\mathfrak{U}_N^u$ if Theorem 1 holds under the condition (4). Hence, for almost all $\omega$, there exists a positive number $\varepsilon$ such that dis $(A, B) < \varepsilon$ implies

$$(10) \qquad |X(A, \omega) - X(B, \omega)| < (\text{dis}\,(A, B))^{1/2}\, \varphi_2(1/\text{dis}\,(A, B)).$$

From (9) and (10), follows the inequality

$$\hat{\varphi}(1/\text{dis}\,(A_n, B_n)) < \varphi_2(1/\text{dis}\,(A_n, B_n))$$

for large $n$'s. By the definition of $\hat{\varphi}(t)$, we obtain

$$(11) \qquad \varphi(1/\text{dis}\,(A_n, B_n)) \leqq \hat{\varphi}(1/\text{dis}\,(A_n, B_n)).$$

Here (9) and (11) show that $\varphi(t)$ belongs to $\mathfrak{L}_N^u$.
Thus Lemma 1 has been proved.

*Proof of Theorem 1*

a) The convergent case

First, we remark that it suffices to prove, for almost all $\omega$, the existence of $\varepsilon'$ such that dis $(A, B) \leqq \varepsilon'$ implies the inequality

$$(12) \qquad X(A, \omega) - X(B, \omega) < (\text{dis}\,(A, B))^{1/2}\, \varphi(1/\text{dis}\,(A, B)).$$

In fact, if this assertion holds then, for almost all $\omega$, there exists a positive $\varepsilon''$ such that dis $(A, B) \leqq \varepsilon''$ implies

$$(13) \qquad -\{X(A, \omega) - X(B, \omega)\} < (\text{dis}\,(A, B))^{1/2}\, \varphi(1/\text{dis}\,(A, B))$$

because the process $\{X(A);\ A \in U_N\}$ is symmetric. Taking min $(\varepsilon', \varepsilon'')$ for $\varepsilon$ *in the definition of* $\mathfrak{U}_N^u$, *we have Theorem 1 in the present case. Therefore*, we may consider only the difference of $X(A)$ and $X(B)$ instead of its absolute value.

By $E^p_{\langle k_1, \dots, k_N; l_1, \dots, l_N\rangle}$ (shortly $E^p_{\langle k_i, l_i\rangle}$), we denote the following event:

$$(14) \qquad X(A) - X(B) > (\text{dis}\,(A, B))^{1/2}\, \varphi(1/\text{dis}\,(A, B)),$$

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where $A = \langle (k_1 + l_1)/2^p, \ldots, (k_N + l_N)/2^p \rangle$ and $B = \langle k_1/2^p, \ldots, k_N/2^p \rangle$ are points in $U_N$. Then we have for large $p$'s that

$$(15) \qquad P(E^p_{\langle k_i, l_i \rangle}) \sim e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} / \varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2}).$$

Summing up the above probability for $p = 1, 2, \ldots$; $k_i = \pm 1, \pm 2, \ldots, \pm 2^p$ $(i = 1, 2, \ldots, N)$ and for all lattice points $\langle (k_1 + l_1)/2^p, \ldots, (k_N + l_N)/2^p \rangle$ satisfying $p/3 < (\sum_{i=1}^N l_i^2)^{1/2} \leqq p$, we obtain

$$\sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \sum_{\langle l_i \rangle} P(E^p_{\langle k_i, l_i \rangle}) = 0(1) \sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \sum_{\langle l_i \rangle} \frac{1}{\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^N l_i^2)^{1/2})}.$$

By the monotony of $\varphi(t)$ and Lemma 1, we have

$$(16) \qquad \begin{aligned} \sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \sum_{\langle l_i \rangle} P(E^p_{\langle k_i, l_i \rangle}) &= 0(1) \sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \sum_{\langle l_i \rangle} \frac{1}{\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} \\ &= 0(1) \sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} \frac{p^N}{\varphi(2^p/p)} e^{-\frac{1}{2}\varphi^2(2^p/p)} \\ &= 0(1) \sum_{p=1}^{\infty} \frac{2^{pN} p^N}{\varphi(2^p/p)} e^{-\frac{1}{2}\varphi^2(2^p/p)} \\ &= 0(1) \sum_{p=1}^{\infty} \left(\frac{2^p}{p}\right)^{N-1} \left(\frac{2^{p+1}}{p+1} - \frac{2^p}{p}\right) \varphi^{4N-1}(2^p/p)\, e^{-\frac{1}{2}\varphi^2(2^p/p)} \\ &= 0(1) \int^{\infty} t^{N-1} \varphi^{4N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)}\, dt < +\infty. \end{aligned}$$

Now let us take an event $E^p_{\langle k_i, l_i \rangle}$ appearing in the summand of (16) and fix it. By $F^{(1)}_{\langle m_1^{(1)}, \ldots, m_N^{(1)};\, n_1^{(1)}, \ldots, n_n^{(1)} \rangle}$ (shortly $F^{(1)}_{\langle m_i^{(1)}, n_i^{(1)} \rangle}$), we denote the following event:

$$(17) \qquad \begin{aligned} &X(A_{\langle m_i^{(1)} \rangle}) - X(B_{\langle n_i^{(1)} \rangle}) > (\mathrm{dis}\,(A_{\langle m_i^{(1)} \rangle}, B_{\langle n_i^{(1)} \rangle}))^{1/2} \\ &\times \left\{ \varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2}) + \frac{2\,NC}{\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} \right\}, \quad m_i^{(1)}, n_i^{(1)} = 0, \pm 1, \pm 2, \ldots, \pm e^c, \end{aligned}$$

where $A_{\langle m_i^{(1)} \rangle} = \langle (k_1 + l_1 + m_1^{(1)} e^{-c})/2^p, \ldots, (k_N + l_N + m_N^{(1)} e^{-c})/2^p \rangle$ and $B_{\langle n_i^{(1)} \rangle} = \langle (k_1 + n_1^{(1)} e^{-c})/2^p, \ldots, (k_N + n_N^{(1)} e^{-c})/2^p \rangle$ are points in $U_N$ and $c$ is a suitably chosen constant which makes $e^c$ an integer. For sufficiently large $c$ and $p$, it follows that

$$(18) \qquad \begin{aligned} \sum_{\langle m_i^{(1)}, n_i^{(1)} \rangle} P(F^{(1)}_{\langle m_i^{(1)}, n_i^{(1)} \rangle}) &= 0(1)\, e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^N l_i^2)^{1/2})} / \varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2}) \\ &= 0(1)\, P(E^p_{\langle k_i, l_i \rangle}). \end{aligned}$$

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Also we define $F^{(k)}_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}$ as follows:

$$X(A_{\langle m_i^{(k)}\rangle})-X(B_{\langle n_i^{(k)}\rangle})>$$

$$(19)\qquad (\mathrm{dis}\,(A_{\langle m_i^{(k)}\rangle},\ B_{\langle n_i^{(k)}\rangle}))^{1/2}\{\varphi(2^p/(\sum_{i=1}^{N}l_i^2)^{1/2})+\frac{2NC}{\varphi(2^p/(\sum_{i=1}^{N}l_i^2)^{1/2})}\sum_{r=0}^{k-1}1/2^r\},$$

$$m_i^{(k)},\ n_i^{(k)},\ =0,\ \pm 1,\ \pm 2,\ \dots,\ \pm e^{kc},$$

where $A_{\langle m_i^{(k)}\rangle}=\langle(k_1+l_1+m_1^{(k)}e^{-kc})/2^p,\ \dots,\ (k_N+l_N+m_N^{(k)}e^{-kc})/2^p\rangle$ and $B_{\langle n_i^{(k)}\rangle}=\langle(k_1+n_1^{(k)}e^{-kc})/2^p,\ \dots,\ (k_N+n_N^{(k)}e^{-kc})/2^p\rangle$. Then we have

$$(20)\qquad P(\bigcup_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}F^{(k)}_{\langle m_i^{(k)},\ n_i^{(k)}\rangle})\leq P(\bigcup_{\langle m_i^{(k-1)},\ n_i^{(k-1)}\rangle}F^{(k-1)}_{\langle m_i^{(k-1)},\ n_i^{(k-1)}\rangle})$$

$$+\sum_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}P\{(\bigcap_{\langle m_i^{(k-1)},\ n_i^{(k-1)}\rangle}F^{(k-1)\prime}_{\langle m_i^{(k-1)},\ n_i^{(k-1)}\rangle})\cap F^{(k)}_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}\},$$

where $F'$ denotes the complement of $F$ for any event $F$, $F\cap G$ denotes the event that both $F$ and $G$ hold, and $F\cup G$ denotes the event that $F$ or $G$ holds, for any pair of events $F$ and $G$.

To estimate the second term in the right side of (20), we use the following:

LEMMA 2. *Let $U$ and $V$ be two random variables whose joint distribution is a* 2*-dimensional Gaussian distribution and each of them is subject to the* 1*-dimensional standard Gaussian distribution, and let $\rho$ denote the correlation coefficient between $U$ and $V$. The function*

$$F(a,\ b;\ \rho)\equiv P(U<a,\ V>b)$$

*is monotone decreasing as a function of $\rho$ for fixed $a$ and $b$ $(0<a<b)$.*

*Proof*. Let $W$ be a random variable independent of $V$ and subject to the 1-dimensional standard Gaussian distribution. Since $(U,\ V)$ and $((1-\rho^2)^{1/2}W+\rho V,\ V)$ are subject to the same distribution, we have

$$F(a,\ b;\ \rho)=P((1-\rho^2)^{1/2}W+\rho V<a,\ V>b)$$
$$=\frac{1}{(2\pi)^{1/2}}\int_b^\infty P(W<(a-\rho v)/(1-\rho^2)^{1/2})\,e^{-\frac{1}{2}v^2}dv.$$

This equality shows Lemma 2, because $(a-\rho v)/(1-\rho^2)^{1/2}$ is monotone decreasing in $\rho$ in the present case.

Let us take a pair of points $(A_{\langle m_{i_0}^{(k-1)}\rangle},\ B_{\langle n_{i_0}^{(k-1)}\rangle})$ satisfying the following

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conditions:

$$(A.1)\qquad \begin{aligned}&\mathrm{dis}\,(A_{\langle m_{i_0}^{(k-1)}\rangle},\ A_{\langle m_i^{(k)}\rangle}) \leqq N^{1/2}e^{-(k-1)c}/2^{p+1},\\ &\mathrm{dis}\,(B_{\langle n_{i_0}^{(k-1)}\rangle},\ B_{\langle n_{i_0}^{(k)}\rangle}) \leqq N^{1/2}e^{-(k-1)c}/2^{p+1}.\end{aligned}$$

From the definition of Brownian motion with an $N$-dimensional parameter, for the correlation coefficient $\rho$ between $(X(A_{\langle m_{i_0}^{(k-1)}\rangle} - X(B_{\langle n_{i_0}^{(k-1)}\rangle}))$ and $(X(A_{\langle m_i^{(k)}\rangle}) - X(B_{\langle n_i^{(k)}\rangle}))$ holds

$$\rho = \{\mathrm{dis}\,(A,\ B') + \mathrm{dis}\,(A',\ B) - \mathrm{dis}\,(A,\ A') - \mathrm{dis}\,(B,\ B')\}/2\{\mathrm{dis}\,(A,\ B)\mathrm{dis}\,(A',B')\}^{1/2},$$

where $A = A_{\langle m_i^{(k)}\rangle}$, $B = B_{\langle n_i^{(k)}\rangle}$, $A' = A_{\langle m_{i_0}^{(k-1)}\rangle}$, and $B' = B_{\langle n_{i_0}^{(k-1)}\rangle}$. Using (A.1) and the condition $\mathrm{dis}\,(A,\ B) > 2^{-p}p/3$, we have

$$\rho > [\mathrm{dis}\,(A,\ B) - \mathrm{dis}\,(A,\ A') - \mathrm{dis}\,(B,\ B')]\,[\mathrm{dis}\,(A,\ B)\,\{\mathrm{dis}\,(A,\ B) + \mathrm{dis}\,(A,\ A') + \mathrm{dis}\,(B,\ B')\}]^{-1/2} > \rho_0,$$

where $\rho_0 = 1 - (9\,N^{1/2})/2\,pe^{(k-1)c}$.

Now we return to the estimation of the right side of (20). In virtue of Lemma 2, we obtain, using $\varphi$ for $\varphi(2^p/(\sum_{i=1}^{N} l_i^2)^{1/2})$,

$$P\{(\bigcap_{\langle m_i^{(k-1)},\ n_i^{(k-1)}\rangle} F^{(k-1)'}_{\langle m_i^{(k-1)},\ n_i^{(k-1)}\rangle}) \cap F_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}\} < P\{F^{(k-1)'}_{\langle m_{i_0}^{(k-1)},\ n_{i_0}^{(k-1)}\rangle} \cap F^{(k)}_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}\}$$

$$(21)\qquad \begin{aligned}&= P\Big[\,X(A_{\langle m_{i_0}^{(k-1)}\rangle}) - X(B_{\langle n_{i_0}^{(k-1)}\rangle}) \leqq (\mathrm{dis}\,(A_{\langle m_{i_0}^{(k-1)}\rangle},\ B_{\langle n_{i_0}^{(k-1)}\rangle}))^{1/2} \times \Big\{\varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-2} 1/2^r\Big\},\\ &\qquad X(A_{\langle m_i^{(k)}\rangle}) - X(B_{\langle n_i^{(k)}\rangle}) > (\mathrm{dis}\,(A_{\langle m_i^{(k)}\rangle},\ B_{\langle n_i^{(k)}\rangle}))^{1/2}\Big\{\varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-1} 1/2^r\Big\}\Big]\\ &< P\Big\{(1-\rho_0^2)^{1/2}X + \rho_0 Y < \varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-2}1/2^r,\ Y > \varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-1}1/2^r\Big\}\\ &< P\Big\{(1-\rho_0^2)^{1/2}X < -\frac{NC}{2^{k-1}\varphi},\ \ Y > \varphi + \frac{2\,NC}{\varphi}\sum_{r=0}^{k-1}1/2^r\Big\}\\ &< e^{-2kc(N+1)}P(\bigcup_{\langle m_i^{(1)},\ n_i^{(1)}\rangle} F^{(1)}_{\langle m_i^{(1)},\ n_i^{(1)}\rangle}),\end{aligned}$$

where $X$ and $Y$ are mutually independent random variables subject to the 1-dimensional standard Gaussian distribution. Combining (20) and (21), we have

$$(22)\qquad P(\bigcup_{\langle m_i^{(k)},\ n_i^{(k)}\rangle} F^{(k)}_{\langle m_i^{(k)},\ n_i^{(k)}\rangle}) < \{1 + e^{-c} + \cdots + e^{-kc}\}\sum_{\langle m_i^{(1)},\ n_i^{(1)}\rangle} P(F^{(1)}_{\langle m_i^{(1)},\ n_i^{(1)}\rangle}).$$

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Let us denote by $\tilde{E}^p_{\langle k_i, l_i\rangle}$ the following event:

$$(23)\qquad \max_{A,B}\{(X(A)-X(B))/(\text{dis}(A,B))^{1/2}\}>\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2}) + 4NC/\varphi(2^p/(\sum_{i=1}^N l_i^2)^{1/2}),$$

where $A$ and $B$ run over the cubes $[(k_1+l_1-1)/2^p, (k_1+l_1+1)/2^p; \ldots; (k_N+l_N-1)/2^p, (k_N+l_N+1)/2^p]$ and $[(k_1-1)/2^p, (k_1+1)/2^p; \ldots; (k_N-1)/2^p, (k_N+1)/2^p]$ respectively. Since $X(A)$ is continuous, we have by (18), (22)

$$(24)\qquad \begin{aligned} P(\tilde{E}^p_{\langle k_i, l_i\rangle}) &\leqq \liminf P(\bigcup_{\langle m_i^{(k)}, n_i^{(k)}\rangle} F^{(k)}_{\langle m_i^{(k)}, n_i^{(k)}\rangle}) \\ &= 0(1)\,P(E^p_{\langle k_i, l_i\rangle}). \end{aligned}$$

From (16) and (24) it follows that

$$(25)\qquad \sum_{p=1}^\infty \sum_{\langle k_i\rangle}\sum_{\langle l_i\rangle} P(\tilde{E}^p_{\langle k_i, l_i\rangle}) < +\infty.$$

According to Borel-Cantelli's lemma in the convergent case, (25) shows that only finitely many events $\tilde{E}^p_{\langle k_i, l_i\rangle}$ appearing in (25) can occur for almost all $\omega$. In other words, for almost all $\omega$, there exists $p_0$ such that no $\tilde{E}^p_{\langle k_i, l_i\rangle}$ can occur for $p$'s larger than $p_0$.

Now, for any pair of points $(A, B)$ of $\text{dis}(A, B) < (p_0 - N^{1/2})/2^{p_0}$, we choose $p$ such that

$$(26)\qquad (p+1-N^{1/2})/2^{p+1} < \text{dis}(A, B) \leqq (p-N^{1/2})/2^p.$$

Evidently $p_0 \leqq p$ and $(p-N^{1/2})/2 < 2^p(\text{dis}(A, B)) \leqq p - N^{1/2}$. For $A$ and $B$, let us choose from all pairs of lattice points $C_p$ and $D_p$ of the form $(k_1/2^p, \ldots, k_N/2^p)$, satisfying $\text{dis}(C_p, D_p) \geqq \text{dis}(A, B)$, a pair $(A_p, B_p)$ which minimizes $\text{dis}(A, C_p) + \text{dis}(B, D_p)$. The event

$$(27)\qquad X(A_p) - X(B_p) > (\text{dis}(A_p, B_p))^{1/2}\varphi(1/\text{dis}(A_p, B_p))$$

is identical with some $E^p_{\langle k_i, l_i\rangle}$ appearing in the summation of (16). Considering the corresponding $\tilde{E}^p_{\langle k_i, l_i\rangle}$, for almost all $\omega$, we obtain

$$(28)\qquad \begin{aligned} X(A)-X(B) &\leqq (\text{dis}(A, B))^{1/2}\left\{\varphi(1/\text{dis}(A_p, B_p)) + \frac{4NC}{\varphi(1/\text{dis}(A_p, B_p))}\right\} \\ &\leqq (\text{dis}(A, B))^{1/2}\left\{\varphi(1/\text{dis}(A, B)) + \frac{4NC}{\varphi(1/\text{dis}(A, B))}\right\} \end{aligned}$$

because $\varphi(t) + 4Nc/\varphi(t)$ is monotone non-decreasing for large $t$'s.

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Hence the function $\varphi(t)+4\,Nc/\varphi(t)$ belongs to $\mathfrak{U}_N^u$ by its definition. Since this result is obtained by assumption of the convergence of the integral (1) only, the same result should also be obtained for $\tilde{\varphi}(t)=\varphi(t)-5\,Nc/\varphi(t)$ because $\tilde{\varphi}(t)$ is non-decreasing for sufficiently large $t$'s and the integral for $\tilde{\varphi}(t)$ is convergent. Moreover, it is easily seen that the inequality

$$\tilde{\varphi}(t)+4\,Nc/\tilde{\varphi}(t)<\varphi(t) \tag{29}$$

holds for large $t$'s. Hence by (29), we see that $\varphi(t)$ belongs to $\mathfrak{U}_N^u$.

Thus Theorem 1 has been proved for the convergent case.

b) The divergent case.

Let $E^p_{\langle k_i, l_i\rangle}$ be the event defined by (14). Because $\varphi(t)$ is monotone non-decreasing, by Lemma 1, we have

$$\begin{aligned}
&\sum_{p=1}^{\infty}\sum_{\langle k_i\rangle}\sum_{\langle l_i\rangle}P(E^p_{\langle k_i, l_i\rangle})\\
&=0(1)\sum_{p=1}^{\infty}\sum_{\langle k_i\rangle}\sum_{\langle l_i\rangle}\frac{1}{\varphi(2^p/(\sum_{i=1}^{N}l_i^2)^{1/2})}e^{-\frac{1}{2}\varphi^2(2^p/(\sum_{i=1}^{N}l_i^2)^{1/2})}\\
&=0(1)\sum_{p=1}^{\infty}\sum_{\langle k_i\rangle}\sum_{\langle l_i\rangle}\frac{1}{\varphi(2^{p+1}/p)}e^{-\frac{1}{2}\varphi^2(2^{p+1}/p)}\\
&=0(1)\sum_{p=1}^{\infty}\frac{2^{pN}p^N}{\varphi(2^{p+1}/p)}e^{-\frac{1}{2}\varphi^2(2^{p+1}/p)}\\
&=0(1)\int^{\infty}t^{N-1}\varphi^{4N-1}(t)e^{-\frac{1}{2}\varphi^2(t)}dt=+\infty,
\end{aligned}\tag{30}$$

where $\sum_{\langle l_i\rangle}$ and $\sum_{\langle k_i\rangle}$ denote the summation for all lattice points $\langle(k_1+l_1)/2^p,\ldots,(k_N+l_N)/2^p\rangle$ satisfying $p/2<(\sum_{i=1}^{N}l_i^2)^{1/2}\leqq p$ and for all lattice points $\langle k_1/2^p,\ldots,k_N/2^p\rangle$ satisfying $\max_{1\leqq i\leqq N}|k_i|\leqq 2^p$, respectively. By the definition of $\mathfrak{L}_N^u$, $\varphi(t)$ belongs to $\mathfrak{L}_N^u$ if $E^p_{\langle k_i, l_i\rangle}$ occurs "*infinitely often*" for almost all $\omega$. To prove that this is the case, we use the following due to K. L. Chung and P. Erdös [7].

LEMMA 3. *Let $\{E_k\}$ be an infinite sequence of events satisfying the following conditions:*

(i) $$\sum_{k=1}^{\infty}P(E_k)=+\infty.$$

(ii) *For every pair of positive integers $h$ and $n$ satisfying $n\geqq h$, there exists $C(h)>0$ and $H(n,h)>n$ such that for every $m\geqq H(n,h)$ holds*

$$P(E_m/E'_h\cap\cdots\cap E'_n)>C(h)\,P(E_m),$$

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*where $P(F/E)$ denote the conditional probability of $F$ on the hypothesis $E$.*

(iii) *There exist two absolute constants $c_1$ and $c_2$ with the following property: to each $E_j$ there corresponds a set of events $E_{j_1}, \ldots, E_{j_s}$ belonging to $\{E_k\}$ such that*

(a) $$\sum_{i=1}^{s} P(E_j \cap E_{j_i}) < c_1 P(E_j)$$

*and that for any other $E_k$ than $E_{j_i}(1 \leqq i \leqq s)$ which stands after $E_j$ in the sequence (viz. $k > j$)*

(b) $$P(E_j \cap E_k) < c_2 P(E_j) P(E_k).$$

*The probability that infinitely many events $E_k$ occur is equal to* 1.

Because (30) shows that the sequence $\{E^p_{\langle k_i, l_i \rangle}\}$ satisfies the condition (i) in Lemma 3, it suffices to prove that the sequence satisfies also (ii) and (iii). For this purpose, we enumerate the events $E^p_{\langle k_i, l_i \rangle}$ in the order that $E^p_{\langle k_i, l_i \rangle}$ stands before $E^{p'}_{\langle k_i', l_i' \rangle}$ if and only if one of the following four conditions holds:

($\alpha$) $p < p'$,

($\beta$) $p = p'$ and $\sum_{i=1}^{N} l_i'^2 < \sum_{i=1}^{N} l_i^2$,

($\gamma$) $p = p'$, $\sum_{i=1}^{N} l'^2_i = \sum_{i=1}^{N} l_i^2$, $k_j = k_j'(j = 1, 2, \ldots, i-1)$ and $k_i < k_i'$ for some $i(\leqq N)$,

($\delta$) $p = p'$, $\sum_{i=1}^{N} l'^2_i = \sum_{i=1}^{N} l_i^2$, $k_i = k_i'(i = 1, 2, \ldots, N)$, $l_j = l_j'(j = 1, 2, \ldots, i-1)$, and $l_i < l_i'$ for some $i(\leqq N)$.

Let $\{E_n\}$ be the newly obtained sequence of events. This special ordering is employed for the convenience of later computations.

Put

$$U_n = X((k_1 + l_1)/2^p, \ldots, (k_N + l_N)/2^p) - X(k_1/2^p, \ldots, k_N/2^p)$$

for $E_n = E^p_{\langle k_i, l_i \rangle}$. Then a simple computation shows that, for any positive integer $n$, we have

(31) $$\lim_{m \to \infty} \rho(U_n, U_m) = 0.$$

If we denote by $E_n(a)$ the event that $U_n + a$ is positive, $P(E_n(a))$ tends to 1 as $a$ increases to infinity. Therefore, for each pair of positive integer $h$ and $n$ satisfying $n \geqq h$, we can choose $a_{h, n}$ such that

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$$(32) \qquad P\{\bigcap_{l=h}^{n}(E_l'\cap E_l(a_{h,n}))\}\geqq P(\bigcap_{l=h}^{n}E_l')/2.$$

Then holds

$$\begin{aligned} P(E_m/E_h'\cap\cdots\cap E_n') &= P(E_m\cap E_h'\cap\cdots\cap E_n')/P(E_h'\cap\cdots\cap E_n') \\ (33) \qquad &\geqq P\{E_m\cap(\bigcap_{l=h}^{n}(E_l'\cap E_l(a_{h,n})))\}/2\,P\{\bigcap_{l=h}^{n}(E_l'\cap E_l(a_{h,n}))\} \\ &= P\{E_m/\bigcap_{l=h}^{n}(E_l'\cap E_l(a_{h,n}))\}/2. \end{aligned}$$

Let $\{X_1, \ldots, X_n, Y_m;\ m\in\mathfrak{M}\}$ be a Gaussian system satisfying the conditions

$$E(X_i)=E(Y_m)=0,\ E(X_i^2)=E(Y_m^2)=1,\ i=1,2,\ldots;\ n,\ m\in\mathfrak{M}.$$

For any bounded Borel sets $B_1, \ldots, B_n$, we define $\varepsilon(m, B_m)=\varepsilon(\rho_{1,m}, \ldots, \rho_{n,m};\ B)$ by

$$P(Y_m\in B_m/X_i\in B_i,\ i=1,2,\ldots,n)=(1+\varepsilon(m, B_m))\,P(Y_m\in B_m),$$

where $\rho_{i,m}=\rho(X_i, Y_m)$, and $B_m$ denotes a Borel set contained in the interval $[-\rho_m^{-s}, \rho_m^{-s}]$ with $s<1$, $\rho_m$ being max $(|\rho_{i,m}|;\ 1\leqq i\leqq n)$. $B_m$ may vary with $m$. Then we have

LEMMA 4. *$\varepsilon(m, B_m)\to 0$ as $\rho_m\to 0$.*

*Proof.* Let $p_m(X_1, \ldots, X_n)$ denote the conditional expectation of $Y_m$ for given values of $X_1, \ldots, X_{n-1}$, and $X_n$. Then the expectation of $p_m(X_1, \ldots, X_n)$ is 0 and its variance tends to 0 with $\rho_m$. Since the Gaussian distribution with mean vector **0** is determined by its covariance matrix, we have

$$\begin{aligned} &P(X_i\in B_i,\ i=1,2,\ldots,n \text{ and } Y_m\in B_m) \\ &\quad = P(X_i\in B_i,\ i=1,2,\ldots,n \text{ and } (1-\alpha^2)^{1/2}Z+p_m(X_1,\ldots,X_n)\in B), \end{aligned}$$

where $\alpha^2=E(p_m^2(X_1, \ldots, X_n))$ and $Z$ denotes the random variable independent of $\langle X_1, \ldots, X_n\rangle$ and subject to the 1-dimensional standard Gaussian distribution. Denoting by $P_{\langle X_i\rangle}$ the probability law of $\langle X_1, \ldots, X_n\rangle$, we have

$$\begin{aligned} &P(X_i\in B_i,\ i=1,2,\ldots,n \text{ and } Y_m\in B_m) \\ (\text{A.2}) \qquad &= \int_{\substack{x_i\in B_i\\ 1\leqq i\leqq n}}\left\{\int_{z\in B_m}\frac{1}{(2\pi(1-\alpha^2))^{1/2}}e^{-\frac{1}{2(1-\alpha^2)}(z-p_m(x_1,\ldots,x_n))^2}dz\right\}P_{\langle X_i\rangle}(dx_1,\ldots,dx_n) \\ &= \int_{\substack{x_i\in B_i\\ 1\leqq i\leqq n}}\left\{\int_{z\in B_m}\frac{1}{(2\pi(1-\alpha^2))^{1/2}}e^{-\frac{1}{2}z^2+0}dz\right\}P_{\langle X_i\rangle}(dx_1,\ldots,dx_n), \end{aligned}$$

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where

$$\theta = -\{\alpha^2 z^2 - 2 z p_m(x_1, \ldots, x_n) + p_m^2(x_1, \ldots, x_n)\}/2(1-\alpha^2). \tag{A.3}$$

$\alpha$ and $p_m(x_1, \ldots, x_n)$ are at most of the same order as $\rho_m$. So, by (A.2), (A.3), and the restriction imposed on $B_m$, we obtain Lemma 4.

Now we apply Lemma 4 to the estimation of the right side of (33). If $E_m = E^{p'}_{\langle k'_i, l'_i\rangle}$ and $E_n = E^{p}_{\langle k_i, l_i\rangle}$, then $\max_{h \leqq l \leqq n} |\rho(U_l, U_m)|$ is at most $(p'/2^{p'-p-1})$. Hence $\varphi(2^{p'}/(\sum_{i=1}^{N} l_i'^2)^{1/2}) < (\max_{h \leqq l \leqq n} |\rho(U_l, U_m)|)^{-2/3}$ for large $m$'s. On the other hand, for large $m$'s, we have

$$P(E_m) < 2\,P(G_m), \tag{A.4}$$

where $G_m$ denotes the event

$$\varphi(2^{p'}/(\sum_{i=1}^{N} l_i'^2)^{1/2}) < U_m/(E(U_m^2))^{1/2} < 2\,\varphi(2^{p'}/(\sum_{i=1}^{N} l_i'^2)^{1/2}).$$

From Lemma 4, it follows that

$$\begin{aligned} P(E_m/\bigcap_{l=h}^{n}(E_l' \cap E_l(a_{h,n}))) &> P(G_m/\bigcap_{l=h}^{n}(E_l' \cap E_l(a_{h,n}))) \\ &> P(G_m)/2\,; \end{aligned} \tag{A.5}$$

we get the last inequality, taking $U_l/(E(U_l^2))^{1/2}$ and $U_m/(E(U_m^2))^{1/2}$ for $X_l$ and $Y_m$ in Lemma 4, respectively. By (33), (A.4), and (A.5), we can see that

$$\lim_{m\to\infty} \frac{P(E_m/E_h' \cap \cdots \cap E_n')}{P(E_m)} \geqq 1/8,$$

which proves (ii).

To verify (iii), we use the following lemma given in [4].

LEMMA 5. *Let $U$ and $V$ be two random variables whose joint distribution is a 2-dimensional Gaussian distribution and each of them is subject to the standard 1-dimensional Gaussian distribution.*

(i) *If $\rho(U, V) < 1/ab$, there exists a positive constant $c$ such that*

$$P(U > a,\ V > b) \leqq c\,P(U > a)\,P(V > b).$$

(ii) *There exist two positive constant $d$ and $\delta$ such that for $a > 0$ holds*

$$P(U > a,\ V > a) \leqq d e^{-\delta(1-\rho^2)a^2} P(U > a),$$

*where $\rho$ denotes $\rho(U, V)$.*

For each $E_j = E^{p}_{\langle k_i, l_i\rangle}$, we choose a sequence $\{E_{j_i} = E^{p'}_{\langle k_{i'}, l_{i'}\rangle}\,;\ i = 1, 2, \ldots, s\}$

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of all the events satisfying $j_i \geqq j$ and $\rho(U_j, U_{j_i}) \geqq \{\varphi(2^p/(\sum_{i=1}^{N} l_i^2)^{1/2}) \times \varphi(2^{p'}/\sum_{i=1}^{N} l_i'^2)^{1/2})\}^{-1}$. For any event $E_k$ other than $E_{j_i}(1 \leqq i \leqq s)$ and standing after $E_j$, by (i) of Lemma 5 and definition of $E_j$ and $E_k$, we have

(34) $$P(E_j \cap E_k) < c\, P(E_j)\, P(E_k),$$

where $c$ is an absolute constant. Thus the sequence $\{E_n\}$ satisfies the condition ($b$) of (iii) in Lemma 3.

In order to verify the condition ($a$) of (iii), we divide the sum of $P(E_j \cap E_{j_i})$ according to the magnitude of the correlation coefficient $\rho(U_j, U_{j_i})$ into two summations as follows:

(35) $$\sum_{i=1}^{s} P(E_j \cap E_{j_i}) = \sum{}' P(E_j \cap E_{j_i}) + \sum{}'' P(E_j \cap E_{j_i}),$$

where $\sum'$ expresses the summation over $i$'s such that $\rho(U_j, U_{j_i})$ is larger than $(1-p^{-1/2})^{1/2}$ and $\sum''$ expresses the summation of the other probabilities. Let $A$, $B$, $A'$, and $B'$ be the parameter points of random variables employed in the definition of $E_j$ and $E_{j_i}$, i.e. $U_j = X(A) - X(B)$ and $U_{j_i} = X(A') - X(B')$. Then, for $E_{j_i}$ summed up in $\sum'$, we can show that there exists a positive integer $k$ less than $p^{1/2}$ and satisfying the following inequality:

(36) $$(1-k/p)^{1/2} \leqq \rho(U_j, U_{j_i}) < (1-(k-1)/p)^{1/2},$$

where $\rho(U_j, U_{j_i})$ can be computed as

$$\rho = \{\text{dis}(A, B') + \text{dis}(A', B) - \text{dis}(A, A') - \text{dis}(B, B')\}/2\{\text{dis}(A, B)\,\text{dis}(A', B')\}^{1/2}.$$

Now, for given $A$ and $B$ we estimate the number of pairs of points $A'$ and $B'$ satisfying the inequality (36). Since the correlation coefficient $\rho(U_j, U_{j_i})$ is less than $[\min\{\text{dis}(A, B), \text{dis}(A', B')\}][\text{dis}(A, B)\,\text{dis}(A', B')]^{-1/2}$, it follows from the definition of the ordering of the sequence $\{E_n\}$ that

(37) $$(1-k/p)\,\text{dis}(A, B) \leqq \text{dis}(A', B') \leqq \text{dis}(A, B).$$

We can also see that $(\text{dis}(A, B') - \text{dis}(B, B'))$ and $(\text{dis}(A', B) - \text{dis}(A, A'))$ are less than $\text{dis}(A, B)$. Hence, by (36) and (37), the inequalities

(38) $$\begin{aligned}(1-2k/p)\,\text{dis}(A, B) &\leqq \text{dis}(A', B) - \text{dis}(A, A'),\\ (1-2k/p)\,\text{dis}(A, B) &\leqq \text{dis}(A, B') - \text{dis}(B, B')\end{aligned}$$

hold for large $p$'s. (37) and (38) show that the corresponding superscript $p'$

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of $E_{j_i}$ is at most $(p+1)$ and also that for given $A$ and $B$, the numbers of such points $A'$ and $B'$ are at most of order $k^N$. Moreover, it follows from Lemma 1 and (ii) of Lemma 5 that for $E_{j_i}=E^{p'}_{\langle k_{i'}, l_{i'}\rangle}$ summed up in $\sum'$ holds

$$
\begin{aligned}
P(E_j\cap E_{j_i}) &= P\{U_j>(\mathrm{dis}(A, B))^{1/2}\varphi(1/\mathrm{dis}(A, B)),\\
&\qquad\qquad U_{j_i}>(\mathrm{dis}(A', B'))^{1/2}\varphi(1/\mathrm{dis}(A', B'))\}\\
&\leqq P\{U_j>(\mathrm{dis}(A, B))^{1/2}\varphi(1/\mathrm{dis}(A, B)),\\
&\qquad\qquad U_{j_i}>(\mathrm{dis}(A', B'))^{1/2}\varphi(1/\mathrm{dis}(A, B))\}\\
&\leqq de^{-\delta(1-\rho^2(U_j, U_{j_i}))p}P(E_j)\\
&\leqq d'e^{-\delta k}P(E_j),
\end{aligned}
\tag{39}
$$

where $d$, $\delta$, and $d'$ are absolute constants. Considering the number of $E_{j_i}$, we see that there exist two positive constants $c_1$ and $c_2$ satisfying

$$
\begin{aligned}
\textstyle\sum' P(E_j\cap E_{j_i}) &< c_1\sum_{k=1}^{\infty}k^{2N}e^{-\delta k}P(E_j)\\
&= c_2P(E_j).
\end{aligned}
\tag{40}
$$

To estimate $\sum''$, we consider first the magnitude of superscript $p'$ of $E_{j_i}=E^{p'}_{\langle k_{i'}, l_{i'}\rangle}$ summed up in $\sum''$. The restriction imposed on $\rho(U_j, U_{j_i})$ implies that

$$p'<p+5\log p. \tag{41}$$

Moreover, simple computation shows that if one of the two distances, dis $(A, A')$ and dis $(B, B')$, between the corresponding parameter points employed in the definitions of $U_j$ and $U_k$ is larger than $p^2/2^p$, then $E_k$ is not among $E_{j_i}(1\leqq i\leqq s)$. Hence, for given $E_j$, the number of $E_{j_i}$ with fixed superscript $p'$ is at most of order $p^{4N}$. By Lemma 1 and Lemma 5, we have

$$
\begin{aligned}
\textstyle\sum'' P(E_j\cap E_{j_i}) &< \textstyle\sum'' P\{U_j>(\mathrm{dis}(A, B))^{1/2}\varphi(1/\mathrm{dis}(A, B)),\\
&\qquad\qquad U_{j_i}>(\mathrm{dis}(A', B'))^{1/2}\varphi(1/\mathrm{dis}(A, B))\}\\
&\leqq dP(E_j)\textstyle\sum'' e^{-\delta(1-\rho^2(U_j, U_{j_i}))p},
\end{aligned}
\tag{42}
$$

where $d$ and $\delta$ are positive constants. Since the correlation coefficient $\rho(U_j, U_{j_i})$ is less than $(1-p^{-1/2})^{1/2}$ in the present case, the estimation for the number of $E_{j_i}$'s shows that

$$
\begin{aligned}
\textstyle\sum'' P(E_j\cap E_{j_i}) &\leqq dP(E_j)\textstyle\sum'' e^{-\delta p^{1/2}}\\
&< dP(E_j)\ (p+5\log p)^{4N+1}e^{-\delta p^{1/2}}\\
&< c_3P(E_j),
\end{aligned}
\tag{43}
$$

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where $c_3$ is a suitably chosen positive constant.

Now $(a)$ of (iii) in Lemma 3 follows from (35), (40), and (43).

Thus we have proved completely the divergent case.

## § 3. Local continuity of Brownian motion with an $N$-dimensional parameter

In this section, we study the continuity of $X(A)$ at the origin $O$ of $E_N$.

THEOREM 2. *Let $\psi(t)$ be a non-negative and monotone non-decreasing function defined for large t's. Then $\psi(t)$ belongs to $\mathfrak{U}_N^\circ$ or $\mathfrak{L}_N^\circ$ according as the integral*

(44) $$\int^{\infty} \frac{1}{t} \psi^{2N-1}(t) e^{-\frac{1}{2}\psi^2(t)} dt$$

*is convergent or divergent.*

COR. 4. *The function*

$$\psi(t) = \{2 \log_{(2)} t + (2N+1) \log_{(3)} t + 2 \log_{(4)} t + \cdots + 2 \log_{(n-1)} t + (2+\delta) \log_{(n)} t\}^{1/2}$$

*belongs to $\mathfrak{U}_N^\circ$ for $\delta > 0$ and belongs to $\mathfrak{L}_N^\circ$ for $\delta \leqq 0$.*

COR. 5. *The function*

$$\psi_\infty(t) = \{2 \log^+_{(2)} t + (2N+1) \log^+_{(3)} t + 2 \sum_{n=4}^{\infty} \log^+_{(n)} t\}^{1/2},$$

*belongs to $\mathfrak{L}_N^\circ$, where $\log^+_{(n)} t$ denotes the function defined in § 2.*

Cor. 4 and Cor. 5 follow from Theorem 2 immediately.

As we remarked in the introduction, Theorem 2 assures the following theorem:

THEOREM 3. *Let $\psi(t)$ be a function given in Theorem 2. Then $\psi(t)$ belongs to $\mathfrak{U}_N^\infty$ or $\mathfrak{L}_N^\infty$ according as the integral* (44) *is convergent or divergent.*

COR. 6. *The function $\psi(t)$ defined in Cor. 4 belongs to $\mathfrak{U}_N^\infty$ for $\delta > 0$ and belongs to $\mathfrak{L}_N^\infty$ for $\delta \leqq 0$.*

COR. 7. *The function $\psi_\infty(t)$ defined in Cor. 5 belongs to $\mathfrak{L}_N^\infty$.*

The proof of Theorem 2 can be given in a parallel way to the proof of Theorem 1.

LEMMA 6. *Theorem 2 holds, if it holds under the following condition:*

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(45) $$(2 \log_{(2)} t)^{1/2} \leqq \varphi(t) \leqq (3 \log_{(2)} t)^{1/2}.$$

*Proof.* We assume that Theorem 2 holds for $\varphi(t)$ satisfying (45) and put

(46) $$\hat{\varphi}(t) = \min \{\max (\varphi(t), \varphi_1(t)), \varphi_2(t)\},$$

where

$$\varphi_1(t) = (2 \log_{(2)} t)^{1/2},$$
$$\varphi_2(t) = (3 \log_{(2)} t)^{1/2}.$$

Evidently, $\hat{\varphi}(t)$ satisfies the condition (45).

If there exists a monotone increasing sequence $\{t_n\}$ such that $\varphi(t_n) < \varphi_1(t_n)$ and $t_n$ tends to infinity with $n$, we have

(47) $$\begin{aligned} \int_{t_1}^{\infty} \frac{1}{t} \varphi^{2N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)} dt &> \int_{t_1}^{t_n} \frac{1}{t} \varphi^{2N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)} dt \\ &\geqq c \log t_n\, \varphi^{2N-1}(t_n)\, e^{-\frac{1}{2}\varphi^2(t_n)} \\ &\geqq c \log t_n\, \varphi_1^{2N-1}(t_n)\, e^{-\frac{1}{2}\varphi_1^2(t_n)} \\ &= c\ (2 \log_{(2)} t_n)^{N-\frac{1}{2}} \end{aligned}$$

because $\varphi(t)$ is monotone non-decreasing, where $c$ is a suitably chosen positive constant. Also (47) holds for $\hat{\varphi}(t)$, because $\hat{\varphi}(t)$ is monotone non-decreasing and $\hat{\varphi}(t_n) = \varphi_1(t_n)$. Hence the integrals (44) for $\varphi(t)$ and $\hat{\varphi}(t)$ diverge simultaneously in the present case. On the contrary, if $\varphi_1(t)$ is less than $\varphi(t)$ for large $t$'s, then $\varphi(t) \geqq \hat{\varphi}(t)$ for large $t$'s, hence there is a positive constant $c$ such that

(48) $$\begin{aligned} \int^{\infty} \frac{1}{t} \varphi^{2N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)} dt &\leqq c \int^{\infty} \frac{1}{t} \hat{\varphi}^{2N-1}(t)\, e^{-\frac{1}{2}\varphi^2(t)} dt \\ &\leqq c \Big\{ \int^{\infty} \frac{1}{t} \varphi^{2N-1}(t)\, e^{-\frac{1}{2}\varphi(t)} dt + \\ &\qquad \int^{\infty} \frac{1}{t} \varphi_2^{2N-2}(t)\, e^{-\frac{1}{2}\varphi_1(t)} dt \Big\}. \end{aligned}$$

So the integrals (44) for $\varphi(t)$ and $\hat{\varphi}(t)$ diverge or converge simultaneously.

First, let us consider the case in which the integral for $\varphi(t)$ is convergent. Considering (47) we see that the set of $t$'s where $\varphi(t)$ is less than $\varphi(t)$ is bounded. Therefore, $\varphi(t) > \varphi_1(t)$ and accordingly $\varphi(t) \geqq \hat{\varphi}(t)$ for sufficiently large $t$'s. So $\varphi(t)$ belongs to $\mathfrak{U}_N^{\circ}$ because $\hat{\varphi}(t)$ belongs to $\mathfrak{U}_N^{\circ}$ by our assumption. Secondly, we consider the case in which the integral for $\varphi(t)$ is divergent. By what has been above stated, the integral for $\hat{\varphi}(t)$ is divergent and so $\hat{\varphi}(t)$ belongs to $\mathfrak{L}_N^{\circ}$ byour as sumption. Hence there exists a sequence

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$\{A_n\}$ such that

(49) $$|X(A_n)| > (\text{dis}(O, A_n))^{1/2}\hat{\psi}(1/\text{dis}(O, A_n)),$$
$$\text{dis}(O, A_n) \to O \text{ as } n \to +\infty.$$

Moreover, $\psi_2(t)$ belongs to $\mathfrak{U}_N^\circ$ because $\psi_2(t)$ satisfies the condition (45). So, for large $n$'s holds

$$\hat{\psi}(1/\text{dis}(O, A_n)) < \psi_2(1/\text{dis}(O, A_n)),$$

hence

(50) $$\psi(1/\text{dis}(O, A_n)) \leqq \hat{\psi}(1/\text{dis}(O, A_n)).$$

Here (49) and (50) show that $\psi(t)$ belongs to $\mathfrak{L}_N^\circ$.

Thus Lemma 6 has been proved.

Proof of Theorem 2.

a) The convergent case.

Let us denote by $E^p_{\langle k_1, \ldots, k_N\rangle}$ (shortly $E^p_{\langle k_i\rangle}$), the following event:

(51) $$X(k/2^p, \ldots, k_N/2^p) > ((\sum_{i=1}^N k_i^2)^{1/2}/2^p)\,\psi(2^p/(\sum_{i=1}^N k_i^2)^{1/2}).$$
$$k_i = \pm 1, \pm 2, \ldots, \pm 2^p, \; i = 1, 2, \ldots, N.$$

Summing up $P(E^p_{\langle k_i\rangle})$ for $p = 1, 2, \ldots$, and for all lattice points $\langle k_1/2^p, \ldots, k_N/2^p\rangle$ satisfying $(\log p)/3 < (\sum_{i=1}^N k_i^2)^{1/2} \leqq \log p$, we have by Lemma 6 that

(52) $$\begin{aligned}\sum_{p=1}^\infty \sum_{\langle k_i\rangle} P(E^p_{\langle k_i\rangle}) &= 0(1) \sum_{p=1}^\infty \sum_{\langle k_i\rangle} \frac{1}{\psi(2^p/(\sum_{i=1}^N k_i^2)^{1/2})} e^{-\frac{1}{2}\psi^2(2^p/(\sum_{i=1}^N k_i^2)^{1/2})} \\ &= 0(1) \sum_{p=1}^\infty \frac{(\log p)^N}{\psi(2^p/\log p)} e^{-\frac{1}{2}\psi^2(2^p/\log p)} \\ &= 0(1) \sum_{p=1}^\infty \psi^{2N-1}(2^p/\log p)\, e^{-\frac{1}{2}\psi^2(2^p/\log p)} \\ &= 0(1) \int^\infty \frac{1}{t}\psi^{2N-1}(t)\, e^{-\frac{1}{2}\psi^2(t)}dt < +\infty.\end{aligned}$$

By $\tilde{E}^p_{\langle k_1, \ldots, k_N\rangle}$ (shortly $\tilde{E}^p_{\langle k_i\rangle}$), we denote the following event:

$$\max_A X(A)/(\text{dis}(O, A))^{1/2} > \psi(2^p/(\sum_{i=1}^N k_i^2)^{1/2}) + \frac{c}{\psi(2^p/(\sum_{i=1}^N k_i^2)^{1/2})},$$

where $A$ runs over the cube $[(k_1 - 1)2^p, (k_1 + 1)/2^p; \ldots; (k_N - 1)/2^p, (k_N + 1)/2^p]$. For sufficiently large $c$ and $p$'s, we have by a similar way as in §2 that

$$P(\tilde{E}^p_{\langle k_i\rangle}) = 0(1)\, P(E^p_{\langle k_i\rangle}).$$

From (52) it follows that

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(53) $$\sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} P(\tilde{E}^p_{\langle k_i \rangle}) < +\infty.$$

According to Borel-Cantelli's lemma in the convergent case, (53) shows that only finitely many events $\tilde{E}^p_{\langle k_i \rangle}$ appearing in (53) can occur for almost all $\omega$. Namely, for almost all $\omega$, there exists $p_0$ such that no $\tilde{E}^p_{\langle k_i \rangle}$ can occur for $p$'s larger than $p_0$.

Now, for any point $A$ of dis $(O, A) < (\log p_0 - N^{1/2})/2^{p_0}$, we choose $p$ such that

$$(\log (p+1) - N^{1/2})/2^{p+1} < \text{dis}\,(O, A) < (\log p - N^{1/2})/2^p.$$

By the same way as in §2, we have

$$X(A) \leqq (\text{dis}\,(O, A))^{1/2} \{\psi(1/\text{dis}\,(O, A)) + 2c/\psi(1/\text{dis}\,(O, A))\}.$$

Thus $\psi(t) + 2c/\psi(t)$ belongs to $\mathfrak{U}^\circ_N$ and we can prove by the same procedure as in §2 that $\psi(t)$ belongs to $\mathfrak{U}^\circ_N$.

b) The divergent case.

Let $E^p_{\langle k_i \rangle}$ be the same event as in the convergent case. By Lemma 6, we have

(54) $$\sum_{p=1}^{\infty} \sum_{\langle k_i \rangle} P(E^p_{\langle k_i \rangle}) = 0(1) \int^{\infty} \frac{1}{t} \psi^{2N-1}(t)\, e^{-\frac{1}{2}\psi^2(t)}\, dt = +\infty,$$

where $\sum_{\langle k_i \rangle}$ denotes the summation for all lattice points $\langle k_1/2^p, \ldots, k_N/2^p \rangle$ satisfying $(\log p)/2 < (\sum_{i=1}^{N} k_i^2)^{1/2} \leqq \log p$. Hence it suffices to prove that the sequence $\{E^p_{\langle k_i \rangle}\}$ satisfies the condition (ii) and (iii) in Lemma 3. To prove that this is the case, we enumerate the events $E^p_{\langle k_i \rangle}$ by the same method as in §2 and denote the new sequence by $\{E_n\}$. Then it is clear that by a similar consideration as in §2, (ii) is satisfied in the present case. Next, for each $E_j = E^p_{\langle k_i \rangle}$, we choose a sequence $\{E_{j_i} = E^{p'}_{\langle k_i' \rangle};\ i = 1, 2, \ldots, s\}$ of the events satisfying $j_i > j$ and

(55) $$\rho(U_j, U_{j_i}) > 1/\{\psi(2^p/(\sum_{i=1}^{N} k_i^2)^{1/2})\, \psi(2^{p'}/(\sum_{i=1}^{N} k_i'^2)^{1/2})\},$$

where $U_j$ and $U_{j_i}$ denote the random variables $X(k_1/2^p, \ldots, k_N/2^p)$ and $X(k_1'/2^{p'}, \ldots, k_N'/2^{p'})$ respectively. For any event $E_k$ other than $E_{j_i} (1 \leqq i \leqq s)$ and standing after $E_j$, we can apply Lemma 5 and accordingly (b) of (iii) holds.

To verify (a) of (iii), we employ the same method as in §2. We divide

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the sum of $P(E_j \cap E_{j_i})$ by the magnitude of the corresponding correlation coefficient $\rho(U_j, U_{j_i})$ into two summations as follows:

$$(56) \qquad \sum_{i=1}^{s} P(E_j \cap E_{j_i}) = \sum{}' P(E_j \cap E_{j_i}) + \sum{}'' P(E_j \cap E_{j_i}).$$

where $\sum'$ expresses the summation over $i$'s such that $\rho(U_j, U_{j_i})$ is larger than $(1-(\log p)^{-1/2})^{1/2}$, and $\sum''$ expresses the summation of the other probabilities. Let $A$ and $B$ be the points $(k_1/2^p, \ldots, k_N/2^p)$ and $(k_1'/2p', \ldots, k_N'/2^{p'})$ respectively. Then, for $E_{j_i}$ summed up $\sum'$, we can show that there exists a positive integer $k$ less than $(\log p)^{1/2}$ and satisfying the following inequality:

$$(57) \qquad (1-k/\log p)^{1/2} \leqq \rho(U_j, U_{j_i}) < (1-(k-1)/\log p)^{1/2},$$

where

$$\rho(U_j, U_{j_i}) = \{\mathrm{dis}\,(O, A) + \mathrm{dis}\,(O, B) - \mathrm{dis}\,(A, B)\}/2\{\mathrm{dis}\,(O, A)\ \mathrm{dis}\,(O, B)\}^{1/2}.$$

Since $\rho(U_j, U_{j_i})$ is less than $\{\min\,(\mathrm{dis}\,(O, A), \mathrm{dis}\,(O, B))\}\,\{\mathrm{dis}\,(O, A)\ \mathrm{dis}\,(O, B)\}^{-1/2}$, it follows from (57) and the definition of ordering of the sequence $\{E_n\}$ that

$$(58) \qquad (1-k/\log p)\ \mathrm{dis}\,(O, A) \leqq \mathrm{dis}\,(O, B) \leqq \mathrm{dis}\,(O, A).$$

From (57) and (58) it follows that for large $p$'s

$$(59) \qquad \mathrm{dis}\,(A, B) < 2\,k\ \mathrm{dis}\,(O, A)/\log p.$$

Now (58) shows that the superscript $p'$ of $E_{j_i} = E^{p'}_{\langle k_i' \rangle}$ summed up in $\sum'$ is at most $p+1$. Also (59) shows that for given $E_j$, the number of such $E_{j_i}$'s is at most of order $k^N$. Therefore, by Lemma 5, Lemma 6, (57), and (58) holds

$$(60) \qquad \begin{aligned} \sum{}' P(E_j \cap E_{j_i}) &\leqq \sum{}' P\{U_j > (\mathrm{dis}\,(O, A))^{1/2}\ \psi(1/\mathrm{dis}\,(O, A)), \\ &\qquad\qquad U_{j_i} > (\mathrm{dis}\,(O, B))^{1/2}\ \psi(1/\mathrm{dis}\,(O, A))\} \\ &\leqq c_1 \sum_{k=1}^{\infty} k^N e^{-\delta(1-\rho^2(U_j, U_{j_i}))\,\psi^2(1/\mathrm{dis}(O, A))} P(E_j) \\ &\leqq c_2 \sum_{k=1}^{\infty} k^N e^{-\delta' k} P(E_j) \\ &= c_3\, P(E_j), \end{aligned}$$

where $c_1$, $c_2$, $c_3$, $\delta$, and $\delta'$ are positive constants. On the other hand, if the superscript $p'$ of $E_n = E^{p'}_{\langle k_i' \rangle}$ is larger than $\log p + 5 \log_{(2)} p$, then $\rho(U_j, U_n)$ is less than $\{\psi(2^p/(\sum_{i=1}^{N} k_i^2)^{1/2})\ \psi(2^p/(\sum_{i=1}^{N} k_i'^2)^{1/2})\}^{-1}$. Hence, by Lemma 5 and Lemma

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6, we have for large $p$'s

$$\begin{aligned}\sum'' P(E_j \cap E_{j_i}) &\leqq \sum'' P\{U_j > (\text{dis}(O, A))^{1/2}\ \psi(1/\text{dis}(O, A)),\\ &\qquad U_{j_i} > (\text{dis}(O, B))^{1/2}\ \psi(1/\text{dis}(O, A))\}\\ &\leqq d\sum'' e^{-\delta(1-\rho^2(U_j, U_{j_i}))\,\psi^2(1/\text{dis}(O, A))} P(E_j) \qquad (61)\\ &\leqq d(\log p + 5\log_{(2)} p)^{2K+1} e^{-\delta'(\log p)^{1/2}} P(E_j)\\ &< P(E_j),\end{aligned}$$

where $d$, $\delta$, and $\delta'$ are positive constants. (60) and (61) show that the sequence $\{E_n\}$ satisfies the consition (a) of (iii).

Thus we have proved Theorem 2.

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