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ON PERMUTATION GROUPS WITH CONSTANT MOVEMENT

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Abstract

Let G be a permutation group on a set Ω with no fixed point in Ω. If for each subset Γ of Ω the size $|\Gamma^g - \Gamma|$ is bounded, for $g \in G$, we define the movement of g as the $\max|\Gamma^g - \Gamma|$ over all subsets Γ of Ω. In particular, if all non-identity elements of G have the same movement, then we say that G has constant movement. In this paper we will first give some families of groups with constant movement. We then classify all transitive permutation groups with a given constant movement m on a set of maximum size.

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1. Introduction

Let G be a permutation group on a set Ω with no fixed points in Ω and let m be a positive integer. If for each subset Γ of Ω and each element $g \in G$, the size $|\Gamma^g - \Gamma|$ is bounded, we define the *movement* of Γ as $\text{move}(\Gamma)=\max_{g\in G}|\Gamma^g - \Gamma|$. If $\text{move}(\Gamma) \leq m$ for all $\Gamma \subseteq \Omega$, then G is said to have *bounded movement* and the *movement* of G is defined as the maximum of $\text{move}(\Gamma)$ over all subsets Γ. This notion was introduced in [6]. Similarly, for each $1 \neq g \in G$, we define the movement of g as the $\max|\Gamma^g - \Gamma|$ over all subsets Γ of Ω. If all non-identity elements of G have the same movement, then we say that G has *constant movement*.

Clearly every permutation group with constant movement has bounded movement. By [6, Theorem 1], if G has bounded movement equal to m, then Ω is finite, and its size is bounded by a function of m.

For transitive groups of movement m, the following bounds on Ω were obtained in [6].

LEMMA 1.1. *Let G be a transitive permutation group on a set Ω such that G has movement m.*

(a) *If G is a 2-group then $|\Omega| \leq 2m$.*

(b) *If G is not a 2-group and p is the least odd prime dividing $|G|$, Then $|\Omega| \leq \lfloor 2mp/(p-1) \rfloor$. (For $x \in R$, $\lfloor x \rfloor$ denotes the integer part of x.)*

There are various types of permutation groups with constant movement for which the bounds in Lemma 1.1 may be attained. For example, let G be either a p-group of exponent p or a 2-group. If we consider G as a permutation group in its regular representation, then we see that all non-identity elements have the same movement.

The purpose of this paper is to classify all transitive permutation groups G of maximum degree n with constant movement m, (where $n = 2m$ if G is a 2-group and otherwise $n = \lfloor 2mp/(p-1) \rfloor$ if p is the least odd prime dividing $|G|$ and by [6] these are the maximum sizes of n).

THEOREM 1.2. *Let m be a positive integer, and let G be a transitive permutation group on a set Ω of maximum size n with constant movement m. Then either G is a 2-group in its regular representation, or for an odd prime p one of the following holds*:

(1) $|\Omega| = p$, $m = (p-1)/2$ *and G is the semi-directed product of $Z_p Z_{2^a}$, where $2^a | (p-1)$ for some $a \geq 1$;*

(2) $G := A_4, A_5$, $|\Omega| = 6$ *and* $m = 2$;

(3) *G is a p-group of exponent p in its regular representation.*

Moreover, all permutation groups listed above have constant movement.

All the groups in Theorem 1.2 are examples (see Section 2). In Section 3, we prove the above theorem, which is a classification theorem for the transitive permutation groups of maximal degree with constant movement.

2. Attaining the bounds: examples

Let G be a transitive permutation group on a finite set Ω. Then by [9, Theorem 3.26], which we shall refer to as Burnside's Lemma, the average number of fixed points in Ω of elements of G is equal to the number of G-orbits in Ω, namely 1, and since 1_G fixes $|\Omega|$ points and $|\Omega| > 1$, it follows that there is some element of G which has no fixed points in Ω. We shall say that such elements are fixed point free on Ω.

Let $1 \neq g \in G$ and suppose that g in its disjoint cycle representation has t nontrivial cycles of lengths $l_1, \ldots, l_t$, say. We might represent g as

$$g = (a_1 a_2 \cdots a_{l_1})(b_1 b_2 \cdots b_{l_2}) \cdots (z_1 z_2 \cdots z_{l_t}).$$

Let $\Gamma(g)$ denote a subset of Ω consisting of $\lfloor l_i/2 \rfloor$ points from i^{th} cycle, for each i, chosen in such away that $\Gamma(g)^g \cap \Gamma(g) = \emptyset$.

For example we could choose $\Gamma(g) = \{a_2, a_4, \dots, b_2, b_4, \dots, z_2, z_4, \dots\}$. Note that $\Gamma(g)$ is not uniquely determined as it depends on the way each cycle is written down. For any set $\Gamma(g)$ of this kind, we say that $\Gamma(g)$ consists of *every second point of every cycle of* g. From the definition of $\Gamma(g)$ we see that

$$|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = \sum_{i=1}^{t} \lfloor l_i/2 \rfloor.$$

In [3] we have shown that this quantity is an upper bound for $|\Gamma^g - \Gamma|$ for an arbitrary subset Γ. Thus the movement of g is $|\Gamma(g)|$.

Now we will show that there certainly are some families of examples of transitive groups with constant movement for which the bound of Lemma 1.1 holds, for any prime p. First we look at groups of exponent p.

LEMMA 2.1. (a) *Let $m := p^{a-1}(p-1)/2$ for some $a \geq 1$, where p is an odd prime and suppose that G is a regular permutation group of exponent p on a set Ω of size $p^a = 2mp/(p-1)$. Then G has constant movement m.*
(b) *Let m be a power of 2, and suppose that G is a 2-group of order $2m$. Then the regular representation of G is a permutation group of constant movement m.*

PROOF. Let $1 \neq g \in G$ and let $\Gamma \subseteq \Omega$. By [3, Lemma 2.1], $|\Gamma^g - \Gamma| \leq m$. Since G is regular, g is fixed point free on Ω. Suppose that $\Gamma(g)$ consists of every second point of every cycle of g. Then by definition $\Gamma(g)^g \cap \Gamma(g) = \emptyset$. If p is an odd prime, then $|\Gamma(g)^g - \Gamma(g)| = |\Gamma(g)| = (|\Omega|/p)(p-1)/2 = p^{a-1}(p-1)/2 = m$.

Thus G has constant movement m. Also with the same argument it can be shown that every 2-group of degree $2m$ in its regular representation has constant movement m. □

In what follows we will see that the regularity condition for each transitive p-group is a necessary condition. Let H be a core-free subgroup of a p-group G and consider the permutation representation by right multiplication on the right cosets of H. If $H \neq 1$, then G is not regular in this action and does not have constant movement. An example of such a core-free subgroup H in a p-group G of exponent p is the cyclic group generated by any non-central element. Such elements exist provided that G is non-abelian.

Let $H = \langle h \rangle \cong Z_n$, and let $K = \langle k \rangle \cong Z_m$ be such that K is a subgroup of $\mathrm{Aut}(H)$. Then $h^k = h^r$ for some positive integer r such that $r^m \equiv 1 \pmod{n}$. Let $G = HK$ be the natural semi-direct product of H by K. Then G is given by the defining relations: $h^n = 1$, $k^m = 1$, $k^{-1}hk = h^r$, with $r^m \equiv 1 \pmod{n}$.

Here every element of G is uniquely expressible as $h^i k^j$, where $0 \leq i \leq n-1$, $0 \leq j \leq m-1$. Certain semi-direct products of this type also provide examples of groups with constant movement where the bound in Lemma 1.1 holds. (We note that, if $n = p$, a prime, then this group G is a subgroup of the group $AGL(1, p) = Z_p Z_{p-1}$.)

LEMMA 2.2. *Let $G := Z_p Z_{2^a}$ denote a group defined as above of order $p.2^a$, where $2^a|(p-1)$ for some $a \geq 1$. Then G acts transitively on a set Ω of size p and in this action G has constant movement $(p-1)/2$.*

PROOF. The group G is a Frobenius group and has up to permutational isomorphism a unique transitive representation of degree p, on a set Ω, say. Let $g \in G$ be such that $o(g) = p$. Then by [3, Lemma 2.1], $|\Gamma^g - \Gamma| \leq m = (p-1)/2$ for all subsets Γ, and if $\Gamma(g)$ consists of every second point of the unique cycle of g, then $|\Gamma(g)^g - \Gamma(g)|$ has size equal to m. Suppose now that $g \in G$ has order $o(g)$ a power of 2. Then g has one fixed point and $(p-1)/o(g)$ cycles of length $o(g)$ in Ω. For each $\Gamma \subseteq \Omega$, $|\Gamma^g - \Gamma|$ consists of at most $o(g)/2$ points from each cycle of g of length $o(g)$,and hence has size at most m. Since each non-identity element of G is either a 2-element or has order p, it follows that G has constant movement equal to m. □

LEMMA 2.3. *The groups A_4, and A_5 acting transitively on a set of size 6 have constant movement equal to 2.*

PROOF. By [2, 4, 5] the groups A_4 and A_5 have bounded movement equal to 2. Using similar argument as in [3, Lemma 3.3], we will show that they also have constant movement 2. Let $1 \neq g \in A_4$. Then g has order 2 or 3. If g has order 2 then g has two cycles of length 2 and hence $|\Gamma(g)^g - \Gamma(g)| = 2$. Similarly, if g has order 3 then g has two cycles of length 3 and again $|\Gamma(g)^g - \Gamma(g)| = 2$. As for A_5, since every non-identity element of A_5 has order 2, 3 or 5, as above it is easy to see that every element of A_5 has movement equal to 2. Hence both of them have constant movement 2. □

3. Proof of Theorem 1.2

Let m be a positive integer. Suppose that G is a transitive permutation group on a set Ω of size n with constant movement m, which have maximal degree. (Where $n = 2m$ if G is a 2-group and otherwise $n = \lfloor 2mp/(p-1) \rfloor$ where p is the least odd prime dividing $|G|$.) By [1, Theorem 1], for some prime q dividing $|G|$, there exists a q-element g of order q^a (for some positive integer a) in G which is fixed point free on Ω. Then g has b_i cycles of length q^i for $i = 1, \dots, a$, where $\sum_{i=1}^{a} b_i q^i = n$ and $b_a > 0$. Now we consider two cases:

Case 1: Suppose that q is odd. Then by the definition of $\Gamma(g)$ we have,

$$m = |\Gamma(g)| = \sum_{i=1}^{a} b_i \frac{q^i - 1}{2}.$$

Suppose that $a \geq 2$, and consider $h = g^{q^{a-1}}$, say. Then h has $b_a q^{a-1}$ cycles of length q, so by the definition of every second point of every cycles of h we have,

$$\begin{aligned} m = |\Gamma(h)| = b_a q^{a-1} \frac{q-1}{2} &= \sum_{i=1}^{a-1} b_i \frac{q^i - 1}{2} + b_a \frac{q^a - 1}{2} \\ &\geq b_a \frac{q^a - 1}{2} = b_a \frac{q-1}{2} (q^{a-1} + \cdots + q + 1) \\ &= b_a q^{a-1} \frac{q-1}{2} + b_a \frac{q-1}{2} (q^{a-2} + \cdots + q + 1) \\ &= b_a q^{a-1} \frac{q-1}{2} + b_a \frac{q^{a-1} - 1}{2} > b_a q^{a-1} \frac{q-1}{2}, \end{aligned}$$

which is a contradiction. Hence $a = 1$ and therefore $b_a = b_1 = n/q$, and $m = (n/q)(q-1)/2$. Suppose there exists an odd prime r dividing $|G|$ such that $r \leq q$, and let $x \in G$, $o(x) = r$. Then

$$m = |\Gamma(x)| \leq \frac{n}{r} \frac{r-1}{2} = \frac{2mq}{q-1} \frac{r-1}{2r}.$$

So $(q-1)r \leq q(r-1)$ and hence $q \leq r$ which is a contradiction. Hence q is the least odd prime dividing $|G|$, that is, we have proved that $q = p$.
Case 2: Now we suppose that $q = 2$, so as above we can assume that $o(g) = 2^a$ for some positive integer a, and g is a fixed point free element on Ω. Then g has b_i cycles of length 2^i for $i = 1, \ldots, a$, where $n = \sum_{i \leq a} b_i 2^i$, $b_a > 0$, and

$$m = |\Gamma(g)| = \sum_{i=1}^{a} b_i 2^{i-1}.$$

Suppose that $a \geq 2$, and consider $g^{2^{a-1}} = h$, say. Then h has $b_a 2^{a-1}$ cycles of length 2, so

$$b_a 2^{a-1} = |\Gamma(h)| = m = \sum_{i=1}^{a-1} b_i 2^{i-1} + b_a 2^{a-1}.$$

The above equality is true if $b_i = 0$ for each $i < a$. So all g-cycles have length 2^a, and hence $2^a | n$.

We first suppose that G is a transitive permutation group on a set of size $n = 2m$ and G is a 2-group. As each $1 \neq g \in G$ has constant movement m, $|\operatorname{supp}(g)| = 2m$, where $\operatorname{supp}(g) = \{\alpha \in \Omega | \alpha^g \neq \alpha\}$. Thus g is a fixed point free element on Ω, that is, $G_\alpha = 1$ for each $\alpha \in \Omega$. Hence G is a regular 2-group.

Now suppose that p is an odd prime. Then G is not a 2-group. Since G is a transitive permutation group with maximal degree, by [7, Theorem 6.4]

$$|\Omega| = \left\lfloor \frac{2mp}{p-1} \right\rfloor = \frac{2mp}{p-1},$$

where p is the least odd prime dividing $|G|$. (Since $2m < 2mp/(p-1)$, so if G is not a 2-group with maximal degree then $|\Omega| \neq 2m$.) Then by [2, 3, 4, 5], one of the following holds:

(1) $|\Omega| = p, m = (p-1)/2$ and G is the semi-direct product $Z_p Z_{2^a}$ where $2^a | (p-1)$ for some $a \geq 1$.

(2) G is the semi-direct product KP with K a 2-group and $P = Z_p$ is fixed point free on Ω; $|\Omega| = 2^s p$, $m = 2^{s-1}(p-1)$, and $2^s < p$, where K has p-orbits of length 2^s, and each element of K moves at most $2^s(p-1)$ points of Ω. (We note that $A_4 \cong (Z_2)^2 Z_3$ is a transitive permutation group of degree 6 which has constant movement 2, this occur in this case where $p = 3$ and $m = 2$.)

(3) G is a p-group.

(4) $p = 3$, $m = 2$, and $G = A_5$.

All groups in part (1) are examples for Theorem 1.2. In parts (2) and (4), except for the groups A_4 and A_5 acting on a set of size 6, the other groups have some elements whose movements are less than m, which contradicts the fact that G has constant movement, (since $G = KP$ has constant movement m, each non-identity element $k \in K$ has $(p-1)$ cycles of length 2^s. We consider the element kk^g of K. This element is fixed point free on Ω and so has movement $p2^{s-1}$, which is a contradiction). In part (3), by Burnside's lemma, G has a fixed point free element, say g, on a set of size p^a for some positive integer a. Since every fixed point free element has order p with movement $p^a(p-1)/2$ (see [3, Proposition 4]), $o(g) = p$ and hence $\operatorname{move}(g) = p^{a-1}(p-1)/2$. But, by our assumption, G has constant movement m and so $m = p^{a-1}(p-1)/2$. Therefore, each non-identity element g of G is a fixed point free element, so that G is a regular p-group of exponent p. This completes the proof of Theorem 1.2. □

4. Intransitive examples

In this section we show that there certainly are families of examples of intransitive permutation groups with constant movement, for any prime p. First for $p = 2$, we have the following example.

EXAMPLE 4.1. Let $m = 2^{r-1} \geq 1$ and let $G := Z_2^r$. Then G has $2^r - 1 = 2m - 1$ subgroups of index 2, say $H_1, \ldots, H_{2m-1}$. For $i = 1, \ldots, 2m-1$, let Ω_i denote the set of two cosets of H_i in G, and set

$$\Omega := \bigcup_{i=1}^{2m-1} \Omega_i.$$

Then G acts faithfully on Ω by right multiplication with $2m-1$ orbits $\Omega_1, \ldots, \Omega_{2m-1}$, each of length 2. Each nontrivial element $g \in G$ lies in exactly $2^{r-1} - 1 = m - 1$ of the subgroups H_i and permutes nontrivially the remaining $m = 2^{r-1}$ points of Ω_i. Thus each nontrivial element of G has $m = 2^{r-1}$ cycles of length 2 in Ω. For any subset $\Gamma \subseteq \Omega$ and any $1 \neq g \in G$, the set $(\Gamma^g - \Gamma)$ consists of at most 1 point from each of the G-orbits on which g acts nontrivially, and hence $max|\Gamma^g - \Gamma| = m$. It follows that G has constant movement m.

The following example shows that intransitive p-groups, p odd, with constant movement do exist.

EXAMPLE 4.2. Let d be a positive integer, let $G := Z_p^d$, let $t := (p^d - 1)/(p-1)$, and let $H_1, \ldots, H_t$ be an enumeration of the subgroups of index p in G. Define Ω_i to be the coset space of H_i in G and $\Omega = \Omega_1 \cup \cdots \cup \Omega_t$. If $g \in G - \{1\}$, then g lies in $(p^{d-1} - 1)/(p-1)$ of the groups H_i and therefore acts on Ω as a permutation with $p(p^{d-1}-1)/(p-1)$ fixed points and p^{d-1} orbits of length p. Taking every second point from each of these p-cycles to form a set Γ we see that move$(g) = m \geq p^{d-1}(p-1)/2$, and it is not hard to prove that in fact move $(g) = m = p^{d-1}(p-1)/2$. Since g is non-trivial, G has constant movement $p^{d-1}(p-1)/2$.

The last example for $p = 3$, inclined to the following example not only are examples of permutation groups with constant movement equal to 3^{d-1} and 2 respectively, but also gives some positive answer to the Question 1.5 in [8].

EXAMPLE 4.3. Let $\Omega = \Omega_1 \cup \Omega_2$ be a set of size 7, such that $\Omega_1 = \{1, 2, 3\}$ and $\Omega_2 = \{1', 2', 3', 4'\}$. Moreover, suppose that $Z_2^2 \cong \langle (1'2')(3'4'), (1'3')(2'4') \rangle$ and $Z_3 \cong \langle (123)(1'2'3') \rangle$

Then the semi-direct product $G := Z_2^2 Z_3$ with normal subgroup Z_2^2 is a permutation group on a set Ω with 2-orbits which has constant movement 2, since each non-identity element of G has two cycle of length 2 or two cycle of length 3.

Finally, one may ask whether there exist further examples of intransitive groups, which have constant movement.

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