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A DOMINATED ERGODIC THEOREM FOR CONTRACTIONS WITH FIXED POINTS

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1. **Introduction.** Let $(X, \mathcal{F}, \mu)$ be a finite measure space, and let T be a contraction in real $L_p(X)$. (i.e. T is linear and $\|T\| \leq 1$). It is said that the Dominated Ergodic Theorem holds for T, if there exists a constant c_p such that, if $M(T)f(x) = \sup_n 1/n \left|\sum_{k=0}^{n-1} T^k f(x)\right|$ then $\|M(T)f\|_p \leq c_p \|f\|_p$ for every f in L_p.

It is well known that if T is a contraction in L_1 and L_∞ then the theorem holds for any p, $1 < p \leq \infty$, [3]. If T is a contraction in only one L_p, $1 < p < \infty$, the theorem is false in general [2], but Akcoglu has shown that if T is positive (i.e. $f \geq 0$ implies $Tf \geq 0$) then the theorem is true [1]. For a non-positive contraction in L_p, it is natural to ask under which extra conditions the Dominated Ergodic Theorem holds. In this note we give a partial answer to this question. Our main result is the following theorem.

THEOREM 1. *Let T be a contraction in L_p and L_∞, $1 < p < \infty$, and let's assume that there exists $g \neq 0$ such that $Tg = g$, then $\|M(T)f\|_{p'} \leq p'/p' - 1 \|f\|_{p'}$, for any f in $L_{p'}$ and $p \leq p' \leq \infty$.*

2. **Proof of the main theorem.** We split the proof in three lemmas.

LEMMA 2.1. *Let T be a contraction in L_p and in L_∞, such that $Th = h$, for same h, $|h| = 1$, then the Dominated Ergodic Theorem holds for T.*

Proof. The operator S, defined by $Sf = hT(hf)$ is a contraction in L_p and L_∞. Also $S1 = hT(h) = 1$. Now if 1_A is the indicator function of a set A, then the identity $1 = S1 = S1_A + S(1 - 1_A)$ together with $\|S\|_\infty \leq 1$, imply that S is positive. Therefore by Akcoglu's result [1] we have $\|M(S)f\|_p \leq p/p - 1 \|f\|_p$, for any f in L_p. But since $T^i f = hS^i(hf)$ it follows that $\|M(T)f\|_p \leq \|M(S)hf\|_p \leq p/p - 1 \|f\|_p$.

LEMMA 2.2. *Let T be a contraction in L_p and L_∞ such that $T^* f = f$, for some $f \neq 0$. Then the Dominated Ergodic Theorem holds for T. (T^* is the adjoint of T).*

Proof. Let h be the signum of f. Since $T^* f = f$ we have

$$0 = \int (hf - hT^* f) = \int (h - Th) f$$

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Now $|Th|\le 1$ implies $(h-Th)f\ge 0$, therefore $(h-Th)f\equiv 0$ or $h=Th$. The lemma now follows from (2.1).

LEMMA 2.3. *Let T be a contraction in L_p, $1<p<\infty$, such that $Tf=f$, f in L_p. Then* $T^*\operatorname{sign} f\,|f|^{P-1}=\operatorname{sign} f\,|f|^{P-1}$.

Proof. From Holder's inequality we know that if $p^{-1}+q^{-1}=1$, then

$$\int f\cdot g=\int |f\cdot g|=\|f\|_p\,\|g\|_q=\|f\|_p^p \quad \text{iff} \quad g=\operatorname{sign} f\,|f|^{p-1}.$$

Using that T^* is a contraction in L_q we obtain

$$\|f\|_p^p=\|f\|_p\,\|\,|f|^{p-1}\|_q\ge\|f\|_p\|\,T^*\operatorname{sign} f\,|f|^{p-1}\|_q\ge\int fT^*(\operatorname{sign} f|f|^{p-1})$$

$$=\int Tf\operatorname{sign} f\,|f|^{p-1}=\|f\|_p^p$$

Therefore all the terms are equal and $T^*(\operatorname{sign} f|f|^{p-1})=\operatorname{sign} f\,|f|^{p-1}$.

Theorem 1 now is immediate since if T is as in Theorem 1, then by (2.3) T^* has an invariant function different from zero and (2.2) gives the theorem for L_p. The rest is standard interpolation between L_p and the obvious L_∞ estimate.

3. **The L_1, $L_\#$ case**. If in Theorem 1 we assume $\|T\|_1\le 1$ instead of $\|T\|_\infty\le 1$ we obtain

$$\|M(T)f\|_p\le p^1/p'-1\,\|f\|_{p'},\qquad 1<p'\le p$$

and

$$\mu\{x;M(T)f(x)>\lambda\}\le\lambda^{-1}\int|f|,\qquad \lambda>0, f \quad \text{in} \quad L_1$$

Proof. $Tf=f$, $|f|\ne 0$ and $\|T^*\|_\infty\le 1$ imply $T^*\operatorname{sign} f=\operatorname{sign} f$. By the proof of (2.1) S^*, defined by $S^*g=\operatorname{sign} fT^*(\operatorname{sign} f\cdot g)$ is positive. Therefore S, the adjoint of S^* is positive. But clearly $Sg=\operatorname{sign} fT(\operatorname{sign} f\cdot g)$. Since S is a contraction in L_p, we can apply Akcoglu's Theorem [1] to get the Dominated Estimate for S. By [4] we have

$$\mu\{X;M(S)f(x)>\lambda\}\le\lambda^{-1}\int|f|,\qquad \lambda>0, f \quad \text{in} \quad L_1.$$

Therefore if $Th=h$, $|h|=1$, the same estimate holds for T and we get the result by interpolation.

4. **Continuous flows.** Let $\{T(t);t\ge 0\}$ be a strongly measurable semigroup of contractions in L_p and L_∞, and let's assume that $T(t)g=g$ for all t and some

fixed g, $g \neq 0$. Then if we define the maximal functions in the usual way i.e.

$$Mf(x) = \sup_t \left| t^{-1} \int_0^t T(s)f(x)\, ds \right|$$

we have $\|Mf\|_p \leq c_p \|f\|_p, f$ in L_p.

It is clear that the methods used in Theorem 1 give the result as soon as we prove a continuous version of Akcoglu's theorem. But this is an easy consequence of the results in [5]. We just observe that $0 \leq s \leq t$, $\lambda_0 = t^{-1}$ imply $t^{-1} \leq e \cdot e^{-\lambda_0 s} \cdot \lambda_0$. This means that if $\{T(t), t \geq 0\}$ is a positive semigroup of contractions in L_p, and f is positive then

$$t^{-1} \int_0^t T(s)f(x)\, dse\lambda_0 \int_0^\infty e^{-\lambda_0 s} T(s)f(x)\, ds \leq e \sup_\lambda \lambda \int_0^\infty e^{-\lambda s} T(s)f(x)\, ds = ef^*(x)$$

where $f^*(x)$ is defined by the last equality. Now by Lemma 5 in [5] we have $\|f^*\|_p \leq p/p-1 \|f\|_p$, and therefore

$$\|Mf\|_p \leq e \|f^*\|_p \leq c_p \|f\|_p.$$

Finally we want to point out, that the same theorems can be obtained by the same methods, using the maximal operator associated to Abel means.

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