ON THE INVARIANCE OF A QUOTIENT GROUP OF THE CENTER OF $F/[R,R]$

Trueman MacHenry

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1. Let F be a free group of rank $\geqslant 2$, let $F/R \cong \Pi$, and let $F_o = F/[R,R]$. Auslander and Lyndon showed that the center F_o^* of F_o is a subgroup of $R/[R,R] = R_o$, and that it is non-trivial if and only if Π is finite [1, corollary 1.3 and theorem 2]. In this paper it will be shown that there is a canonically defined (and not always trivial) quotient group of the center of F_o which depends only on Π. This result provides a dual to the well-known result of Baer [2] and Hopf [6] that $H_2(\Pi,J) \cong R \cap F'/[R,F]$, where J is the ring of integers and $F' = [F,F]$. $H_2(\Pi,J)$ is a quotient group of $Z = R_o \cap F_o'$ while the group discussed here is a quotient group of $D = R_o \cap F_o^* = F_o^*$.

In order to state the main results we let Π be a finite group and denote by P the subgroup of R_o whose elements are products of all conjugates of an element in R_o by distinct coset representatives of R_o in F_o. Thus, regarding R_o as a Π-module under the operation induced by the inner automorphisms of F_o acting on R_o, $P = \{r \in R_o | r = \sum_{\alpha\in\Pi} \alpha r_o,\ r_o \in R_o\}$. Clearly, $P \leqslant F_o^*$. For an arbitrary group Π we define

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$$K = \begin{cases} D/P & \text{if } \Pi \text{ is finite} \\ D & \text{if } \Pi \text{ is infinite.} \end{cases}$$

THEOREM 1. $K \cong H_1(\Pi, J)$ <u>if</u> Π <u>is finite</u>;

$K = \langle 1 \rangle$ <u>if</u> Π <u>is infinite and</u> rank $F > 1$.

Thus, for finite Π, $K \cong \Pi/\Pi'$.

Next, let Π be finite and let $T: F \to R_0$ be the transfer map of F to R_0 [5, p. 201]. If TR is the image of the restriction of T to R, then

THEOREM 2.[1)] i) $TF = D$,

ii) $TR = P$,

iii) $TF/TR = K$.

Finally, with Z as defined above,

THEOREM 3. $Z \cap D = \langle 1 \rangle$.

Thus no central element in F_0 can be written as a non-trivial product of commutators.

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1) That $TF = D$ has been proved independently by A. Karrass and D. Solitar, by H. Neumann, and by M. Ojanguren [9, Satz 6.2] using different methods from ours. The first two of these proofs have not been published.

2. Proofs of theorems 1 and 2. In the course of these proofs we use the following notation:

$J\Pi$ is the integral group ring of Π;

$\mathfrak{f}$ the fundamental ideal of $J\Pi$, i.e., $\mathfrak{f} = \mathrm{id}\langle 1 - \alpha, \alpha \in \Pi \rangle$;

$\mathfrak{s}$ the trace ideal, i.e., $\mathfrak{s} = \mathrm{id}\langle \Sigma\alpha, \alpha \in \Pi \rangle$;

T is a right transversal of R in F with 1 representing R. T is chosen to be a (two-sided) Schreier system [8, p. 93].

For any group G, $G' = [G,G]$ and $G^* =$ center of G.

LEMMA 1. $\mathcal{T}F/\mathcal{T}R \cong \Pi/\Pi'$, where Π is a finite group.

Proof. We suppose Π finite and let $F = \langle x_i \rangle_{i \in I} = X$, where X is a free generating set of F of cardinality greater than 1, and let T be a right Schreier transversal of R in F, then

$$\mathcal{T}x = \prod_{t \in T} tx\overline{tx}^{-1} \bmod R' = \prod_{t \in T} (t,x) \quad , \quad x \in X,$$

where $\overline{tx}$ is the representative in T of tx, and $(t,x) = tx\overline{tx}^{-1} \bmod R'$, [see, e.g., 5, 14.2.4]. Since $\mathcal{T}$ is a homomorphism, $\mathcal{T}x_i$ generates $\mathcal{T}F$. Since R_o is free abelian, so is $\mathcal{T}F$. Moreover, since exactly $|T| - 1$ of the elements of (t,x), $t \in T$, $x \in X$, are the identity [8, theorem 2.10], $\mathcal{T}x_i \neq 1$ for any $i \in I$. Because $\{(t,x) | t \in T, x \in X, (t,x) \neq 1\}$ is a free generating set for R [8, theorem 2.9], $\{\mathcal{T}x_i\}_{i \in I}$ is a free-abelian generating set for $\mathcal{T}F$. Hence the free-abelian rank of TF is equal to the free rank of F and so $TF \cong F/F'$. The mapping $Tx_i \overset{\theta}{\to} x_i \bmod F'$ determines such an isomorphism; call it θ also. θ sends R to RF'/F', and with the aid of the third isomorphism theorem we have finished the proof of the lemma.

Next let $\psi: F \to \Pi$ be an epimorphism with Kernel R. In order to prove theorem 2, it is convenient to choose a particular representation for F_o, namely, the Magnus representation: If M is the free Π-module with a free generating set $\{s_x | x \in X\}$, then the set of matrices of the form $\begin{pmatrix}\alpha & m\\ 0 & 1\end{pmatrix}$, $\alpha \in \Pi$, $m \in M$, form a group E which is the splitting extension of the Π-module M by the group Π. The matrices of the form $\begin{pmatrix}\Psi x & s_x\\ 0 & 1\end{pmatrix}$ generate a subgroup of E isomorphic to F_o [7]. The subgroup of E representing R_o belongs to M. With this representation of F_o in mind we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & R_o & \to & F_o & \to & \Pi & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & M & \to & E & \to & \Pi & \to & 1 \\ & & & & & & \downarrow & & \\ & & & & & & 1 & & \end{array} .$$

We shall abbreviate the matrices $\begin{pmatrix}\alpha & m\\ 0 & 1\end{pmatrix}$ to (α, m).

Now M is a Π-derivation module for F (in fact, for JF) determined by a Π-derivation Δ of F to M, i.e., a map $\Delta: F \to M$ such that $\Delta x = s_x$, $x \in X$, and $\Delta(fg) = \Psi f \Delta g + \Delta f$, $f, g \in F$ [cf. 4 and 3, chapter 14, problems 11-13]. Given an element $f \in F$, its Magnus representative[2)] will be $(\Psi f, \Delta f)$.

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2) The element Δf is a homomorphic image of the Fox derivative of Δf, and the coefficient of s_x in f is a homomorphic image of the partial of f with respect to x (see [4]).

LEMMA 2. $\Delta tx\overline{tx}^{-1} = -\Delta\overline{tx} + \Psi t\ s_x + \Delta t,\ t \in T,\ x \in X.$

Thus the Magnus representation of R_o is generated by $\{(1, -\Delta tx + \Psi t\ s_x + \Delta t) \mid x \in X,\ t \in T\}$.

LEMMA 3. TF *is represented in* M *by elements of the form* (1,m) *where* $m \in \mathfrak{s}M$, *i.e.*, m *belongs to the submodule of* M *whose coefficients lie in* $\mathfrak{s}$.

Proof. Using lemma 2,

$$\Delta \prod_{t\in T} tx\overline{tx}^{-1} = \sum_{t\in T} (-\Delta\overline{tx} + \Psi t\ s_x + \Delta t) = \sum_{\alpha\in\Pi} \alpha s_x\ .$$

We have proved more, namely,

COROLLARY L3. $\mathfrak{s}M$ *lies in* R_o.

LEMMA 4. *Let* $1 \to A \to B \to C \to 1$ *be a splitting extension of an abelian group* A *by a group* C. *If* $b \in B$ *is written canonically as* $b = (c,a)$, *then the center of* B *consists just of those elements* $b^* = (c^*,a^*)$ *such that* $c^* \in C^*$, *the center of* C, *and* $c^*\cdot a = a$ *for all* $a \in A$ *and* $c\cdot a^* = a^*$ *for all* $c \in C$, A *being regarded as a left C-module whose action is determined by the extension*.

The proof is straightforward and will therefore be omitted.

LEMMA 5. *The annihilator in* $J\Pi$ *of* $\mathfrak{f}$ *is* $\mathfrak{s}$.

Proof. Plainly $\mathfrak{s}$ < annihilator of $\mathfrak{f}$. On the other hand $(\sum_{\alpha\in\Pi} k_\alpha \alpha)(1 - \gamma) = o$, $k_\alpha \in J$, implies that $k_\alpha = k_{\alpha\gamma^{-1}}$, for all $\alpha \in \Pi$.

Denoting the center of E by E*, we have

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LEMMA 6. $E^* = \mathcal{S}M$.

Proof. Elements of E can be represented canonically in the form (γ, m), $\gamma \in \Pi$, $m \in M$ with multiplication

$$(\gamma, m)(\gamma', m') = (\gamma\gamma', m + \gamma m').$$

Since M is free, by lemma 4 $E^* \leqslant M$ and consists of those elements $m^* = \sum u_x s_x$, $u_x \in J\Pi$, $x \in X$, such that $\gamma m^* = m^*$ for all $\gamma \in \Pi$. Thus we demand that $\gamma u_x = u_x$. By lemma 5, $u_x \in \mathcal{S}$. Since $\sum_{\alpha \in \Pi} \alpha\, s_x \in E^*$, the proof is complete.

Combining corollary L3 and lemma 6, we have

COROLLARY L6. $E^* \leqslant R_o$.

But E^* consists precisely of those elements of R_o left fixed by the action of Π. Hence $E^* = D$ [1, corollary 1.4]. By lemma 3 $\mathsf{T}F = D$.

The proof of theorem 2 will be complete if we can show that $\mathsf{T}R = P$. However this follows easily from [5, p. 206, lemma 14.4.1] or we may compute directly, using the Magnus representation, that $\Delta t r \overline{tr}^{-1} = -\Delta \overline{tr} + \Psi t \Delta r + \Delta t$. Hence $\Delta \prod_{t \in T} t r \overline{tr}^{-1} = \sum_{\alpha \in \Pi} \alpha \Delta r$. But P is represented in M by $\{(1, \sum_{\alpha \in \Pi} \alpha \Delta r),\ r \in R\}$. Thus theorem 2 is proved.

If Π is finite and the rank of $F \geqslant 2$, theorem 1 follows from theorem 2 and lemma 1. If the rank of F is 1 and Π is finite, then the result is obvious; however, theorem 2 now holds only with the weaker conclusion that $\mathsf{T}F/\mathsf{T}R \simeq K$. If Π is infinite, then $K = D = \langle 1 \rangle$ [1, theorem 2] if the rank of F is greater than 1. This completes the proof of theorem 1.

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3. Proof of theorem 3.

LEMMA 7. $\Delta F' \leqslant \mathfrak{f}M$, *i.e.*, $\Delta F'$ *is contained in the submodule of* M *whose coefficients lie in* $\mathfrak{f}$.

Proof. First we notice that if $f, g \in F'$, then

$$\Delta fg = \Psi f \Delta g + \Delta f \in \mathfrak{f}M.$$

Thus it is sufficient to show that $[f,g] \in \mathfrak{f}M$, $f, g \in F$. But direct computation shows that

$$\Delta[f,g] = (1 - \Psi(fgf^{-1}))\Delta f + (\Psi f - \Psi[f,g])\Delta g.$$

LEMMA 8. $\mathfrak{f} \cap \mathfrak{d} = \langle o \rangle$.

To prove theorem 3 we first observe that $Z = R_o \cap F'_o \leqslant \Delta F'$ and then that the map $s_x \to 1$ determines a Π-homomorphism $M \to J\Pi$, where $J\Pi$ is regarded as a left Π-module. Under this homomorphism $\mathfrak{f}M \to \mathfrak{f}$ and $\mathfrak{d}M \to \mathfrak{d}$. Since $Z \leqslant \mathfrak{f}M$ (lemma 7) and $D = \mathfrak{d}M$ (lemma 6 and remark ff. lemma 6), by lemma 8 it follows that $\mathfrak{f}M \cap \mathfrak{d}M = \langle o \rangle$, and hence that $Z \cap D = \langle 1 \rangle$.

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Adelphi University
Garden City
New York

and

York University
Toronto

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