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BOUND SETS IN PARTIAL ORDERS AND THE FIXED POINT PROPERTY

BY
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ABSTRACT. In this paper we introduce several properties closely related to the fixed point property of a partially ordered set P: the comparability property, the fixed point property for cones, and the fixed point extension property. We apply these properties to the sets of common bounds of the minimal (maximal) elements of the partially ordered set P in order to derive fixed point theorems for P.

In this paper we develop sufficient and also an equivalent condition for the fixed point property of posets. These conditions will be expressed in terms of certain sets of common bounds. Earlier results in this direction which imply the fixed point property imposed a variety of restrictions on the poset; the set of minimal elements is finite and such that each non-empty subset has a supremum, Theorem 2 in [4]; the poset is finite and all common bound sets of non-empty subsets of the set of maximal elements have the fixed point property, Theorem 2 in [3]; the poset is finite and admits a cutset such that every non-empty subset has an infimum or a supremum, Corollary 2.6 in [1].

A straightforward argument which goes back to Theorem 2 in [2] shows that in a poset which satisfies the fixed point property every maximal chain is complete; in particular, every element of the poset is in between a minimal and a maximal element of the poset. Therefore, it seems natural to start with the sets of extremal elements of a poset in order to gain insight into the fixed point property.

1. **Bound sets and fixed point properties.** Let P be a poset. $\mathrm{Max}(P) = \{p \in P | \forall q \in P\ q \geqq p \Rightarrow q = p\}$ denotes the set of *maximal* elements and $\mathrm{Min}(P) = \{p \in P | \forall q \in P\ q \leqq p \Rightarrow q = p\}$ the set of *minimal* elements of P. We shall call P *chain closed* if every maximal chain C in P is a complete lattice; *chain bounded* if every chain in P has a lower and an upper bound; and *element bounded* if for every $p \in P$ there is $u \in \mathrm{Max}(P)$ and $1 \in \mathrm{Min}(P)$ such that $1 \leqq p \leqq u$. It is obvious that the following implications hold: chain closed $\Rightarrow$

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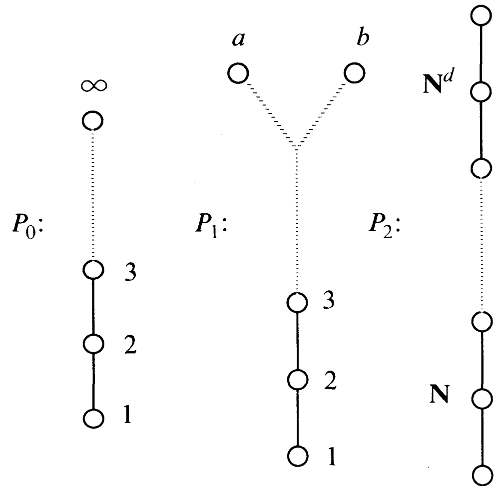
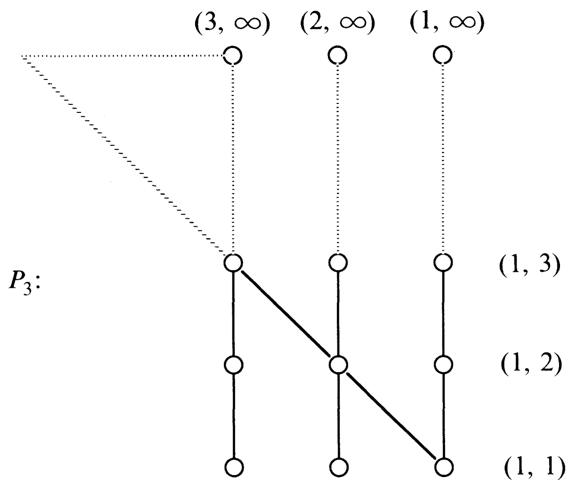
chain bounded $\Rightarrow$ element bounded. The example below shows that the reverse implications are false.

EXAMPLE 1.1. Let $\mathbf{N}$ denote the natural numbers, $\mathbf{N}^d$ their dual and define $P_0 = \mathbf{N} \cup \{\infty\}$ where $n < \infty$, for all $n \in \mathbf{N}$; $P_1 = \mathbf{N} \cup \{a, b\}$ where $n < a$, $n < b$, $a \| b$, and $a \| b$ means $a \nleq b$; and $a \ngeq b$; $P_2 = \mathbf{N} \cup \mathbf{N}^d$ where for all $m \in \mathbf{N}$ and $n \in \mathbf{N}^d$ $m < n$; $P_3 = \cup \{ \{n\} \times P_0 | n \in \mathbf{N}\}$ is a disjoint union with the order inherited in each fiber P_0 and where the only other order relations between elements are given through the "diagonal" elements, $(m, m) < (n, n)$ in P_3 whenever $m < n$.

P_3 is element bounded, but not chain bounded, and P_2 is chain bounded, but not chain closed, while P_1 is chain closed.





For an arbitrary subset $Q \subset P$ we define the following kinds of *bound sets* of Q:

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(1) $I(Q) = \{p \in P | \exists q \in Q\ p \leqq q\}$ the *ideal* generated by Q,
(2) $F(Q) = \{p \in P | \exists q \in Q\ p \geqq q\}$ the *filter* generated by Q,
(3) $\text{cone}(Q) = I(Q) \cup Q \cup F(Q)$ the *cone* generated by Q,
(4) $B^{<}(Q) = \{p \in P | \forall q \in Q\ p \leqq q\}$ the *lower bound set* of Q,
(5) $B^{>}(Q) = \{p \in P | \forall q \in Q\ p \geqq q\}$ the *upper bound set* of Q,

and if $L, U \subset P$ are two subsets:

(6) $B(L, U) = \{p \in P | \forall l \in L\ \forall u \in U\ l \leqq p \leqq u\}$ the *bound set* of the pair of sets L and U.

Using the notion of bound set we define the following setsystems for the poset P:

$$\mathscr{L}(P) = \{B^{<}(U) | \emptyset \neq U \subset \text{Max}(P), B^{<}(U) \neq \emptyset\}$$

$$\mathscr{U}(P) = \{B^{>}(L) | \emptyset \neq L \subset \text{Min}(P), B^{>}(L) \neq \emptyset\}$$

$$\mathscr{B}(P) = \{B(L, U) | \emptyset \neq L \subset \text{Min}(P), \emptyset \neq U \subset \text{Max}(P), B(L, U) \neq \emptyset\}.$$

Each of the setsystems is a partial order under set inclusion. Every map $f:P \to P$ induces a map $\mathscr{B}(f):\mathscr{B}(P) \to \mathscr{B}(P)$ in a natural manner as follows. For $S \in \mathscr{B}(P)$, put $L_f = B^{<}(f(S)) \cap \text{Min}(P)$ and $U_f = B^{>}(f(S)) \cap \text{Max}(P)$; then neither set is empty and $\emptyset \neq f(S) \subset B(L_f, U_f)$. Thus $\mathscr{B}(f)(S) := B(L_f, U_f)$ is a well-defined map. Using only the appropriate half of the construction above we obtain the maps $\mathscr{L}(f):\mathscr{L}(P) \to \mathscr{L}(P)$ and $\mathscr{U}(f):\mathscr{U}(P) \to \mathscr{U}(P)$.

Straightforward calculations establish

LEMMA 1.1. *If $f:P \to P$ is order preserving then so are $\mathscr{B}(f)$, $\mathscr{L}(f)$, and $\mathscr{U}(f)$.*

In the context of fixed points of order preserving maps we need the following properties of posets. Poset P has the *fixed point property* (fpp) if every order preserving map $f:P \to P$ has a fixed point, i.e. there is $p \in P$ so that $f(p) = p$. Poset P has the *comparability property* (cpp) if every order preserving map $f:P \to P$ has a point of comparability, i.e. there is $p \in P$ so that $f(p) \sim p$, i.e. $f(p) \leqq p$ or $f(p) \geqq p$. Poset P has the *cone fixed point property* (cfpp) if for every element $p \in P$ each order preserving map $f:I(p) \to I(p)$ has a fixed point and each order preserving map $f:F(p) \to F(p)$ has a fixed point. A subset $Q \subset P$ has the *fixed point extension property* (fpep) if for every order preserving map $f:Q \to Q$ each order preserving extension $g:P \to P$ of f, i.e. $g|Q = f$, has a fixed point.

We collect some simple facts about these notions.

LEMMA 1.2. *Let P be a poset.*

(a) *P has* fpp *if and only if $\emptyset$ has* fpep.

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(b) *P has* fpp *then P has* cpp.
(c) *P has* fpp *then P has* cfpp.

PROOF. Parts (a) and (b) are obvious. For (c), let $p \in P$ and let $f:I(p) \to I(p)$ be order preserving. Define the map $g:P \to P$ by $g(x) = f(x)$, for $x \in I(p)$, and $g(x) = p$ otherwise. Then g is order preserving and its fixed points are those of f. □

THEOREM 1.1. *Let P be a poset. P has* fpp *if and only if P has* cpp *and P has* cfpp.

PROOF. The necessity of the two conditions follows from Lemma 1.2. Let now $f:P \to P$ be order preserving. Property cpp guarantees $p \in P$ so that $p \sim f(p)$. Suppose that $p < f(p)$, then $f(F(p)) \subset F(p)$ so that by cfpp f has a fixed point in $F(p)$. □

THEOREM 1.2. *Let P be a poset. If P has* cfpp *then P is chain closed.*

PROOF. Let $C \subset P$ be a maximal chain and let $\emptyset \neq D \subset C$ be a subchain. Pick a well-ordered cofinal subchain $\bar{D} \subset D$ with first element d_0 and a dually well-ordered cofinal subchain $\bar{E} \subset C - B^{>}(D)$. Assume that $B(\bar{D}, \bar{E}) = \emptyset$. Define a map $f:I(d_0) \to I(d_0)$ as follows

$$f(x) = \begin{cases} d, \text{ if } d \in \bar{D} \text{ is the smallest } p \in \bar{D} \text{ such that } p \nleqq x \\ e, \text{ if } e \in \bar{E} \text{ is the largest } p \in \bar{E} \text{ such that } p \ngeqq x, \text{ and} \\ \quad \text{if for all } d \in \bar{D}\ d \leqq x. \end{cases}$$

Obviously, f is order preserving and fixed point free which contradicts the hypothesis. Thus, $B(\bar{D}, \bar{E}) \neq \emptyset$ holds so that the maximality of C implies $B(\bar{D}, \bar{E}) \subset C$. Hence $B(\bar{D}, \bar{E})$ contains a single element which is the supremum $\sup_C D$. The dual argument shows the existence of $\inf_C D$.

If $D = \emptyset$, then let $\bar{C} \subset C$ be a well-ordered cofinal subchain of C with first element c_0. As with f above, define $g:I(c_0) \to I(c_0)$ as $g(x) = c$, if $c \in \bar{C}$ is the smallest $p \in \bar{C}$ such that $p \nleqq x$. Following the argument above we see that C has a largest element. The dual construction establishes a smallest element for C. □

LEMMA 1.3. *Let P be a poset and let $Q \subset P$ be arbitrary.*

(a) *Q has* fpp *then Q has* fpep.
(b) *P has* fpp *then Q has* fpep.
(c) *cone*(*Q*) *has* fpp *then Q has* fpep.

PROOF. Parts (a) and (b) are obvious. For (c) let $f:Q \to Q$ be order preserving and $g:P \to P$ be an extension of f. Then $g(\text{cone}(Q)) \subset \text{cone}(Q)$ and thus has a fixed point by hypothesis. Hence Q has fpep. □

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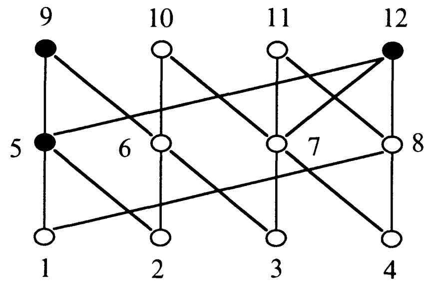
Simple examples show that the converses of Lemma 1.2 (b) and (c) and of Lemma 1.3 (a), (b) and (c) are false. The converse of Lemma 1.3 (c) is false even if Q is a bound set.

EXAMPLE 1.2. Let P be the poset drawn below and let $Q = \{5, 9, 12\} = B^{>}(\{1, 2\}) \in \mathscr{U}(P)$. Then Q has fpep since it has fpp. On the other hand, cone(Q) $= I(Q)$ has the 4-element crown $\{5, 3, 9, 12\}$ as a retract and thus fails to have the fpp. The retraction is indicated by arrows.



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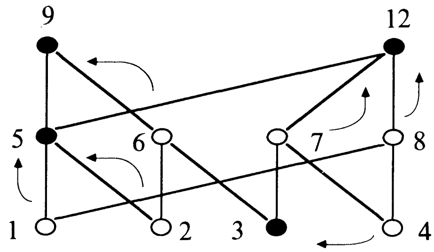

cone(Q)

An interesting connection between the three fixed point properties is given in the next lemma. We shall call a subset $Q \subset P$ *order autonomous*

$$\text{if } \forall p \in P - Q(\exists q \in Q\, p < q \Rightarrow \forall q \in Q\, p < q)$$

$$\text{and } \forall p \in P - Q(\exists q \in Q\, p > q \Rightarrow \forall q \in Q\, p > q) \text{ holds.}$$

Thus a subset $Q \subset P$ is order autonomous exactly when it is a lexicographic component in a representation of P as a lexicographic sum with one-element components in $P - Q$.

LEMMA 1.4. *Let $Q \subset P$ be order autonomous and suppose that Q has* cfpp. *Then the following statements are equivalent*

(1) *Q has* fpp *or $B^{<}(Q)$ has* fpp *or $B^{>}(Q)$ has* fpp
(2) *Q has* fpep.

PROOF. If $Q = \emptyset$ then the equivalence follows from Lemma 1.2 (a) and Lemma 1.3 (b), and we may assume that $Q \neq \emptyset$.
(1) $\Rightarrow$ (2). Let $f:Q \to Q$ be order preserving and let $g:P \to P$ be an extension of f. Suppose, for example, that $g(B^{>}(Q)) \not\subset B^{>}(Q)$, i.e. for some $x \in B^{>}(Q)$, $g(x) \notin B^{>}(Q)$. Then for all $y \in Q$, $g(x) \geqq f(y)$ implies $g(x) \in Q$ since Q is order autonomous. Thus, $x > g(x)$ and $g(E) \subset E$ where $E = I(\{g(x)\}) \cap Q$. Since Q has cfpp, g has a fixed point. Likewise we produce a fixed point of g if $g(B^{<}(Q)) \not\subset B^{<}(Q)$.
(2) $\Rightarrow$ (1). Suppose that $f:Q \to Q$, $f^{<}:B^{<}(Q) \to B^{<}(Q)$ and $f^{>}:B^{>}(Q) \to B^{>}(Q)$ are order preserving and without fixed points. Pick some element $q \in Q$

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and define $g:P \to P$ as

$$g(p) = \begin{cases} f(p) & \text{if } p \in Q \\ f^{<}(p) & \text{if } p \in B^{<}(Q) \\ f^{>}(p) & \text{if } p \in B^{>}(Q) \\ q & \text{otherwise.} \end{cases}$$

Since Q is order autonomous, g is order preserving. By construction g extends f and has no fixed points so that fpep fails for Q.

2. **Fixed point theorems.** One of the limitations of Theorem 2 in [3] aside from its finiteness hypothesis is that it fails as soon as one of the bound sets is empty, in particular if this applies to $B^{<}(\mathrm{Max}(P))$. The systems of bound sets introduced in section 1 sidestep this problem since they contain only non-empty bound sets.

THEOREM 2.1. *Let P be a poset satisfying*

(1) *P is element bounded*
(2) *$\mathscr{L}(P)$ has* fpp, *and*
(3) *for each $S \in \mathscr{L}(P)$, S has* fpp

then P has fpp.

PROOF. Let $f:P \to P$ be order preserving and let $\mathscr{L}(f):\mathscr{L}(P) \to \mathscr{L}(P)$ be the induced map. By hypothesis (2), $\mathscr{L}(f)$ has a fixed point $\emptyset \neq S = \mathscr{L}(f)(S) = B^{<}(U_f)$. Now, for all $u \in U_f$ and all $s \in S$, $f(s) \leqq u$ holds, but that means $f(s) \in B^{<}(U_f)$, for all $s \in S$. Therefore, $f(s) \in B^{<}(U_f) = S$ holds and hypothesis (3) establishes a fixed point for f. □

Obviously, the theorem is also true for the operators $\mathscr{B}$ and $\mathscr{U}$; in addition, for the operators $\mathscr{L}$ and $\mathscr{U}$ one only needs to demand the appropriate one-sided version of "element bounded". Note that if P is finite then it is element bounded and if, in addition, $B^{<}(\mathrm{Max}(P)) \neq \emptyset$, then $\mathscr{L}(P)$ has a least element and hence fpp.

COROLLARY 1. (Duffus, Poguntke, Rival)
If P is finite, $B^{<}(\mathrm{Max}(P)) \neq \emptyset$, and for each $S \in \mathscr{L}(P)$, S has fpp, *then also P has* fpp.

If $\mathrm{Min}(P)$ is finite then so is $\mathscr{U}(P)$, and if $B^{>}(\mathrm{Min}(P)) \neq \emptyset$ then $\mathscr{U}(P)$ has fpp. Therefore Theorem 2.1 generalizes Corollary 5.3 in [1].

COROLLARY 2. (Baclawski, Björner)
If P is element bounded, $\mathrm{Min}(P)$ *finite, $B^{>}(\mathrm{Min}(P)) \neq \emptyset$, and S has* fpp, *for each $S \in \mathscr{U}(P)$, then P has* fpp.

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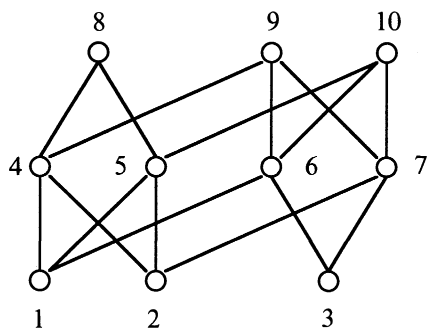
On the other hand, Theorem 2.1 for operator $\mathscr{U}$ is a special case of Theorem 5.2 in [1] for the class of those posets P which satisfy $B^{>}(\mathrm{Min}(P)) \neq \emptyset$. In order to see this, let L be the dual of the power set of poset P and let $F: P \to L$ be defined as $F(p) = B^{>}(I(p) \cap \mathrm{Min}(P))$. Then $\mathscr{U}(P)$ is isomorphic to the (join) sub-semilattice L_M of L generated by $F(\mathrm{Min}(P))$. Now the conditions (a)-(d) of Theorem 5.2 in [1] are the same as the conditions (1)-(3) of Theorem 2.1.

The connections between the two theorems for the class of posets P which satisfy $B^{>}(\mathrm{Min}(P)) = \emptyset$ are not clear.

Condition (3) of Theorem 2.1 is too strong as the following example shows.

EXAMPLE 2.1. Let P be the partial order drawn below. P has fpp and $\mathscr{L}(P)$ has a least element $B^{<}(\{8, 9, 10\}) = \{1, 2\}$ which does not have fpp. All other bound sets in $\mathscr{L}(P)$ have fpp. It is easy to check that $\{1, 2\}$ has fpep. □



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This observation together with Lemma 1.3 (a) leads to the following generalization of Theorem 2.1.

THEOREM 2.2. *Let P be a poset satisfying*

(1) *P is element bounded,*
(2) *$\mathscr{L}(P)$ has* fpp, *and*
(3) *for each $S \in \mathscr{L}(P)$, S has* fpep,

then P has fpp.

PROOF. As in the proof of Theorem 2.1 there is $S \in \mathscr{L}(P)$ such that $f(S) \subset S$. Since f is an extension of $f|S$, f has a fixed point by hypothesis (3). □

As before, analogous versions of Theorem 2.2 hold for the operators $\mathscr{U}$ and $\mathscr{B}$. Applying Lemma 1.3 (b), we obtain the following characterization of the fixed point property.

COROLLARY. *Let P be element bounded and suppose that $\mathscr{L}(P)$ has* fpp. *Then the following statements are equivalent*

(1) P has fpp
(2) each bound set $S \in \mathscr{L}(P)$ has fpep.

Note that because of Theorem 1.2 the hypothesis "P is element bounded" in the theorems of this section imposes no restrictions whatsoever.

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