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# On the Comaximal Graph of a Commutative Ring

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*Abstract.* Let $R$ be a commutative ring with 1. In a 1995 paper in J. Algebra, Sharma and Bhatwadekar defined a graph on $R$, $\Gamma(R)$, with vertices as elements of $R$, where two distinct vertices $a$ and $b$ are adjacent if and only if $Ra + Rb = R$. In this paper, we consider a subgraph $\Gamma_2(R)$ of $\Gamma(R)$ that consists of non-unit elements. We investigate the behavior of $\Gamma_2(R)$ and $\Gamma_2(R)\setminus \mathrm{J}(R)$, where $\mathrm{J}(R)$ is the Jacobson radical of $R$. We associate the ring properties of $R$, the graph properties of $\Gamma_2(R)$, and the topological properties of $\mathrm{Max}(R)$. Diameter, girth, cycles and dominating sets are investigated, and algebraic and topological characterizations are given for graphical properties of these graphs.

## 1 Introduction

The study of algebraic structures by way of graph theory has become an exciting research topic in the last decade. There are many papers on assigning a graph to a ring. In [5], Beck introduced the idea of a zero-divisor graph of a commutative ring $R$ with 1. He defined $\Gamma_0(R)$ to be the graph whose vertices are elements of $R$ and in which two vertices $a$ and $b$ are adjacent if and only if $ab = 0$. In [4], Anderson and Livingston introduced and studied the subgraph $\Gamma_0(R)$ whose vertices are the non-zero zero-divisors, and the authors studied the interplay between the ring-theoretic properties of a commutative ring and the graph-theory properties of its zero-divisor graph. The total graph of a ring was introduced in [2] and was also investigated in [1].

In [14], Sharma and Bhatwadekar defined a graph $\Gamma(R)$, with elements of $R$ as vertices and where two distinct vertices $a$ and $b$ are adjacent if and only if $Ra + Rb = R$. With this definition, they showed that $\chi(\Gamma(R)) < \infty$ if and only if $R$ is a finite ring, where $\chi(G)$ is the chromatic number of a graph $G$. Later, Maimani et al. [10] characterized the connectedness and the diameter of the graph $\Gamma_2(R) \setminus J(R)$, where $\Gamma_2(R)$ is the subgraph of $\Gamma(R)$ induced by non-unit elements and $\mathrm{J}(R)$ is the Jacobson radical of $R$. Recently, Wang [15] characterized those rings $R$ for which $\Gamma_2(R)$ is a forest and those rings $R$ for which $\Gamma_2(R) \setminus \mathrm{J}(R)$ is Eulerian. He found all finite rings $R$ such that the genus of $\Gamma_2(R)$ (resp., $\Gamma(R)$) is at most one. He also studied the comaximal graph of a non-commutative ring [16]. The goal of this paper is to study the behavior of $\Gamma_2(R)$ (resp., $\Gamma_2(R)\setminus \mathrm{J}(R)$). Inasmuch as the subgraph of $\Gamma_2(R)$ whose vertices are in $\mathrm{J}(R)$ is the empty graph, some results $\Gamma_2(R)$ are valid for $\Gamma_2(R) \setminus \mathrm{J}(R)$.

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For example, when $R$ is a Gelfand ring,

$$\operatorname{diam}\Gamma_2(R) = \operatorname{diam}(\Gamma_2(R) \setminus \mathrm{J}(R)) = \min\{|\operatorname{Max}(R)|, 3\}.$$

In the third section we study cycles in $\Gamma_2(R) \setminus \mathrm{J}(R)$ and characterize when $\Gamma_2(R) \setminus \mathrm{J}(R)$ is triangulated or hypertriangulated. We prove that $\Gamma_2(R) \setminus \mathrm{J}(R)$ is a triangulated graph if and only if $\operatorname{Max}(R)$ has no isolated points. Also, when $R$ has no strongly isolated maximal ideal, every cycle in $\Gamma(R)$ has length 3 or 4 and every edge of $\Gamma(R)$ is an edge of a cycle with length 3 or 4.

It is interesting that some results of this paper for the graph $\Gamma_2(R) \setminus \mathrm{J}(R)$ are similar to the results in [12] for the zero divisor graph of $R$. But note that these two graphs usually are not isomorphic (even the number of vertices can be different). In fact if the zero divisor graph of $R$ is isomorphic to $\Gamma_2(R) \setminus \mathrm{J}(R)$, then $R$ must be a quasi regular ring, *i.e.*, every element of $R$ is either a unit or a zero divisor.

Throughout this paper, $R$ is a commutative ring, $|R| \neq 4$, and it is not a local ring (when $R$ is local, $\Gamma_2(R) \setminus \mathrm{J}(R) = \varnothing$). We say $R$ is *semiprimitive* if $\bigcap \operatorname{Max}(R) = (0)$. For any ideal $I$ of $R$ and $a \in R$, we set

$$\mathrm{M}(a) = \{M \in \operatorname{Max}(R) : a \in M\}, \quad \mathrm{D}(a) = \operatorname{Max}(R) \setminus \mathrm{M}(a).$$

Then the sets $\mathrm{M}(I) = \bigcap_{a\in I} \mathrm{M}(a)$, where $I$ is an ideal of $R$, satisfy the axioms for the closed sets of a topology on $\operatorname{Max}(R)$, called the *Stone topology* (see [9, 7M]). The operators cl and int denote the closure and the interior in $\operatorname{Max}(R)$.

A ring $R$ is called *Gelfand* (*pm*-ring) if every prime ideal of $R$ is contained in a unique maximal ideal. When the Jacobson radical and the nilradical of a ring $R$ coincide, DeMarco and Orsatti [6] show that $R$ is Gelfand if and only if $\operatorname{Max}(R)$ is Hausdorff and if and only if $\operatorname{Spec}(R)$ is normal (in general, not Hausdorff). This class of rings contains the class of von Neumann regular ring, local rings, zero dimensional rings, and the rings $C(X)$ of continuous functions.

A maximal ideal $M$ of $R$ is called *isolated* if $\{M\}$ is a clopen (closed and open) subset of $\operatorname{Max}(R)$. If $R$ is semiprimitve, $M$ is isolated if and only if $M = (e)$, for some idempotent $e \in R$ (see [13, Theorem 2.6]).

We first need the following lemmas.

***Lemma 1.1*** *Let $R$ be a ring. If $A$ and $B$ are disjoint closed subsets of* $\operatorname{Max}(R)$*, then there exists $a \in R$ such that $A \subseteq \mathrm{M}(a)$ and $B \subseteq \mathrm{M}(a-1)$. Furthermore, if $A$ is clopen, then there exists $a \in R$ such that $A = \mathrm{M}(a)$, $A^c = \mathrm{M}(a-1)$ and $a^2 - a \in \mathrm{J}(R)$.*

**Proof** There are the ideals $I$ and $J$ such that $A = \mathrm{M}(I)$ and $B = \mathrm{M}(J)$. Obviously, $I + J = R$ (for $A \cap B = \varnothing$), so $a + b = 1$ for some $a \in I$ and $b \in J$. Thus $A \subseteq \mathrm{M}(a)$ and $B \subseteq \mathrm{M}(a-1)$. The second part is trivial. ■

The following lemma is well known.

***Lemma 1.2*** *Let $R$ be a semiprimitive ring. Then $R$ is a zero dimensional ring if and only if* $\mathrm{M}(a)$ *is a clopen subset of* $\operatorname{Max}(R)$ *for each $a \in R$.*

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**Proof** Suppose that $R$ is a zero dimensional ring, hence $R$ is von Neumann regular. So for every $a \in R$, there exists $b \in R$ such that $a = a^2b$. Therefore $\mathrm{M}(a) \cup \mathrm{M}(1-ab) = \mathrm{Max}(R)$, *i.e.*, $\mathrm{M}(a)$ is clopen. Conversely, suppose that $\mathrm{M}(a)$ is clopen. Inasmuch as $\mathrm{J}(R) = 0$, Lemma 1.1 implies that $\mathrm{M}(a) = \mathrm{M}(e)$ for some idempotent $e \in R$. Hence $Ra = Re$, *i.e.*, $R$ is von Neumann regular. ∎

## 2 Distance in $\Gamma_2(R)$

Recall that for two vertices $a$ and $b$ of a graph $G$, $\mathrm{d}(a,b)$ is the length of the shortest path from $a$ to $b$. The diameter of $G$ is denoted by $\mathrm{diam}\, G$ and is defined by $\mathrm{diam}\, G = \sup\{\mathrm{d}(a,b) : a,b \in G\}$. The girth of $G$, $\mathrm{gr}\, G$, is defined as the length of the shortest cycle in $G$ ($\mathrm{gr}\, G = \infty$ if $G$ contains no cycles). The reader is referred to [7] for undefined terms and notations.

The following fact is [10, Theorem 3.1].

***Theorem 2.1*** *The graph $\Gamma_2(R) \setminus \mathrm{J}(R)$ is connected and $\mathrm{diam}(\Gamma_2(R) \setminus \mathrm{J}(R)) \leq 3$.*

The following proposition characterizes the concept of distance in $\Gamma_2(R) \setminus \mathrm{J}(R)$.

***Proposition 2.2*** *Let $a,b,c \in \Gamma_2(R) \setminus \mathrm{J}(R)$ be distinct elements.*

(i) *$c$ is adjacent to both $a$ and $b$ if and only if $\mathrm{M}(c) \subseteq \mathrm{D}(ab)$.*
(ii) *$\mathrm{d}(a,b) = 1$ if and only if $\mathrm{M}(a) \cap \mathrm{M}(b) = \varnothing$.*
(iii) *$\mathrm{d}(a,b) = 2$ if and only if $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \notin \mathrm{J}(R)$.*
(iv) *$\mathrm{d}(a,b) = 3$ if and only if $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \in \mathrm{J}(R)$.*

**Proof** (i) $c$ is adjacent to both $a$ and $b$ if and only if $\mathrm{M}(a) \cap \mathrm{M}(c) = \mathrm{M}(b) \cap \mathrm{M}(c) = \varnothing$, if and only if $\mathrm{M}(c) \subseteq \mathrm{D}(ab)$.

(ii) is evident.

(iii) We note that $ab \notin \mathrm{J}(R)$ if and only if there exists $c \in \Gamma_2(R) \setminus \mathrm{J}(R)$ such that $\mathrm{M}(c) \subseteq \mathrm{D}(ab)$. To see this, let $M \in \mathrm{D}(ab)$. Hence $abr + c = 1$, for some $c \in M$ and $r \in R$. Therefore $\mathrm{M}(ab) \cap \mathrm{M}(c) = \varnothing$, *i.e.*, $\mathrm{M}(c) \subseteq \mathrm{D}(ab)$.

(iv) By Theorem 2.1, $\mathrm{d}(a,b) = 3$ if and only if $\mathrm{d}(a,b) \neq 1,2$, if and only if $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \in \mathrm{J}(R)$, by (ii) and (iii). ∎

The following theorem characterizes the diameter and the girth of $\Gamma_2(R) \setminus \mathrm{J}(R)$ according to the number of maximal ideals of $R$.

***Theorem 2.3*** *Let $R$ be a Gelfand ring.*

(i) $\mathrm{diam}(\Gamma_2(R) \setminus \mathrm{J}(R)) = \min\{|\mathrm{Max}(R)|, 3\}$.
(ii) *If $|\mathrm{Max}(R)| = 2$, then $\mathrm{gr}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 4$ or $\infty$; otherwise, $\mathrm{gr}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 3$.*

**Proof** (i) First we prove that $|\mathrm{Max}(R)| \geq 3$ if and only if $\mathrm{diam}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 3$. Suppose that $|\mathrm{Max}(R)| \geq 3$, and $M_1, M_2, M_3$ are distinct maximal ideals in $R$. Inasmuch as $\mathrm{Max}(R)$ is Hausdorff, there are $a_i \in R$ such that $M_i \in \mathrm{D}(a_i)$ and $a_i a_j \in \mathrm{J}(R)$, for $i \neq j$ and $i,j = 1,2,3$. Thus $M_3 \in \mathrm{M}(a_1) \cap \mathrm{M}(a_2) \neq \varnothing$ and $a_1 a_2 \in \mathrm{J}(R)$. Therefore $\mathrm{d}(a_1,a_2) = 3$ by Proposition 2.2(iv), and this shows that $\mathrm{diam}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 3$. Also by Lemma 1.1, there are $a_i' \in \Gamma_2(R) \setminus \mathrm{J}(R)$, i=1,2,3 such that

$$M_i \in \mathrm{M}(a_i') \quad \text{and} \quad \mathrm{M}(a_i) \subseteq \mathrm{M}(a_i' - 1).$$

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Hence $\mathrm{M}(a_i') \subseteq \mathrm{D}(a_i)$, so $\mathrm{M}(a_i') \cap \mathrm{M}(a_j') \subseteq \mathrm{D}(a_i) \cap \mathrm{D}(a_j) = \varnothing$, and this implies that $\mathrm{d}(a_i', a_j') = 1$. This shows that $\mathrm{gr}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 3$.

Conversely, if $\mathrm{diam}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 3$, then there are $a, b \in \Gamma_2(R) \setminus \mathrm{J}(R)$ such that $\mathrm{d}(a, b) = 3$. By Proposition 2.2(iv), $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \in \mathrm{J}(R)$. So there are maximal ideals $M_1, M_2, M_3$ such that

$$M_1 \in \mathrm{D}(b) \setminus \mathrm{D}(a) \quad \text{and} \quad M_2 \in \mathrm{D}(a) \setminus \mathrm{D}(b) \quad \text{and} \quad M_3 \in \mathrm{M}(a) \cap \mathrm{M}(b).$$

Thus $M_1, M_2, M_3$ are distinct maximal ideals in $R$, *i.e.*, $|\mathrm{Max}(R)| \geq 3$.

Now suppose that $|\mathrm{Max}(R)| = 2$. Since $|R| > 4$, we can consider the maximal ideal $M$ and $a, b \in M \setminus \mathrm{J}(R)$. Consequently, $\mathrm{d}(a, b) > 1$, *i.e.*, $\mathrm{diam}(\Gamma_2(R) \setminus \mathrm{J}(R)) > 1$. Thus $\mathrm{diam}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 2$, and (i) holds.

(ii) By the proof of part (i), $|\mathrm{Max}(R)| \geq 3$ implies that $\mathrm{gr}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 3$. Now suppose $|\mathrm{Max}(R)| = 2$, hence $R/\mathrm{J}(R) \simeq F_1 \times F_2$, where $F_1$ and $F_2$ are fields. If either $|F_1| = 2$ or $|F_2| = 2$, then $\mathrm{gr}(\Gamma_2(R)) = \mathrm{gr}(\Gamma_2(R/\mathrm{J}(R)) = \infty$. Otherwise, it is easy to see that $\mathrm{gr}(\Gamma_2(R)) = \mathrm{gr}(\Gamma_2(R/\mathrm{J}(R)) = 4$; see [15, Lemma 3.3]. ■

***Corollary 2.4*** *Let R be a Gelfand ring.*

(i) $\mathrm{diam}\,\Gamma_2(R) = \min\{|\mathrm{Max}(R)|, 3\}$.
(ii) *If* $|\mathrm{Max}(R)| = 2$, *then* $\mathrm{gr}\,\Gamma_2(R) = 4$ *or* $\infty$; *otherwise,* $\mathrm{gr}\,\Gamma_2(R) = 3$.

The associated number $\mathrm{e}(a)$ of a vertex $a$ of a graph $G$ is defined to be $\mathrm{e}(a) = \max\{\mathrm{d}(a, b) : a \neq b\}$. A center of $G$ is defined to be a vertex $a_0$ with the smallest associated number. The associated number $\mathrm{e}(a_0)$ is called the radius of $G$ and is denoted by $\rho(G)$.

***Remark*** Suppose that $R$ is a commutative semiprimitive ring. By [13, Lemma 2.12], for every $a \in R$, $\mathrm{int}\,\mathrm{M}(a) = \mathrm{D}(\mathrm{Ann}(a))$. Thus for any ring $R$ we have $(\mathrm{J}(R) : a) \neq \mathrm{J}(R)$ if and only if $a + \mathrm{J}(R)$ is a zero divisor in $R/\mathrm{J}(R)$, if and only if $\mathrm{int}\,\mathrm{M}(a + \mathrm{J}(R)) \neq \varnothing$, if and only if $\mathrm{int}\,\mathrm{M}(a) \neq \varnothing$.

***Theorem 2.5*** *Let R be a ring and* $a \in \Gamma_2(R) \setminus \mathrm{J}(R)$.

(i) $\mathrm{e}(a) = 1$ *if and only if* $Ra \in \mathrm{Max}(R)$ *and* $|Ra| = 2$.
(ii) $\mathrm{e}(a) = 2$ *if and only if* $(\mathrm{J}(R) : a) = \mathrm{J}(R)$ *or* $Ra \notin \mathrm{Max}(R)$.
(iii) $\mathrm{e}(a) = 3$ *if and only if* $(\mathrm{J}(R) : a) \neq \mathrm{J}(R)$ *and* $|\mathrm{M}(a)| > 1$.

**Proof** (i) Suppose that $\mathrm{e}(a) = 1$. Hence $\mathrm{M}(a) \cap \mathrm{M}(b) = \varnothing$, for all $b \in (\Gamma_2(R) \setminus \mathrm{J}(R))$ with $b \neq a$. This shows that $\mathrm{M}(a) = \{M\}$ and $|M| = 2$.

(ii) and (iii) By hypothesis and Theorem 2.1, $\mathrm{e}(a) = 2$ or 3. We consider two cases.

*Case 1.* Suppose that $(\mathrm{J}(R) : a) \neq \mathrm{J}(R)$. If $|\mathrm{M}(a)| > 1$, then by the above remark there are $M \in \mathrm{int}\,\mathrm{M}(a)$ and $M' \in \mathrm{M}(a) \setminus \{M\}$. Therefore by Lemma 1.1, there exists $b \in \Gamma_2(R) \setminus \mathrm{J}(R)$ such that

$$M \in \mathrm{M}(b - 1) \quad \text{and} \quad \big(\mathrm{Max}(R) - \mathrm{int}\,\mathrm{M}(a)\big) \cup \{M'\} \subseteq \mathrm{M}(b).$$

Thus $ab \in \mathrm{J}(R)$ and $M' \in \mathrm{M}(a) \cap \mathrm{M}(b)$, and Proposition 2.2(iv) implies that $\mathrm{d}(a, b) = 3$, *i.e.*, $\mathrm{e}(a) = 3$. Now if $|\mathrm{M}(a)| = 1$, then $\mathrm{M}(a) = \{M\}$. So for every

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$c \in M \setminus \mathrm{J}(R)$, $M \in \mathrm{M}(a) \cap \mathrm{M}(c)$ and $ac \notin \mathrm{J}(R)$. Thus $\mathrm{d}(a, c) = 2$, by Proposition 2.2(iii), *i.e.*, $\mathrm{e}(a) \leq 2$. Now if $Ra \notin \mathrm{Max}(R)$, then $\mathrm{e}(a) \neq 2$, hence $\mathrm{e}(a) = 2$.

*Case 2.* Suppose that $(\mathrm{J}(R) : a) = 0$, then Proposition 2.2(iii) implies that $\mathrm{e}(a) \leq 2$. If $\mathrm{e}(a) = 1$, then $|Ra| = 2$, so $a$ is idempotent. Consequently, $1 - a \in (\mathrm{J}(R) : a)$, *i.e.*, $a = 1$, and this is impossible. Therefore $\mathrm{e}(a) = 2$. ■

***Corollary 2.6*** *Let $R$ be a ring and $a \in \Gamma_2(R)$.*

(i) $\mathrm{e}(a) = 0$ *if and only if* $a \in \mathrm{J}(R)$.
(ii) $\mathrm{e}(a) = 1$ *if and only if* $Ra \in \mathrm{Max}(R)$ *and* $|Ra| = 2$.
(iii) $\mathrm{e}(a) = 2$ *if and only if* $(\mathrm{J}(R) : a) = \mathrm{J}(R)$ *or* $Ra \notin \mathrm{Max}(R)$.
(iv) $\mathrm{e}(a) = 3$ *if and only if* $(\mathrm{J}(R) : a) \neq \mathrm{J}(R)$ *and* $|\mathrm{M}(a)| > 1$.

***Corollary 2.7*** *Let $R$ be a semiprimitive ring and $a \in \Gamma_2(R)$.*

(i) $\mathrm{e}(a) = 1$ *if and only if* $Ra \in \mathrm{Max}(R)$ *and* $|Ra| = 2$.
(ii) $\mathrm{e}(a) = 2$ *if and only if $a$ is a non-zero divisor or* $Ra \notin \mathrm{Max}(R)$.
(iii) $\mathrm{e}(a) = 3$ *if and only if $a$ is a zero divisor and* $|\mathrm{M}(a)| > 1$.

A ring $R$ is called *quasi regular* if every element of $R$ is either a unit or a zero divisor. Clearly every von Neumann regular ring is a quasi regular ring, but a quasi regular ring is not necessary a regular ring (see [11, Proposition 2.3]).

***Corollary 2.8*** *Let $R$ be a ring.*

(i) $\rho(\Gamma_2(R) \setminus \mathrm{J}(R)) = 1$ *if and only if $R$ has a maximal ideal of cardinal* 2.
(ii) $\rho(\Gamma_2(R) \setminus \mathrm{J}(R)) = 3$ *if and only if $R/\mathrm{J}(R)$ is a quasi regular ring and $R$ has no isolated maximal ideal.*

*Otherwise,* $\rho(\Gamma_2(R) \setminus \mathrm{J}(R)) = 2$.

**Proof** (i) follows from Theorem 2.5(i).

(ii) By Lemma 1.1, Theorem 2.5(iii), and [11, Proposition 2.3(2)], we have $\rho(\Gamma_2(R) \setminus \mathrm{J}(R)) = 3$ if and only if for each $a \in \Gamma_2(R) \setminus \mathrm{J}(R)$, $(\mathrm{J}(R) : a) \neq \mathrm{J}(R)$ and $|\mathrm{M}(a)| > 1$, if and only if $R/\mathrm{J}(R)$ is a quasi regular ring and $R$ has no isolated maximal ideal. ■

## 3 Cycles in $\Gamma_2(R) \setminus \mathrm{J}(R)$

A graph $G$ is called triangulated (hypertriangulated) if each vertex (edge) of $G$ is a vertex (edge) of a triangle.

***Theorem 3.1*** *Let $R$ be a ring.*

(i) $\Gamma_2(R) \setminus \mathrm{J}(R)$ *is a triangulated graph if and only if $R$ has no isolated maximal ideals.*
(ii) $\Gamma_2(R) \setminus \mathrm{J}(R)$ *is a hypertriangulated graph if and only if $R$ has no non-trivial idempotent elements.*

**Proof** (i) Let $\Gamma_2(R) \setminus \mathrm{J}(R)$ be a triangulated graph and suppose $R$ has an isolated maximal ideal $M$. Hence $\mathrm{D}(a) = \{M\}$, for some $a \in \Gamma_2(R) \setminus \mathrm{J}(R)$. By hypothesis there are $b, c \in \Gamma_2(R) \setminus \mathrm{J}(R)$ such that $\mathrm{M}(a) \cap \mathrm{M}(b) = \mathrm{M}(a) \cap \mathrm{M}(c) = \mathrm{M}(b) \cap \mathrm{M}(c) = \varnothing$. This implies that $\mathrm{M}(b) = \mathrm{M}(c) = \{M\}$, a contradiction. Conversely, suppose that

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$R$ does not contain an isolated maximal ideal, and take $a \in \Gamma_2(R) \setminus \mathrm{J}(R)$. Therefore there exist two different points $M, M' \in \mathrm{D}(a)$. By Lemma 1.1 there exists $b \in R$ such that

$$M \in \mathrm{M}(b) \quad \text{and} \quad \mathrm{M}(a) \cup \{M'\} \subseteq \mathrm{M}(b-1).$$

Thus $\mathrm{M}(a) \cap \mathrm{M}(b) = \varnothing$, *i.e.*, $a$ and $b$ are adjacent. There also exists $c \in R$ such that

$$M' \in \mathrm{M}(c) \quad \text{and} \quad \mathrm{M}(a) \cup \mathrm{M}(b) \subseteq \mathrm{M}(c-1).$$

This implies that $\mathrm{M}(c) \subseteq \mathrm{D}(ab)$, *i.e.*, $c$ is a vertex adjacent to both $a$ and $b$. Therefore $a$ is a vertex of the triangle with vertices $a, b$, and $c$.

(ii) Let $\Gamma_2(R) \setminus \mathrm{J}(R)$ be a hypertriangulated graph. If $R$ has an non-trivial idempotent $e$, then $\mathrm{D}(e(1-e)) = \mathrm{D}(0) = \varnothing$, so by Proposition 2.2(i), there is no vertex adjacent to both $e$ and $e-1$, a contradiction.

Conversely, let $a-b$ be an edge in $\Gamma_2(R) \setminus \mathrm{J}(R)$. Since $\mathrm{M}(a) \cap \mathrm{M}(b) = \varnothing$ and $\mathrm{Max}(R)$ is connected, $\mathrm{M}(a) \cup \mathrm{M}(b) \neq \mathrm{Max}(R)$, *i.e.*, $\mathrm{D}(ab) \neq \varnothing$. Thus by Proposition 2.2(i), there exists a vertex adjacent to both $a$ and $b$, *i.e.*, $\Gamma_2(R) \setminus \mathrm{J}(R)$ is a hypertriangulated graph. ■

***Definition 3.2*** It follows from Lemma 1.1 that if $M$ is an isolated maximal ideal, then $\mathrm{M}(a) = \{M\}$, for some $a \in R$. In this case, if $a$ is unique, then $M$ is called a *strongly isolated* maximal ideal of $R$.

The most important rings have no strongly isolated maximal ideals. For example, rings for which 2 is a unit, non-semiprimitve rings (for which $\mathrm{M}(a) = \mathrm{M}(a + r)$ for each $a \in R$ and $r \in \mathrm{J}(R)$), and semiprimitive rings, as follows.

***Proposition 3.3*** *Let $R$ be a semisimple ring, then $R$ has strongly isolated maximal ideals if and only if $R \simeq F \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $F$ is a field.*

**Proof** Suppose that $R$ has a strongly isolated maximal ideal $M$. Inasmuch as $M$ is semisimple and it is generated by an idempotent, $|\,\mathrm{Max}(R)|$ is finite. Hence $R \simeq F_1 \times F_2 \times \cdots \times F_n$, where the $F_i$ are fields. With less generality, we can consider $M = 0 \times F_2 \times \cdots \times F_n$. If for example $F_2 \neq \mathbb{Z}_2$, then there exists a unit $u \neq 1$ in $F_2$. Put $a = (0, 1, 1, \ldots, 1)$ and $a' = (0, u, 1, \ldots, 1)$. Then $\mathrm{M}(a) = \mathrm{M}(a') = \{M\}$, a contradiction. So $F_i = \mathbb{Z}_i$, for all $i \geq 2$. Conversely, $M = 0 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ is always a strongly isolated maximal ideal. ■

***Corollary 3.4*** *Let $R$ be a semiprimitive ring and $|\,\mathrm{Max}(R)| < \infty$. Then $R$ has strongly isolated maximal ideals if and only if $R \simeq F \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where $F$ is a field.*

***Theorem 3.5*** *Let $a \in \Gamma_2(R)$. Then $a$ is an endpoint if and only if $\mathrm{D}(a) = \{M\}$, where $M$ is a strongly isolated maximal ideal.*

**Proof** ($\Leftarrow$) Suppose that $\mathrm{D}(a) = \{M\}$, and $M$ is a strongly isolated maximal ideal. If $a$ is adjacent to both $b$ and $c$, then $\mathrm{M}(a) \cap \mathrm{M}(b) = \mathrm{M}(a) \cap \mathrm{M}(c) = \varnothing$, *i.e.*, $\mathrm{M}(b) = \mathrm{M}(c) = \{M\}$. Therefore $b = c$, by hypothesis.

($\Rightarrow$) If $|\,\mathrm{D}(a)| > 1$, then there are the distinct maximal ideals $M_1, M_2 \in \mathrm{D}(a)$ and $M \in \mathrm{M}(a)$. Set $F_1 = \mathrm{M}(a) \cup \{M_2\}$ and $F_2 = \mathrm{M}(a) \cup \{M_1\}$. Then there are $a_i \in \Gamma_2(R)$

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($i = 1, 2$) such that $M_i \in \mathrm{M}(a_i)$ and $F_i \subseteq \mathrm{M}(a_i - 1)$. Thus $\mathrm{M}(a) \cap \mathrm{M}(a_1) = \mathrm{M}(a) \cap \mathrm{M}(a_2) = \varnothing$. Hence $a$ is adjacent to both $a_1$ and $a_2$, *i.e.*, $a$ is not an endpoint. Now suppose that $\mathrm{D}(a) = \{M\}$, and $M$ is not a strongly isolated maximal ideal. Hence there are $b, c \in R$ such that $\mathrm{M}(b) = \mathrm{M}(c) = \{M\}$. This implies that $a$ is adjacent to both $b$ and $c$. ∎

If $M$ is a strongly isolated maximal ideal, then there is an $a \in R$ such that $\mathrm{D}(a) = \{M\}$, *i.e.*, $a$ is an endpoint. Thus we have the following corollary.

***Corollary 3.6*** *$\Gamma_2(R)$ (also $\Gamma_2(R) \setminus \mathrm{J}(R)$) has an endpoint if and only if $R$ has a strongly isolated maximal ideal.*

***Lemma 3.7*** *Let $R$ be a ring that has no strongly isolated maximal ideals, and let $a, b, c \in \Gamma_2(R)$. If $a$ is adjacent to both $b$ and $c$, then there exists $a \neq a' \in \Gamma_2(R)$ such that $a'$ is adjacent to both $b$ and $c$.*

**Proof** There exists $M \in \mathrm{M}(a)$. Suppose that $|\mathrm{M}(a)| = 1$. If $a^2 \neq a$, then we put $a' = a^2$, otherwise $M$ is isolated, so by hypothesis there exists $a \neq a' \in M$ such that $\mathrm{M}(a') = \mathrm{M}(a) = \{M\}$. Hence $a'$ is adjacent to both $b$ and $c$ by Proposition 2.2(i). If $|\mathrm{M}(a)| > 1$, then there exists $M' \in \mathrm{M}(a) \setminus \{M\}$. Put $F = \mathrm{M}(bc) \cup \{M\}$. Hence there exists $a' \in R$ such that $M' \in \mathrm{M}(a')$ and $F \subseteq \mathrm{M}(a' - 1)$. Thus $\mathrm{M}(a') \subseteq \mathrm{D}(bc)$. This shows that $a \neq a'$ and $a'$ is adjacent to both $b$ and $c$. ∎

***Corollary 3.8*** *Let $R$ be a ring that has no strongly isolated maximal ideals. Then every vertex of $\Gamma_2(R) \setminus \mathrm{J}(R)$ is a 4-cycle-vertex.*

**Proof** By Theorem 3.5, no vertex $a$ is an endpoint, so the proof follows from Lemma 3.7. ∎

If $a$ and $b$ are two vertices in $\Gamma_2(R) \setminus \mathrm{J}(R)$, then by $\mathrm{c}(a, b)$ we mean the length of the smallest cycle containing $a$ and $b$. For every two vertices $a$ and $b$, all possible cases for $\mathrm{c}(a, b)$ are given in the following theorem.

***Theorem 3.9*** *Let $R$ be a ring that has no strongly isolated maximal ideals and $a, b \in \Gamma_2(R) \setminus \mathrm{J}(R)$.*

(i) *$\mathrm{c}(a, b) = 3$ if and only if $\mathrm{M}(a) \cap \mathrm{M}(b) = \varnothing$ and $ab \notin \mathrm{J}(R)$.*
(ii) *$\mathrm{c}(a, b) = 4$ if and only if $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \notin \mathrm{J}(R)$, or $\mathrm{M}(a) \cap \mathrm{M}(b) = \varnothing$ and $ab \in \mathrm{J}(R)$.*
(iii) *$\mathrm{c}(a, b) = 6$ if and only if $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \in \mathrm{J}(R)$.*

**Proof** (i) We have $\mathrm{c}(a, b) = 3$ if and only if $\mathrm{d}(a, b) = 1$ and there exists $c \in R$ such that $c$ is adjacent to both $a$ and $b$, and this holds if and only if $\mathrm{D}(ab) \neq \varnothing$ by Proposition 2.2(i).

(ii) If $\mathrm{M}(a) \cap \mathrm{M}(b) = \varnothing$ and $ab \in \mathrm{J}(R)$, then there exists $a' \in R$ such that $M(a') \subseteq \mathrm{M}(a)$. Hence $b$ is adjacent to both $a$ and $a'$. So by Lemma 3.7, there is $c \in R$ such that $c$ is adjacent to both $a$ and $a'$. Therefore the path with vertices $a, b, a'$, and $c$ is a cycle with length 4, *i.e.*, $\mathrm{c}(a, b) \leq 4$. Now (i) implies that $\mathrm{c}(a, b) = 4$. If $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \notin \mathrm{J}(R)$, then by Proposition 2.2(i), there exists $c \in \Gamma_2(R) \setminus \mathrm{J}(R)$ such that $c$ is adjacent to both $a$ and $b$. Thus by Lemma 3.7, there is $c' \in R$ such that the

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path with vertices $a, c, b$, and $c'$ is a cycle with length 4. The converse follows from Proposition 2.2.

(iii) Inasmuch as $\mathrm{c}(a, b) = 6$, then parts (i) and (ii) imply that $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \in \mathrm{J}(R)$. Conversely, since $\mathrm{M}(a) \cap \mathrm{M}(b) \neq \varnothing$ and $ab \in \mathrm{J}(R)$, by Proposition 2.2(iv) $\mathrm{d}(a, b) = 3$ and this implies that $\mathrm{c}(a, b) > 5$. Hence there are vertices $c$ and $d$ such that $Ra + Rc = Rc + Rd = Rb + Rd = R$. By Lemma 3.7, there is $c' \in R$ such that $c'$ is adjacent to both $a$ and $d$. Therefore $\mathrm{M}(a) \cap \mathrm{M}(c') = \varnothing$, so $\mathrm{M}(c') \subseteq \mathrm{M}(b)$, by hypothesis. Thus $\mathrm{M}(c') \cap \mathrm{M}(d) \subseteq \mathrm{M}(b) \cap \mathrm{M}(d) = \varnothing$, *i.e.*, $c'$ is adjacent to $d$. Again by Lemma 3.7, there is $d' \in R$ such that $d'$ is adjacent to both $b$ and $c'$. Thus the path with vertices $a, c, d, b, d'$, and $c'$ is a cycle with length 6, *i.e.*, $\mathrm{c}(a, b) = 6$. ■

As in [3], for distinct vertices $a$ and $b$ in a graph $G$ we say that $a$ and $b$ are orthogonal, written $a \perp b$, if $a$ and $b$ are adjacent and there is no vertex $c$ of $G$ which is adjacent to both $a$ and $b$. A graph $G$ is called complemented if for each vertex $a$ of $G$, there is a vertex $b$ of $G$ (called a complement of $a$) such that $a \perp b$, and that $G$ is uniquely complemented if $G$ is complemented and whenever $a \perp b$ and $a \perp c$, then $b \sim c$ (*i.e.*, $b$ and $c$ are adjacent to exactly the same vertices). By Proposition 2.2(i), for the distinct vertices $a$ and $b$ in $\Gamma_2(R) \setminus \mathrm{J}(R)$, $a \perp b$ if and only if $\mathrm{M}(a) \cap \mathrm{M}(b) = \varnothing$ and $\mathrm{D}(ab) = \varnothing$ if and only if $\mathrm{M}(a) = \mathrm{D}(b)$. Thus we have the following propositions.

***Proposition 3.10*** *$\Gamma_2(R) \setminus \mathrm{J}(R)$ is complemented if and only if $\Gamma_2(R) \setminus \mathrm{J}(R)$ is uniquely complemented.*

***Proposition 3.11*** *$\Gamma_2(R) \setminus \mathrm{J}(R)$ is complemented if and only if $R/\mathrm{J}(R)$ is a zero dimensional ring.*

**Proof** By Lemmas 1.1 and 1.2 we have that $\Gamma_2(R) \setminus \mathrm{J}(R)$ is complemented if and only if for all $a \in \Gamma_2(R) \setminus \mathrm{J}(R)$, $\mathrm{M}(a)$ is an clopen subset of $\mathrm{Max}(R)$, if and only if for all $a \in \Gamma(R)$, $\mathrm{M}(a + \mathrm{J}(R))$ is an clopen subset of $\mathrm{Max}(R/\mathrm{J}(R))$, if and only if $R/\mathrm{J}(R)$ is a zero dimensional ring. ■

## 4 Dominating Sets and Complete Subgraphs

In a graph $G$, a dominating set is a set of vertices $D$ such that every vertex outside $D$ is adjacent to at least one vertex in $D$. The dominating number of $G$ denoted by $\mathrm{dt}\, G$ is the smallest cardinal number of the form $|D|$ where $D$ is a dominating set. A complete subgraph of $G$ is a subgraph in which every vertex is adjacent to every other vertex. The smallest cardinal number $\alpha$ such that every complete subgraph $G$ has cardinality $\leq \alpha$, denoted by $\omega G$, is called the clique number of $G$.

The set of all cardinal numbers of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base for the open sets of a topological space $X$, has a smallest element; this cardinal number is called the weight of the topological space $X$ and is denoted by $\mathrm{w}(X)$. The density of a space $X$ is defined as the smallest cardinal number of the form $|Y|$, where $Y$ is a dense subspace of $X$; this cardinal number is denoted by $\mathrm{d}(X)$. For every topological space $X$ we have $\mathrm{d}(X) \leq \mathrm{w}(X)$; see [8, Theorem 1.3.7].

The next proposition follows from [15, Corollary 3.6].

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***Proposition 4.1*** *The following are equivalent for* $\Gamma_2(R) \setminus \mathrm{J}(R)$.

(i) $\mathrm{dt}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 1$.
(ii) $\Gamma_2(R) \setminus \mathrm{J}(R)$ *is a star graph.*
(iii) $R$ *is isomorphic to* $\mathbb{Z}_2 \times F$*, where* $F$ *is a field.*
(iv) $\Gamma_2(R) \setminus \mathrm{J}(R)$ *is a tree.*

***Theorem 4.2*** *The following are equivalent.*

(i) $R \not\cong \mathbb{Z}_2 \times F$*, where* $F$ *is a field.*
(ii) $\mathrm{d}(\mathrm{Max}(R)) \leq \mathrm{dt}(\Gamma_2(R) \setminus \mathrm{J}(R)) \leq \mathrm{w}(\mathrm{Max}(R))$.

**Proof** (i) ⇒ (ii) Suppose that $D$ is a minimal dominating set. For every $a \in D$, take $M_a \in \mathrm{M}(a)$. We show that the set $\mathcal{A} = \{M_a : a \in D\}$ is dense in $\mathrm{Max}(R)$. Otherwise, Lemma 1.1 implies that $\mathcal{A} \subsetneq \mathrm{M}(a')$, for some $a' \in \Gamma_2(R) \setminus \mathrm{J}(R)$. Inasmuch as $D$ is a minimal dominating set, then $a' \notin D$ (for if $a' \in D$, then by Proposition 4.1, $D \setminus \{a'\} \neq \varnothing$ is a dominating set, and this is impossible). Therefore $a'$ is not adjacent to any element of $D$, a contradiction. Thus $\mathcal{A}$ is dense in $\mathrm{Max}(R)$. This implies that $\mathrm{d}(\mathrm{Max}(R)) \leq |\mathcal{A}| \leq |D|$ for every dominating set $D$ and hence $\mathrm{d}(\mathrm{Max}(R)) \leq \mathrm{dt}(\Gamma_2(R) \setminus \mathrm{J}(R))$. In order to show that $\mathrm{dt}(\Gamma_2(R) \setminus \mathrm{J}(R)) \leq \mathrm{w}(\mathrm{Max}(R))$, suppose $\mathfrak{B} = \{B_\lambda : \lambda \in \Lambda\}$ is a base for the open subsets of $\mathrm{Max}(R)$. By Lemma 1.1 there are $a_\lambda \in \Gamma_2(R) \setminus \mathrm{J}(R)$ such that $\mathrm{M}(a_\lambda) \subseteq B_\lambda$. We claim that $D = \{a_\lambda : \lambda \in \Lambda\}$ is a dominating set. To see this, let $b \in \Gamma_2(R) \setminus \mathrm{J}(R)$; then there exists $B_\lambda \in \mathfrak{B}$ such that $B_\lambda \subseteq \mathrm{D}(b)$. Therefore $\mathrm{M}(a_\lambda) \subseteq \mathrm{D}(b)$, *i.e.*, $b$ is adjacent to $a_\lambda$. Consequently $D$ is a dominating set. Now $\mathrm{dt}(\Gamma_2(R) \setminus \mathrm{J}(R)) \leq |D| \leq |\mathfrak{B}|$ for every base $\mathfrak{B}$ for open subsets of $\mathrm{Max}(R)$. This means that $\mathrm{dt}(\Gamma_2(R) \setminus \mathrm{J}(R)) \leq \mathrm{w}(\mathrm{Max}(R))$.

(ii) ⇒ (i) If $R$ is isomorphic to $\mathbb{Z}_2 \times F$, where $F$ is a field, then $\mathrm{dt}(\Gamma_2(R) \setminus \mathrm{J}(R)) = 1$. Hence $\mathrm{d}(\mathrm{Max}(R)) = 1$, and this implies that $R$ is a local ring, a contradiction. ∎

It is well known that $\omega(\Gamma_2(R) \setminus \mathrm{J}(R)) = |\mathrm{Max}(R)|$ [15, Theorem 3.9]. Also if $|\mathrm{Max}(R)|$ is finite, $\mathrm{d}(\mathrm{Max}(R)) = |\mathrm{Max}(R)|$. Thus we have the following corollary.

***Corollary 4.3*** *If* $|\mathrm{Max}(R)|$ *is finite, then*

$$\omega\big(\Gamma_2(R) \setminus \mathrm{J}(R)\big) = \mathrm{dt}\big(\Gamma_2(R) \setminus \mathrm{J}(R)) = \mathrm{d}(\mathrm{Max}(R)\big) = \mathrm{w}\big(\mathrm{Max}(R)\big) = |\mathrm{Max}(R)|.$$

**Proof** Suppose that $\mathrm{Max}(R) = \{M_1, M_2, \dots, M_n\}$. For any $1 \leq i \leq n$, there is $a_i \in R$ such that $\mathrm{D}(a_i) = \{M_i\}$. Hence the set $\{\mathrm{D}(a_1), \mathrm{D}(a_2), \dots, \mathrm{D}(a_n)\}$ is a base for $\mathrm{Max}(R)$, *i.e.*, $\mathrm{w}(\mathrm{Max}(R)) \leq |\mathrm{Max}(R)|$. Therefore the proof follows from Theorem 4.2. ∎

***Theorem 4.4*** *Let* $R$ *be a ring that has no maximal ideal of order* 2*. Then the following statements are equivalent.*

(i) $\Gamma_2(R) \setminus \mathrm{J}(R)$ *is not triangulated and the set of centers of* $\Gamma_2(R) \setminus \mathrm{J}(R)$ *is a dominating set.*
(ii) *The set of isolated points of* $\mathrm{Max}(R)$ *is dense in* $\mathrm{Max}(R)$.
(iii) *Every intersection of essential ideals of* $R/\mathrm{J}(R)$ *is an essential ideal in* $R/\mathrm{J}(R)$.

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**Proof** (i) $\Rightarrow$ (ii) Since $\Gamma_2(R)\setminus \mathrm{J}(R)$ is not triangulated, then by Theorem 3.1, $\mathrm{Max}(R)$ has at least one isolated point $M_0$. By Lemma 1.1 there exists $a_0 \in \Gamma_2(R)\setminus \mathrm{J}(R)$ such that $\mathrm{M}(a_0) = \{M_0\}$ and $a_0^2 - a_0 \in \mathrm{J}(R)$. Therefore $\mathrm{e}(a_0) = 2$, by Theorem 2.5. If we denote the set of centers of $\Gamma_2(R)\setminus \mathrm{J}(R)$ by $D$, we have $D = \{a \in \Gamma(R) : \mathrm{e}(a) = 2\}$. For every $a \in D$, take $M_a \in \mathrm{M}(a)$, and put $\mathcal{A} = \{M_a : a \in D\}$. We claim that $|\mathcal{A}| > 1$. To see this, suppose $|\mathrm{Max}(R)| < \infty$; then there exists an isolated maximal ideal $M \in D(a_0)$. So $\mathrm{M}(a) = \{M\}$ for some $a \in \Gamma_2(R)\setminus \mathrm{J}(R)$. Thus $\mathrm{e}(a) = 2$, *i.e.*, $a \in D$ and $M_a \in \mathcal{A}\setminus\{M_0\}$. If $|\mathrm{Max}(R)| = \infty$, then there are the distinct maximal ideals $M, M' \in D(a_0)$ and $a \in M \cap \mathrm{M}' \setminus M_0$. Hence $a \in \Gamma_2(R)\setminus \mathrm{J}(R)$ and $Ra \notin \mathrm{Max}(R)$, consequently, $\mathrm{e}(a) = 2$, by Theorem 2.5. This implies that $a \in D$ and $M_a \in \mathcal{A}\setminus\{M_0\}$. Now we show that $\mathcal{A} = \{M_a : a \in D\}$ is a dense subset in $\mathrm{Max}(R)$. Otherwise, Lemma 1.1 implies that $\mathcal{A} \subseteq \mathrm{M}(a')$, for some $a' \in \Gamma_2(R)\setminus \mathrm{J}(R)$. Since

$$\mathrm{Max}(R) = \mathrm{M}(a_0) \cup \mathrm{M}(a_0 - 1) \subseteq \mathrm{M}(a') \cup \mathrm{M}(a_0 - 1),$$

we have $a_0 - 1 \in (\mathrm{J}(R) : a')$, and Theorem 2.5 implies that $\mathrm{e}(a') = 3$, *i.e.*, $a' \notin D$. On the other hand, for each $a \in D$ we have $M_a \in \mathrm{M}(a) \cap \mathrm{M}(a')$, and this implies that $a'$ is not adjacent to any element of $D$, a contradiction. Thus $\mathcal{A}$ is dense in $\mathrm{Max}(R)$.

(ii) $\Rightarrow$ (i) Let $\mathcal{A} = \{M_\lambda : \lambda \in \Lambda\}$ be the set of isolated points of $\mathrm{Max}(R)$. By Theorem 2.5, $\Gamma(R)$ is not triangulated. Consider $D = \{a_\lambda \in R : \mathrm{M}(a_\lambda) = \{M_\lambda\}\}$. By Theorem 2.5, $\mathrm{e}(a) \geq 2$ for all $a \in \Gamma(R)$, and $\mathrm{e}(a_\lambda) = 2$ for all $\lambda \in \Lambda$. Hence every element of $D$ is a center of $\Gamma_2(R)\setminus \mathrm{J}(R)$. Now suppose that $b \in (\Gamma_2(R)\setminus \mathrm{J}(R))\setminus D$. Since $\mathcal{A}$ is dense in $\mathrm{Max}(R)$, there exists $M_\lambda \in \mathrm{D}(b) \cap \mathcal{A}$. Therefore $\mathrm{M}(a_\lambda) \subseteq \mathrm{D}(b)$, which implies that $a_\lambda$ is adjacent to $b$, *i.e.*, $D$ is a dominating set.

(ii) $\Leftrightarrow$ (iii) follows from [13, Proposition 2.9]. ■

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