J. Austral. Math. Soc. **22** (Series A) (1976), 362–370.

ON LATTICE-ORDERED RINGS IN WHICH THE SQUARE OF EVERY ELEMENT IS POSITIVE

STUART A. STEINBERG

(Received 4 April 1974; revised 10 September 1975)

Abstract

It is shown that a unital lattice-ordered ring in which the square of every element is positive is embeddable in a product of totally ordered rings provided it is archimedean, semiperfect, or π-regular. Also, some canonical examples of unital l-domains with squares positive that are not totally ordered are discussed.

1. Introduction

Diem (1968) has shown that a lattice-ordered ring (l-ring) which satisfies the identity $x^+x^-=0$ and has no nilpotent l-ideals is an f-ring. In this paper it is shown that a unital l-ring in which the square of every element is positive is an f-ring provided it is either archimedean, semiperfect, or an algebraic l-algebra over a partially-ordered field.

Diem proved the theorem mentioned above by showing that an l-prime l-ring that satisfies $x^+x^-=0$ is a (totally ordered) domain. Birkhoff and Pierce (1958, Theorem 15) have shown that an l-ring with a positive unit satisfies this identity if and only if 1 is a weak order unit (i.e., $1\wedge x=0$ implies $x=0$). Since the identity $x^+x^-=0$ implies that all squares are positive [Birkhoff and Pierce (1958), p. 59, Lemma 2], the question of whether or not there exists a unital l-prime l-ring with squares positive that is not totally ordered, i.e., in which 1 is not a weak order unit, arises naturally from Diem's result. We exhibit some canonical examples of unital l-domains with squares positive that are not totally ordered.

The reader is referred to Birkhoff and Pierce (1958) and Johnson (1960) for the general theory of l-rings. If M is a partially-ordered abelian group (*po-group*), then $M^+=\{x\in M: x\geqq 0\}$ will denote its *positive cone*; and if M is an *l-group* (i.e., M is also a lattice), the *positive part*, the *negative part*, and the *absolute value* of $x\in M$ are $x^+=x\vee 0$, $x^-=(-x)\vee 0$, and $|x|=x\vee -x=$

$x^+ + x^-$, respectively. By a *convex l-subgroup* of the l-group M we mean a subgroup N which is convex (i.e., $0 \leqq a \leqq b$ with $b \in N$ implies $a \in N$) and a sub-lattice of M. By a *po-ring* we mean a *direct* partially-ordered ring, and by an l-ring we mean a po-ring which is also a lattice. An *l-ideal* of an l-ring is a convex l-subgroup that is also an ideal. The direct sum of a family $\{M_\alpha \mid \alpha \in A\}$ of po-groups is the group direct sum $\Sigma \bigoplus M_\alpha$ supplied with the positive cone $\Sigma \bigoplus M_\alpha^+ \cdot \boldsymbol{Z}$ and $\boldsymbol{Q}$ will denote the totally ordered rings of integers and rational numbers, respectively. A ring will be called *unital* if it has an identity element.

The class of l-rings in which all squares are positive is the variety determined by the identity $(x^2)^- = 0$. It has already been mentioned that this variety contains that determined by the identity $x^+x^- = 0$, which in turn contains the variety of f-rings [Birkhoff and Pierce (1958), pp. 55–57]: An *f-ring* is an l-ring that is a subring and a sublattice of a product of totally ordered rings, or, equivalently, which satisfies the identity $(x^+a^+ \wedge x^-) \vee (a^+x^+ \wedge x^-) = 0$. We will often use the following characterization of a unital f-ring [Birkhoff and Pierce (1958), Corollary 1, p. 59]: *A unital l-ring is an f-ring if and only if it satisfies the identities* $x^+y^+ = (xy^+)^+ = (x^+y)^+$.

Portions of this paper formed part of the author's dissertation written at the University of Illinois under the direction of Elliot Weinberg, and portions developed while the author held a University of Toledo Summer Faculty Fellowship.

2. A canonical construction

Let F be a po-ring and let M be an l-group. M is called a left *l-module over F* if M is a left F-module and $F^+M^+ \subseteq M^+$. If F is unital we also require that $1 \cdot x = x$ for each $x \in M$. If M is a left l-module over F and if $\alpha x \wedge y = 0$ whenever $x \wedge y = 0$ in M and $\alpha \in F^+$, M is called an *f-module*. Over a totally ordered division ring every l-module is an f-module. This is a consequence of

LEMMA 1. *Let M be an l-module over the po-division ring F. Then M is an f-module over F if and only if* $\alpha^{-1}M^+ \subseteq M^+$ *for each nonzero* $\alpha \in F^+$.

PROOF. If M is an f-module over F, then scalar multiplication by $0 \neq \alpha \in F^+$ is an automorphism of the l-group M. Since the inverse of this automorphism is scalar multiplication by α^{-1}, $\alpha^{-1}M^+ \subseteq M^+$.

Conversely, suppose $\alpha^{-1}M^+ \subseteq M^+$ for all $0 < \alpha \in F$. If $x \wedge y = 0$ in M and $0 < \alpha \in F$, then $0 \leqq \alpha(x \wedge y) \leqq \alpha x \wedge \alpha y$ implies

$$0 \leqq x \wedge y \leqq \alpha^{-1}(\alpha x \wedge \alpha y) \leqq \alpha^{-1}(\alpha x) \wedge \alpha^{-1}(\alpha y) = x \wedge y = 0.$$

Thus $\alpha x \wedge \alpha y = 0$. Since F is directed there exists $\beta \in F^+$ with $\beta \geqq 1, \alpha$. Then the inequalities $0 \leqq \alpha x \wedge y \leqq \beta x \wedge \beta y = 0$ show that M is an f-module.

If a and b are two elements of the f-module M, then a is called *infinitely smaller than* b *with respect to* F, written $a \ll b$, if $\alpha |a| \leqq |b|$ for each $\alpha \in F$ (since F is directed, this is equivalent to $\alpha |a| \leqq |b|$ for each $\alpha \in F^+$). If $a \ll b$ and $b \neq 0$, then $\alpha |a| < |b|$ for each $\alpha \in F$. For if $|b| = |\alpha_0| a$, then $2\alpha_0 |a| \leqq \alpha_0 |a|$ implies $|b| = \alpha_0 |a| \leqq 0$; so $b = 0$. M is called *archimedean over* F if $a = 0$ whenever $a \ll b$. Note that if F is unital and M is F-archimedean (or $a \ll b$ with respect to F), then M is $\boldsymbol{Z}$-archimedean ($a \ll b$ with respect to $\boldsymbol{Z}$). When no confusion is likely we will suppress the phrase "over F."

Let F be a commutative unital po-ring. By an *l-algebra over F* we mean an algebra R over F which is also an f-module over F. If R is an l-algebra and an f-ring it will be called an *f-algebra*. If the unital l-algebra R has squares positive, then each nilpotent element of R is, in absolute value, $\leqq 1$ [Diem (1968), Theorem 3.3]. Since, for $\alpha \in F$, αa is nilpotent whenever a is, we have $a^2 \ll a$ for each nilpotent element a of R. The elements disjoint from 1 behave in just the opposite way.

LEMMA 2. *If the unital l-algebra R has squares positive, then $1 \wedge a = 0$ implies $a \ll a^2$.*

PROOF. For each $\alpha \in F^+$, $0 \leqq (\alpha - a)^2 = \alpha^2 - 2\alpha a + a^2$ yields $2\alpha a \leqq \alpha^2 + a^2$. Hence

$$2\alpha a = 2\alpha a \wedge (\alpha^2 + a^2) \leqq 2\alpha a \wedge \alpha^2 + 2\alpha a \wedge a^2 = 2\alpha a \wedge a^2.$$

Thus $\alpha a \leqq a^2$.

In Birkhoff and Pierce (1958), Corollary 3, p. 61 it is shown that a unital archimedean l-ring is an f-ring provided 1 is a weak order unit. This result, together with Lemma 2, gives

COROLLARY 1. *An archimedean l-algebra with an identity element is an f-algebra if and only if it has squares positive.*

The following example [see Example 2.3 of Diem (1968)] shows that Corollary 1 is false for an l-ring without an identity element. In fact, this example can serve as a counterexample to many of the results of this paper if the identity element is dropped. Let $R = \boldsymbol{Q}a \oplus \boldsymbol{Q}b$ as an l-group with multiplication defined by $a^2 = ab = ba = b^2 = a$.

An *l-domain* is an l-ring R in which the semigroup R^+ has no zero divisors. Note that a unital l-domain R with squares positive must be a domain. For if $C(1)$ is the convex l-subgroup of R generated by 1, then by Diem's theorem $C(1)$ contains all the nilpotent elements of R. But $C(1)$, being an f-ring and an l-domain, is a domain.

We present next some canonical examples of unital l-domains with squares positive in which 1 is not a weak order unit. First we need some lemmas.

LEMMA 3. *Let M be an f-module over the unital po-ring F. If $x_1 \ll x_2 \ll x_3 \ll \cdots$ in M, then for each $n \in \mathbf{Z}^+$ and for all $\alpha_1, \cdots, \alpha_n \in F$, $\alpha_1 x_1 + \cdots \alpha_n x_n \ll x_{n+1}$.*

PROOF. We prove this for $n = 2$. An easy induction argument will then complete the proof. Since F is directed $\alpha_1 = \beta_1 - \beta_2$ with $\beta_j \in F^+$. So $|\alpha_i x_i| \leqq (\beta_1 + \beta_2)|x_i| = \gamma_i |x_i|$ with $\gamma_i \in F^+$. Thus, for any $\beta \in F^+$ there exist $\gamma_1, \gamma_2 \in F^+$ with

$$\beta\,|\alpha_1 x_1 + \alpha_2 x_2| \leqq \beta\gamma_1 |x_1| + \beta\gamma_2 |x_2| \leqq |x_2| + \beta\gamma_2 |x_2| = (1 + \beta\gamma_2)|x_2| \leqq |x_3|.$$

Let M be a module over the commutative integral domain F. An element $x \in M$ is called *torsion* (or *F-torsion*) if $\alpha x = 0$ for some nonzero $\alpha \in F$. The set $T = T(M)$ of torsion elements of M is a submodule of M, called the *torsion submodule*, and M/T is *torsion-free* in the sense that $T(M/T) = 0$. If, in addition, F is totally ordered and M is an f-module over F, then T is a convex l-submodule of M. (More generally, if F is merely partially ordered and $T_1 = \{x \in M : \alpha x = 0 \text{ for some } 0 < \alpha \in F\}$, then T_1 is a convex l-submodule of M and $T_1(M/T_1) = 0$.) Let Q be the totally ordered field of quotients of the totally ordered integral domain F and let M be a torsion-free f-module over F; then the module of quotients of M with respect to $S = F\backslash\{0\}$,

$$M_S = \left\{\frac{x}{a} : x \in M,\ a \in S\right\},$$

can be made in a unique way into an f-module over Q that contains M. The Q-f-module M_S is constructed, of course, exactly as in the case $F = \mathbf{Z}$ and can be identified with the tensor product $M \otimes_F Q$. We summarize this discussion in

LEMMA 4. *Let M be an f-module over the commutative totally ordered domain F, and let Q be the totally ordered quotient field of F. Then the torsion submodule of M is a convex l-submodule of M. If M is torsion-free, then the module of quotients of M with respect to $S = F\backslash\{0\}$ is an f-module over Q containing M.*

The partially-ordered module $_FM$ is called *semi-closed* (or *F-semi-closed*) if $\alpha x \in M^+$ implies $x \in M^+$, where $0 \neq \alpha \in F^+$ and $x \in M$. If M is a torsion-free f-module over F, then M is semi-closed. For if $\alpha x \in M$ with $0 \neq \alpha \in F^+$ then $0 = (\alpha x)^- = \alpha x^-$; so $x^- = 0$ and $x \in M^+$. This will be used in the next theorem.

Let S be a totally ordered domain and let $T = S[x]$ be the polynomial ring over S in the indeterminate x. Let

$$P_0 = P_0(S) = \left\{ \sum_{i=0}^{n} \alpha_i x^i : \alpha_0 \geqq 0 \text{ and if } n > 1, \alpha_n > 0 \right\}$$

and let

$$P_1 = P(S) = \left\{ \sum_{i=0}^{n} \alpha_i x^i : n > 1 \text{ and } \alpha_n > 0 \right\}$$

$$\cup \{\alpha_0 + \alpha_1 x : \alpha_0 \geqq 0 \text{ and } \alpha_1 \geqq 0\}.$$

Note that $P_0 \subseteq P_1$.

THEOREM 1. (a) *P_0 and P_1 are partial orders for the ring $T = S[x]$. Moreover, (T, P_0) and T, P_1) are l-domains with squares positive in a which the identity element (if it exists) is not a weak order unit.*

(b) *Let R be a unital l-algebra with squares positive over the commutative totally ordered domain F. Suppose that a is.a positive element of R that is disjoint from 1 and that a is not F-torsion. Then*

(i) *($F[a]$, $F[a]^+$) is isomorphic to (T, P) where $T = F[x]$ and P is a partial order contained in P_1.*

(ii) *If If ($F[a]$, $F[a]^+$) is F-semi-closed (this is true, in particular, if R is a torsion-free F-module), then P contains P_0.*

PROOF. That (T, P_0) and (T, P_1) have the stated properties is a straightforward calculation which we will omit. To prove (b) we first assume that R is a torsion-free F-module. Let Q be the totally ordered field of quotients of F, and let R_1 be the module of quotients of R, as in Lemma 4. Then R_1 is an l-algebra over Q with squares positive which contains R. By Lemma 2, $a \ll a^2 \ll a^3 \ll \cdots$ with respect to Q. Thus, for $0 \neq \alpha_n$, $\alpha_1 a + \cdots + \alpha_n a^n \in Q[a]^+$ if and only if $\alpha_n > 0$. For if $\alpha_n > 0$, then by Lemma 3 $a^n > -\alpha_n^{-1}(\alpha_{n-1}a^{n-1} + \cdots + \alpha_1 a)$, so $\alpha_1 a + \cdots + \alpha_n a^n > 0$. And if $\alpha_n < 0$, then $-(\alpha_1 a_1 + \cdots + \alpha_n a^n) > 0$. But then a is transcendental over Q; for if $p(x) \in Q[x]$ is any nonzero polynomial, then we have just seen that either $ap(a) > 0$ or $ap(a) < 0$.

Let $P_i(a) = \{f(a) : f(x) \in P_i(Q)\}$. We claim that $P_0(a) \subseteq Q[a]^+ \subseteq P_1(a)$. To see the first inclusion, take $p(a) = \alpha_0 + \cdots + \alpha_n a^n \in Q[a]$ with $\alpha_0 \geqq 0$ and $\alpha_n > 0$. Then $\alpha \geqq 0 \geqq -(\alpha_1 a + \cdots + \alpha_n a^n)$, so $p(a) \in Q[a]^+$. To see the second inclusion, suppose that $p(a) = \alpha_0 + \alpha_1 a + \cdots + \alpha_n a^n \in Q[a]^+$ with $\alpha_n \neq 0$. Since $ap(a) \in Q[a]^+$, $\alpha_n > 0$ by the previous paragraph. If $n > 1$, then $p(a) \in P_1(a)$. If $n = 1$, then $\alpha_0 < 0$ implies $-\alpha_0 \wedge \alpha_1 a = 0$. This contradicts $\alpha_1 a > -\alpha_0$, and hence $\alpha_0 \geqq 0$. It is now easy to see that (b) is true if R is torsion-free.

For the general case let A be the torsion submodule of R. Then A is an l-ideal of R (Lemma 4), $\bar{R} = R/A$ is torsion-free, and $1 \wedge \bar{a} = 0$ ($\bar{a}$ is the image of a in $\bar{R}$). So (b) is true for ($F[\bar{a}]$, $F[\bar{a}]^+$). By the first paragraph of the proof $\bar{a}$ is transcendental over F, and hence so is a. Furthermore, if $p(a) \in F[a]^+$, then

$p(\bar{a}) \in F[\bar{a}]^+ \subseteq P_1(\bar{a})$. Hence if $(F[a], F[a]^+)$ is isomorphic to (T, P), then $P \subseteq P_1$. This establishes (i).

Since $F^+ \cdot 1 \subseteq R^+$, to prove (ii) it suffices to show that $\alpha_n > 0$ implies $b = \alpha_1 a + \cdots + \alpha_n a^n \in F[a]^+$. But $\bar{b} \in F[\bar{a}]^+$, so there exists $t \in A$ with $b + t \geqq 0$. If $0 < \alpha \in F$ with $\alpha t = 0$, then $\alpha b = \alpha(b + t) \geqq 0$. Since $(F[a], F[a]^+)$ is semi-closed, $b \in F[a]^+$.

REMARK. The construction which appears in Theorem 1 may be generalized. An l-algebra ${}_F R$ is called *supertessimal* if for each $x \in R$ $x \ll x^2$ with respect to F. The class of supertessimal l-algebras is a variety each member of which has no nonzero nilpotent elements. If F is an f-ring and R is a supertessimal l-algebra with squares positive over F, let S be the l-algebra obtained by freely adjoining F to R. Thus, as an f-module over F, $S = F \oplus R$; and multiplication is given by $(\alpha, x)\ (\beta, y) = (\alpha\beta,\ \alpha y + \beta x + xy)$. Then ${}_F S$ is a unital l-algebra with squares positive in which 1 is not a weak order unit.

Note that S could contain nonzero nilpotent elements. To be explicit, let G be a totally ordered field and let $G[t]$ be the ring of polynomials over G in the indeterminate t, ordered lexicographically so that the constant term dominates. Because of the homomorphism $F_n = G[t]/(t^n) \to G$ any l-algebra over G can be made into an l-algebra over F_n. If F_n is used above with $n \geqq 2$, then an S will be produced with nontrivial nilpotent elements.

In general, the set of nilpotent elements of S will be an l-ideal (as is the case for an l-ring that satisfies the identity $x^+ x^- = 0$ [Diem (1968)]). For if $\alpha \in F$ is nilpotent and $x \in R$, then $\alpha x = 0$ since R has no nilpotent elements. So if $(\alpha, x) \in S$ is nilpotent with $0 = (\alpha, x)^n = \left(\alpha^n, \sum_{k \geqq 1} \binom{n}{k} \alpha^{n-k} x^k\right) = (\alpha^n, x^n)$, then $x = 0$ and $\alpha^n = 0$. Thus the set A of nilpotent elements of S is precisely the set of nilpotent elements of F, and hence A is a convex l-subgroup of S (F is an f-ring). Also, if $(\alpha, 0)$ is nilpotent and $(\beta, x) \in S$, then $(\alpha, 0)(\beta, x) = (\alpha\beta, 0)$; so A is an ideal, whence an l-ideal.

3. Semiperfect l-rings

Let R be a unital ring with Jacobson radical N. R is called *semiperfect* if R/N is left artinian and idempotents may be lifted modulo N, and R is called *local* if R/N is a division ring. In a semiperfect ring a finite set of orthogonal idempotents may be lifted modulo N [Lambek (1966), p. 73]. The next lemma is known for f-rings.

LEMMA 5. *If R is a unital l-ring with squares positive, then every idempotent element is central. Consequently, a right (left) ideal generated by an idempotent is an l-ideal.*

PROOF. Let $S = C(1)$ be the convex l-subgroup generated by 1. If e is an idempotent, then so is $1 - e$; hence $e \in S$. Since S is an f-ring the idempotents of S are central elements of S [Henriksen and Isabell (1962), 2.1]. Thus all the idempotents of R commute and so they are all central [Divinsky (1965), p. 25].

Let $A = Re$ be an ideal of R where $e = e^2$, and let $f = 1 - e$. Suppose $|x| \leqq |re| = |r|e$ for some $r \in R$. Then $|xf| \leqq |r|ef = 0$. Hence $xf = 0$ and $x = xe + xf = xe$. Thus A is an l-ideal.

THEOREM 2. *A semiperfect l-ring R with squares positive is an f-ring.*

PROOF. We first reduce to the case that R is local. Since the idempotents of R, and hence of R/N, are central, $R/N = D_1 \oplus \cdots \oplus D_n$ (ring direct sum), where each D_i is a division ring. Let $\{e_i\}$ be an orthogonal set of idempotents of R such that $e_i + N$ is the identity of D_i. Then $1 = e_1 + \cdots + e_n$, so, by Lemma 5, R is a direct sum of local l-rings.

Now assume that R is local. Suppose that $x \wedge y = 0$ and $a \in R^+$. Let $b = a \vee 2$. If $b \notin N$, then $b^{-1} \in R$ and $b^{-1} = bb^{-2} \in R^+$. Since b and b^{-1} are both positive, multiplication by b is a lattice homomorphism of R [Steinberg (1972), Lemma 1], so $bx \wedge by = 0$. If $b \in N$, then $(b-1)^{-1} \in R^+$, whence

$$(b-1)x \wedge (b-1)y = 0.$$

So

$$0 \leqq (b-1)x \wedge y \leqq (b-1)x \wedge (b-1)y = 0.$$

Hence

$$0 \leqq bx \wedge y = [(b-1)x + x] \wedge y \leqq (b-1)x \wedge y + x \wedge y = 0.$$

In either case, $bx \wedge by = 0$. Thus $ax \wedge ay = 0$, and similarly $xa \wedge ya = 0$; i.e., R is an f-ring.

Birkhoff and Pierce [(1968), p. 62, Corollary 5] have shown that R is an f-algebra provided it is a finite dimensional real l-algebra with an identity element that is a weak order unit. Since an artinian ring is semiperfect we get the following generalization of this result.

COROLLARY 2. *A finite dimensional unital l-algebra over a totally ordered field that has squares positive is an f-algebra.*

Note that the l-algebra (T, P_1) of Theorem 1, where $T = \boldsymbol{Q}[x]$, is a commutative l-algebra with squares positive and an identity element. It has the maximum condition on ideals and is l-simple, but is not an f-ring.

Next we consider algebraic l-algebras. The element a in the ring R is called *regular* if there exists x in R with $a = axa$; equivalently, the right (left) ideal

https://doi.org/10.1017/S1446788700014804 Published online by Cambridge University Press

generated by a has an idempotent generator. R is called *regular* if each of its elements is regular, and it is called π-regular if a power of each of its elements is regular. It is well-known (and easily verified) that an algebraic algebra over a field is π-regular.

COROLLARY 3. *A unital π-regular l-ring R that has squares positive is an f-ring*

PROOF. Since the conditions of the corollary are inherited by each l-homomorphic image of R, and since R is a subdirect product of subdirectly irreducible l-rings, we may assume that R itself is subdirectly irreducible. But then R is local. To see this, let L be the set of non-units of R and let N be the Jacobson radical of R. If $a \in R$, then there exists a positive integer n and an idempotent e such that $Ra^n = Re$. By Lemma 5 $e = 0$ or $e = 1$. If $a \in L$, then $e = 0$ and a is nilpotent. If $x \in R$, then xa is also nilpotent; otherwise xa, and hence a, is a unit. Thus $Ra \subseteq N$ and $L = N$; i.e., R is local. Whence R is an f-ring by Theorem 2.

An algebra over a field is *locally finite* is each of its finitely generated subalgebras is finite dimensional. As an analogue of the fact that an algebraic algebra that satisfies a polynomial identity is locally finite [Herstein (1968), p. 167] we have

COROLLARY 4. *A unital algebraic l-algebra R (over a po-field) that has squares positive is a locally finite f-algebra. It is commutative modulo its Jacobson radical.*

PROOF. By Corollary 3 and the remarks preceding it, R is an f-algebra. Recall that in an f-ring the set $Z_n = \{x : x^n = 0\}$ is a nilpotent l-ideal [Birkhoff and Pierce (1968), Theorem 16, p. 63]. Since the Jacobson radical N of R is nil [(1964), p. 19], N is the set of nilpotent elements of R and thus is locally finite. It is well-known [Arens and Kaplansky (1948), Theorem 3.3] (and can easily be seen) that an algebraic algebra without nilpotent elements is strongly regular. Thus $\bar{R} = R/N$ is a regular f-algebra, whence each one-sided ideal of $\bar{R}$ is an l-ideal. If $\bar{P}$ is a prime ideal of $\bar{R}$, then $\bar{R}/\bar{P}$ is totally ordered division ring. Since $\bar{R}/\bar{P}$ is algebraic over its center, a theorem of Albert (1940) or Herstein (1968), p. 103 tells us that $\bar{R}/\bar{P}$ is a field. Thus R/N is commutative, and hence locally finite. Finally, since N and R/N are locally finite, so is R [Jacobson (1964), p. 241].

The ring R is *left π-regular* if for each $a \in R$ there exists an integer n and an $x \in R$ with $a^n = xa^{n+1}$; equivalently, each chain of principal left ideals $Ra \supseteq Ra^2 \supseteq \cdots$ is finite. It is not surprising that *a unital left π-regular l-ring R with squares positive is an f-ring*: To see this let $a \in R^+$ and let $b = a \vee 1$. If

$x \in R$ with $b^n = xb^{n+1}$, then $(1 - xb)b^n = 0$. Since b^n is not a zero divisor in R^+ and since $(1 - xb)^2 b^n = 0$, $(1 - xb)^2 = 0$. Thus $xb = 1 - (1 - xb)$ is invertible and hence so is b. But then left (right) multiplication by b, and hence a, is a lattice homomorphism of R.

Added in proof: The example $(Z[x], P_1)$ of Theorem 1 appears as Example 1.7 in [T. M. Viswanathan (1969), 'Ordered Modules of Fractions', *J. f. d. reine u. angew. Math.* **235**, 78–107].

References

A. A. Albert (1940), 'On ordered algebras', *Bull. Amer. Math. Soc.* **46**, 521–522.

R. F. Arens and I. Kaplansky (1948), 'Topological representation of algebras', *Trans. Amer. Math. Soc.* **63**, 467–48.

G. Birkhoff and R. S. Pierce (1958), 'Lattice-ordered rings', *An. Acad. Brasil. Ci.* **28**, 41–69.

J. E. Diem (1968), 'A radical for lattice-ordered rings', *Pacific J. Math.* **25**, 71–82.

N. Divinsky (1965), '*Rings and Radicals*', (University of Toronto Press, Toronto).

M. Henriksen and J. Isabell (1962), 'Lattice-ordered rings and function rings', *Pacific J. Math.* **12**, 533–565.

I. N. Herstein (1968), 'Noncommutative Rings', *Carus Mathematical Monographs* **15**, Math. Assoc. of America (Wiley).

N. Jacobson (1964), '*Structure of Rings*', rev. ed. Colloquium Publications No. **36**, (Amer. Math. Soc., Providence).

D. G. Johnson (1960), 'A Structure theory for a class of lattice-ordered rings', *Acta. Math.* **104**, 163–215.

J. Lambek (1966), *Lectures on Rings and modules*', (Blaisdell Publishing Co., Waltham, Mass.)

S. A. Steinberg (1972), 'An embedding theorem for commutative lattice-ordered domains', *Proc. Amer. Math. Soc.* **31**, 409–416.

University of Toledo,
Toledo,
Ohio, 43606,
U.S.A.

https://doi.org/10.1017/S1446788700014804 Published online by Cambridge University Press