Glasgow Math. J. **48** (2006) 217–229. © 2006 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089506003016. Printed in the United Kingdom

A GENERALIZATION OF PP-RINGS AND p.q.-BAER RINGS*

LIU ZHONGKUI and ZHAO RENYU

Department of Mathematics, Northwest Normal University, Lanzhou 730070, Gansu, People's Republic of China
e-mail: liuzk@nwnu.edu.cn; renyuzhao026@sohu.com

(Received 6 June, 2005; revised 2 February, 2006; accepted 3 February, 2006)

Abstract. We introduce the concept of left APP-rings which is a generalization of left p.q.-Baer rings and right PP-rings, and investigate its properties. It is shown that the APP property is inherited by polynomial extensions and is a Morita invariant property.

2000 *Mathematics Subject Classification.* 16D40.

1. Introduction. Throughout this paper, R denotes a ring with unity. Recall that R is (*quasi-*) *Baer* if the right annihilator of every nonempty subset (every right ideal) of R is generated by an idempotent of R. In [**13**] Kaplansky introduced Baer rings to abstract various properties of AW^*-algebras and von Neumann algebras. Clark defined quasi-Baer rings in [**9**] and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. As a generalization of quasi-Baer rings, Birkenmeier, Kim and Park in [**6**] introduced the concept of principally quasi-Baer rings. A ring R is called *left principally quasi-Baer* (or simply *left p.q.-Baer*) if the left annihilator of a principal left ideal of R is generated by an idempotent. Similarly, right p.q.-Baer rings can be defined. A ring is called *p.q.-Baer* if it is both right and left p.q.-Baer. Observe that biregular rings and quasi-Baer rings are p.q.-Baer. For more details and examples of left p.q.-Baer rings, see [**3**], [**4**], [**5**], [**6**], and [**15**]. We say a ring R is a *left APP-ring* if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$. This concept is a common generalization of left p.q.-Baer rings and right PP-rings. In this paper we investigate left APP-rings. In section 2 we provide several basic results. In section 3 we discuss various constructions and extensions under which the class of left APP-rings is closed.

For a nonempty subset Y of R, $l_R(Y)$ and $r_R(Y)$ denote the left and right annihilator of Y in R, respectively.

2. Left APP-rings. An ideal I of R is said to be *right s-unital* if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. Note that if I and J are right s-unital ideals, then so is $I \cap J$ (if $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$). It follows from [**22**, Theorem 1] that I is right s-unital if and only if for any finitely many elements $a_1, a_2, \ldots, a_n \in I$ there exists an element $x \in I$ such that $a_i = a_i x$, $i = 1, 2, \ldots, n$. A submodule N of a left R-module M is called a *pure submodule* if $L \otimes_R N \longrightarrow L \otimes_R M$ is a monomorphism

*Supported by National Natural Science Foundation of China (10171082), TRAPOYT and the Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China.

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

for every right R-module L. By [**19**, Proposition 11.3.13], an ideal I is right s-unital if and only if R/I is flat as a left R-module if and only if I is pure as a left ideal of R.

DEFINITION 2.1. A ring R is called a *left APP-ring* if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$.

Right APP-rings may be defined analogously. Clearly every left p.q.-Baer ring is a left APP-ring (thus the class of left APP-rings includes all biregular rings and all quasi-Baer rings).

A ring R is called a *right* (resp. *left*) *PP-ring* if the right (resp. left) annihilator of an element of R is generated by an idempotent. R is called a *PP-ring* if it is both right and left PP. Clearly every Baer ring is a PP-ring. The following result appeared in Fraser and Nicholson [**10**, Proposition 1].

LEMMA 2.2. *The following conditions are equivalent for a ring R.*
(1) *R is a right PP-ring.*
(2) *If $\emptyset \neq X \subseteq R$ then for all $a \in l_R(X)$, $a \in al_R(X)$.*

From [**1**], a ring R is called an *Armendariz* ring if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0$ for every i and j. From [**11**], a ring R is called a *quasi-Armendariz* ring if whenever $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ satisfy $f(x)R[x]g(x) = 0$, we have $a_i R b_j = 0$ for every i and j. Armendariz rings are quasi-Armendariz rings. Results and examples of quasi-Armendariz rings appeared in [**11**].

PROPOSITION 2.3. *For any ring, we have the following implications:*
(1) *right PP $\Rightarrow$ left APP.*
(2) *quasi-Baer $\Rightarrow$ left p.q.-Baer $\Rightarrow$ left APP $\Rightarrow$ quasi-Armendariz.*

Proof. (1). This follows from Lemma 2.2.
(2). If R is a left APP-ring, then, by [**11**, Theorem 3.9], R is a quasi-Armendariz ring. Other implications are clear. □

All of the converses in Proposition 2.3 do not hold. In fact, left p.q.-Baer $\not\Rightarrow$ quasi-Baer follows from [**6**, Example 1.5]. Some examples were given in [**6**, Examples 1.3 and 1.5] to show that the class of left p.q.-Baer rings is not contained in the class of right PP-rings and, the class of right PP-rings is not contained in the class of left p.q.-Baer rings. By Proposition 2.3, it is clear that both of these classes are contained in the class of left APP-rings. This shows that left APP $\not\Rightarrow$ left p.q.-Baer and left APP $\not\Rightarrow$ right PP. Quasi-Armendariz $\not\Rightarrow$ left APP follows from the following example.

EXAMPLE 2.4. Use the ring in [**4**, Example 2.3]. For a given field F, let

$$S = \left\{ (a_n)_{n=1}^{\infty} \in \prod F \,|\, a_n \text{ is eventually constant} \right\},$$

which is a subring of the countably infinite direct product $\prod F$. Then the ring S is a commutative ring. Let $R = S[[x]]$. Clearly S is a reduced ring. Suppose that $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots \in S[[x]]$ are such that $f(x)g(x) = 0$. Then, from [**1**, p. 2269], it follows that $a_i b_j = 0$ for all i and j. Thus R is a reduced ring. From [**2**], R is an Armendariz ring, and so it is a quasi-Armendariz ring.

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

Suppose that R is an APP-ring. Let $f(x) = f_0 + f_1x + f_2x^2 + \cdots$ and $g(x) = g_0 + g_1x + g_2x^2 + \cdots \in R$, where $f_0 = (0, 1, 0, 0, \ldots)$, $f_1 = (0, 1, 0, 1, 0, 0, \ldots)$, $f_2 = (0, 1, 0, 1, 0, 1, 0, 0, \ldots), \ldots,$ and $g_0 = (1, 0, 0, 0, \ldots)$, $g_1 = (1, 0, 1, 0, 0, 0, \ldots)$, $g_2 = (1, 0, 1, 0, 1, 0, 0, 0, \ldots), \ldots$. Then $g(x) \in l_R(Rf(x))$. Thus there exists $h(x) \in l_R(Rf(x))$ such that $g(x) = g(x)h(x)$. Suppose that $h(x) = h_0 + h_1x + h_2x^2 + \cdots$. Now from $h(x)f(x) = 0$ and from [**1**, p. 2269] it follows that $h_if_j = 0$ for all i and j and, so there exists $n_i \in \mathbb{N}$ such that h_i has the form $(b_1^i, 0, b_3^i, 0, \ldots, b_{2n_i+1}^i, 0, 0, 0, \ldots)$, where $b_k^i \in F$, $i = 0, 1, 2, \ldots$. From $g(x)(1 - h(x)) = 0$ it follows that $g_i(1 - h_0) = 0$ and $g_ih_j = 0$ for all i and $j \geq 1$ and, so there exists $m_i \in \mathbb{N}$ such that h_i has the form $(0, b_2^i, 0, b_4^i, 0, \ldots, b_{2m_i}^i, 0, 0, 0, \ldots)$, where $b_k^i \in F$, $i = 1, 2, \ldots$. Thus $h_1 = h_2 = \cdots = 0$ and so $h(x) = h_0$. This contradicts with $g_i = g_ih_0$, $i = 0, 1, \ldots$. Thus R is not APP.

The following is an example of commutative APP-rings which are neither PP nor p.q.-Baer. Recall that a ring R is called a left Bezout ring if every finitely generated left ideal of R is principal. We denote by w.g.dim(R) the weak global dimension of a ring R, which is defined as $sup\{fd(A) | A$ is a left R-module$\}$. Note that w.g.dim$(R) \leq 1$ if and only if every left ideal of R is flat.

Example 2.5. (see, [**8**, p. 64]) Let $\mathbb{Z}$ be the ring of integers and let

$$S = \left(\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}\right) \bigg/ \left(\bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}\right).$$

Then S is clearly a Boolean ring and, by [**8**, p. 64], the weak global dimension of $S[[x]]$ is one and $S[[x]]$ is not semihereditary. Let $R = S[[x]]$. Then every principal ideal of R is flat, and so $R/l_R(Ra) = R/l_R(a) \cong Ra$ is flat. Thus $l_R(Ra)$ is pure as a left ideal of R for every $a \in R$. Hence R is an APP-ring. In [**8**, Theorem 43], it was shown that the power series ring $A[[x]]$ over a von Neumann regular ring A is semihereditary if and only if $A[[x]]$ is a Bezout ring in which all principal ideals are projective. On the other hand, by [**8**, Theorem 42], $S[[x]]$ is a Bezout ring since the weak global dimension of $S[[x]]$ is one. Thus R is not PP, and so is not p.q.-Baer.

Proposition 2.6. *The following conditions are equivalent for a ring R.*
(1) *R is a left APP-ring.*
(2) *If I is a finitely generated left ideal of R then for all $a \in l_R(I)$, $a \in al_R(I)$.*

Proof. Clearly (2) implies (1). Now suppose that R is a left APP-ring and $I = Ra_1 + \cdots + Ra_n$ is a finitely generated left ideal of R. Then $l_R(I) = \cap_{i=1}^n l_R(Ra_i)$. Let $a \in l_R(I)$. Then $a \in l_R(Ra_i)$ for each i. Hence there exists $x_i \in l_R(Ra_i)$ such that $ax_i = a$ for each i. Then $ax = a$, where $x = x_1x_2 \cdots x_n \in l_R(I)$. □

Proposition 2.7. *Suppose that R satisfies the ascending chain condition on principal left ideals. Then the following conditions are equivalent.*
(1) *R is a left APP-ring.*
(2) *R is a left p.q.-Baer ring.*

Proof. Clearly (2) implies (1). Suppose that R is a left APP-ring. For every $a \in R$, denote $L = l_R(Ra)$. Take a maximal principal ideal Rb contained in L. Since $b = bl$ for some $l \in L$, $Rb \subseteq Rl$, so maximality of Rb implies that $Rb = Rl$. Hence $l = xb$ for some $x \in R$ and $b = bxb$ and $Rb = Re$, where $e = xb = e^2$. Clearly $L = Le + L(1 - e)$. Note that if $t \in L(1 - e)$, then $Re \subseteq R(e + t - et) \subseteq L$. Hence $Re = R(e + t - et)$ and,

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

since $Re \cap R(t - et) = 0$, we get that $t - et = 0$. However $(et)^2 = 0$, so $t^2 = 0$. On the other hand, for every $u \in L(1 - e)$, $u = ul$ for some $l \in L$. Consequently $u = ul(1 - e)$. Now $w = l(1 - e) \in L(1 - e)$, so $w^2 = 0$. Consequently $u = uw = uw^2 = 0$. Thus $L(1 - e) = 0$, so $L = Re$ and we are done. □

Note that this reasoning shows in fact that in rings satisfying ascending chain condition on principal left ideals, right s-unital ideals are generated by idempotents (as left ideals).

PROPOSITION 2.8. *Let R be a commutative Bezout ring. Then the following conditions are equivalent.*

(1) *R is an APP-ring.*

(2) *$w.g.dim(R) \leq 1$.*

Proof. If R is a commutative Bezout ring, then w.g.dim$(R) \leq 1$ if and only if every ideal of R is flat if and only if every finitely generated ideal of R is flat if and only if every principal ideal of R is flat if and only if $R/l_R(Ra)$ is flat for every $a \in R$ if and only if R is an APP-ring. □

Note that Baer rings have no nonzero central nilpotent elements, and so commutative Baer rings are reduced. Huh, Kim and Lee in [**12**, Proposition 4] extended this property onto right PP-rings by showing that right PP-rings have no nonzero central nilpotent elements. For left APP-rings we have the following more general result.

PROPOSITION 2.9. *Let R be a left APP-ring. If $0 \neq a \in R$ is such that $l_R(Ra) \subseteq r_R(a)$, then $aRa \neq 0$.*

Proof. Suppose that $aRa = 0$. Then $a \in l_R(Ra)$. Since R is a left APP-ring, there exists $b \in l_R(Ra)$ such that $a = ab$. Thus $b \in r_R(a)$ and so $a = ab = 0$. □

As a corollary we have that left APP-rings have no nonzero central nilpotent elements.

COROLLARY 2.10. *Let R be a left APP-ring. Then R is semiprime if and only if $l_R(Ra) \subseteq r_R(a)$ for all $a \in R$.*

Proof. Suppose that R is semiprime. Note that $((Ra)l_R(Ra)R)^2 = 0$ for all $a \in R$. Thus $Ral_R(Ra)R = 0$ and so $l_R(Ra) \subseteq r_R(a)$ for all $a \in R$. Conversely if $l_R(Ra) \subseteq r_R(a)$ for all $a \in R$, then, by Proposition 2.9, R is semiprime. □

COROLLARY 2.11. *Commutative APP-rings are reduced.*

In [**12**, Example 3], an example was given to show that commutative reduced rings need not be PP. In fact, there exist commutative reduced rings which need not be APP. For example, let R be the ring as in Example 2.4. Then R is a commutative reduced ring. But R is not an APP-ring.

3. Extensions of left APP-rings. In this section we discuss various constructions and extensions under which the class of left APP-rings is closed. We deal with the direct sums as rings without identity when the index sets are infinite. In this case the definitions of right PP-rings, left p.q.-Baer rings and left APP-rings are also valid.

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

Note that the direct sums of right PP-rings need not be right PP. Consider the following example. Let F be a field and $R_i = F$, $i = 1, 2, \dots$. Suppose that $R = \oplus_{i=1}^{\infty} R_i$ is a right PP-ring. Then for $a = (1, 0, 0, \cdots) \in R$, there exists $e \in R$ such that $r_R(a) = eR$. Write $e = (e_1, e_2, \dots, e_n, 0, 0, \dots)$. Denote $x = (x_i)_{i=1}^{\infty}$ where $x_{n+1} = 1$ and $x_i = 0$ for $i = 1, 2, \dots, n, n+2, \dots$. Clearly $ax = 0$ but $x \notin eR$. So R is not a PP-ring. This example also shows that the direct sums of left p.q.-Baer rings need not be left p.q.-Baer.

From [**12**], a ring R is called a *generalized right PP-ring* if for any $x \in R$ the right ideal $x^n R$ is projective for some positive integer n, depending on x, or equivalently, if for any $x \in R$ the right annihilator of x^n is generated by an idempotent for some positive integer n, depending on x. By [**12**, Lemma 1(iv)], R is a generalized right PP-ring if and only if R is a right PP-ring when R is reduced. Note that in the above example, the ring $R = \oplus_{i=1}^{\infty} R_i$ is reduced. So above example also shows that the direct sums of generalized right PP-rings need not be generalized right PP. Hence Proposition 7(ii) of [**12**] is incorrect.

But for left APP-rings we have the following result.

PROPOSITION 3.1. *Let R_i, $i \in I$ be rings. Then we have the following:*
(1) *$R = \prod_{i\in I} R_i$ is a left APP-ring if and only if R_i is a left APP-ring for each $i \in I$.*
(2) *$R = \bigoplus_{i\in I} R_i$ is a left APP-ring if and only if R_i is a left APP-ring for each $i \in I$.*

If $|I| < \infty$, then the result is clear. If $|I| = \infty$, then Proposition 3.1 is a direct corollary of the following more general result. Let $\aleph$ be an infinite cardinal number. Suppose that I is a set and $\{R_i | i \in I\}$ is a family of rings. Let $x = (x_i)_{i\in I} \in \prod_{i\in I} R_i$. We define the support of x as $supp(x) = \{i \in I | x_i \neq 0\}$. For an infinite cardinal number $\aleph$, define the $\aleph$-product of the R_i's as

$$\prod_{i\in I}^{\aleph} R_i = \left\{ x \in \prod_{i\in I} R_i \mid |supp(x)| < \aleph \right\}.$$

Clearly one may view the direct sum and the direct product of a family of rings as two special cases of the same object, namely, the $\aleph$-product of the family of rings. $\aleph$-products of some families of modules have been studied by [**17**], [**20**] and [**21**].

PROPOSITION 3.2. *Let R_i, $i \in I$ be rings. Then $R = \prod_{i\in I}^{\aleph} R_i$ is a left APP-ring if and only if R_i is a left APP-ring for each $i \in I$.*

Proof. If the ring R is a left APP-ring, then clearly so is each R_i. Conversely suppose that every R_i is a left APP-ring. Let $a = (a_i)_{i\in I}$ and $b = (b_i)_{i\in I}$ be in R such that $aRb = 0$. Then $a_i R_i b_i = 0$ for every $i \in I$. Thus, for every $i \in supp(b)$, there exists $c_i \in R_i$ such that $a_i = a_i c_i$ and $c_i R_i b_i = 0$. Now define $x = (x_i)_{i\in I}$ via

$$x_i = \begin{cases} c_i & i \in supp(b) \\ 1 & i \in supp(a) - supp(b) \\ 0 & i \notin supp(a) \cup supp(b). \end{cases}$$

Then $x \in R$ since $|supp(x)| < \aleph$, and $a = ax$, $xRb = 0$. Thus R is a left APP-ring. $\square$

Let A be a ring, B be a unitary subring of A, $\{A_i\}_{i=1}^{\infty}$ be a countable set of copies of A, D be the direct product of all rings A_i, and let $R = R(A, B)$ be the subring of D generated by the ideal $\bigoplus_{i=1}^{\infty} A_i$ and by the subring $\{(b, b, \cdots) | b \in B\}$ (see [**23**]). Then we have the following result.

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

PROPOSITION 3.3. *If A is a commutative ring, then the ring $R(A, B)$ is an APP-ring if and only if A and B are APP-rings.*

Proof. Denote $R = R(A, B)$. Let $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty} \in R$ be such that $(x_i)_{i=1}^{\infty} R (y_i)_{i=1}^{\infty} = 0$. We note that there exists n such that $x_n = x_{n+1} = \cdots \in B$ and $y_n = y_{n+1} = \cdots \in B$. Clearly we have $x_i A y_i = 0$ for $i = 1, 2, \ldots, n$. Since A is an APP-ring, there exists $w_i \in A$ such that $x_i = x_i w_i$ and $w_i A y_i = 0$, $i = 1, 2, \ldots, n-1$. Since B is an APP-ring and $x_n B y_n = 0$, there exists $w_n \in B$ such that $x_n = x_n w_n$ and $w_n B y_n = 0$. Since A is commutative, we have $w_n A y_n = 0$. Thus $(x_i)_{i=1}^{\infty} = (x_i)_{i=1}^{\infty}(w_1, w_2, \ldots, w_{n-1}, w_n, w_n, \ldots)$ and $(w_1, w_2, \ldots, w_{n-1}, w_n, w_n, \ldots) R (y_i)_{i=1}^{\infty} = 0$. Hence R is an APP-ring.

Conversely, if R is an APP-ring, then it is easy to see that A and B are APP-rings by noting that A is commutative. □

Note that if $R(A, B)$ is a left APP-ring, then A is a left APP-ring. But Example 3.9(2) shows that B need not be a left APP-ring in general.

PROPOSITION 3.4. *Let A be a left APP-ring. If $l_B(Ab) = 0$ for every $0 \neq b \in B$, then the ring $R(A, B)$ is a left APP-ring.*

Proof. In the proof of Proposition 3.3, if $y_n = 0$, then take $w_n = 1 \in B$. If $y_n \neq 0$, then $l_B(Ay_n) = 0$. Thus $x_n = 0$. If we take $w_n = 0$, then $x_n = x_n w_n$ and $w_n A y_n = 0$. Thus $(x_i)_{i=1}^{\infty} = (x_i)_{i=1}^{\infty}(w_1, w_2, \ldots, w_{n-1}, w_n, w_n, \ldots)$ and $(w_1, w_2, \ldots, w_{n-1}, w_n, w_n, \ldots) R (y_i)_{i=1}^{\infty} = 0$. Hence R is a left APP-ring. □

Let n be a positive integer. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R.

PROPOSITION 3.5. *R is a left APP-ring if and only if $M_n(R)$ is a left APP-ring.*

Proof. Let R be a left APP-ring and $A = (a_{ij}) \in M_n(R)$. Suppose that $B = (b_{ij}) \in M_n(R)$ is such that $B \in l_{M_n(R)}(M_n(R)A)$. Then $BM_n(R)A = 0$. Let E_{ij} denote the (i, j)-matrix unit. Then $(\sum_{p,q} b_{pq}E_{pq}) r E_{ij} (\sum_{s,t} a_{st}E_{st}) = 0$ for any $r \in R$ and any i and j. Thus $\sum_{p,t} b_{pi} r a_{jt} E_{pt} = 0$, which implies that $b_{pi} r a_{jt} = 0$ for any p and t. Hence $b_{pi} \in l_R(Ra_{jt})$ for all i, j, p and t. So $b_{pq} \in l_R(\sum_{i,j} Ra_{ij})$ for all p, q. By Proposition 2.6, there exists $c \in l_R(\sum_{i,j} Ra_{ij})$ such that $b_{pq} = b_{pq}c$ for all p, q. Thus

$$B = B \begin{pmatrix} c & & & \\ & c & & \\ & & \ddots & \\ & & & c \end{pmatrix}$$

and it is easy to see that

$$\begin{pmatrix} c & & & \\ & c & & \\ & & \ddots & \\ & & & c \end{pmatrix} M_n(R)A = 0.$$

Thus $M_n(R)$ is a left APP-ring.

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

Conversely suppose that $M_n(R)$ is a left APP-ring and $a, b \in R$ is such that $a \in l_R(Rb)$. Set

$$A = \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

Then $AM_n(R)B = 0$. Thus there exists $C = (c_{ij}) \in M_n(R)$ such that $A = AC$, $CM_n(R)B = 0$. Now it is easy to see that $a = ac_{11}$ and $c_{11}Rb = 0$. Thus R is a left APP-ring. □

PROPOSITION 3.6. *R is a left APP-ring if and only if the upper triangular matrix ring $T_n(R)$ over R is a left APP-ring.*

Proof. Let R be a left APP-ring and $A = (a_{ij}) \in T_n(R)$. Suppose that $B = (b_{ij}) \in T_n(R)$ is such that $B \in l_{T_n(R)}(T_n(R)A)$. Then $BT_n(R)A = 0$. By analogy with the proof of Proposition 3.5, we obtain that $b_{pi} \in l_R(Ra_{jt})$ for all i, j, p and t with $p \le i \le j \le t$. Thus $b_{11} \in l_R(\sum_{1\le i\le j\le n} Ra_{ij})$, $b_{12}, b_{22} \in l_R(\sum_{2\le i\le j\le n} Ra_{ij}), \ldots, b_{1,n-1}, b_{2,n-1}, \ldots, b_{n-1,n-1} \in l_R(\sum_{n-1\le i\le j\le n} Ra_{ij})$, $b_{1n}, b_{2n}, \ldots, b_{nn} \in l_R(Ra_{nn})$. Since R is a left APP-ring, by Proposition 2.6, there exist $c_1, c_2, \ldots, c_n$ such that

$$c_1 \in l_R\Bigg(\sum_{1\le i\le j\le n} Ra_{ij}\Bigg), \quad b_{11} = b_{11}c_1,$$

$$c_2 \in l_R\Bigg(\sum_{2\le i\le j\le n} Ra_{ij}\Bigg), \quad b_{12} = b_{12}c_2, b_{22} = b_{22}c_2,$$

$$\cdots\cdots\cdots\cdots$$

$$c_n \in l_R(Ra_{nn}), \qquad b_{kn} = b_{kn}c_n, k = 1, 2, \ldots, n.$$

Now it is easy to see that

$$B = B\begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix}, \quad \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix} T_n(R)A = 0.$$

Hence $T_n(R)$ is a left APP-ring.

Conversely if $T_n(R)$ is a left APP-ring, then, by analogy with the proof of Proposition 3.5, we can show that R is left APP. □

PROPOSITION 3.7. *Let $e \in R$ be an idempotent. If R is a left APP-ring then eRe is a left APP-ring.*

Proof. Let $x \in eRe$ and $a \in l_{eRe}(eRex)$. Then $aRexe = (ae)Rexe = a(eRe)xe = 0$. Thus $a \in l_R(R(exe))$. Since $l_R(R(exe))$ is pure as a left ideal of R, there exists $b \in l_R(R(exe))$ such that $a = ab$. Thus $a = ae = abe = (eae)be = (eae)(ebe) = a(ebe)$ and $(ebe)(eRe)x = eb(eRe)x = eb(eRe)xe = eb(eR)(exe) \subseteq ebR(exe) = 0$. Hence $ebe \in l_{eRe}(eRex)$. This means that $l_{eRe}(eRex)$ is pure as a left ideal of eRe and so eRe is a left APP-ring. □

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

From [**6**, Theorem 2.2], the concept of left p.q.-Baer rings is a Morita invariant property. But the concept of right PP-rings is not a Morita invariant property because $\mathbb{Z}[x]$ is Baer but the 2×2 full matrix ring over $\mathbb{Z}[x]$ is not a right PP-ring ([**2**]). From Propositions 3.5 and 3.7, for left APP-rings we have the following result.

THEOREM 3.8. *The endomorphism ring of a finitely generated projective module over a left APP-ring is left APP. In particular, the left APP condition is a Morita invariant property.*

EXAMPLE 3.9. (1). Subrings of a left APP-ring need not be left APP. Let $A = M_2(F)$ where F is a field. Then A is a left APP-ring by Proposition 3.5. Let

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} | a, b \in F \right\}.$$

Then B is not a left APP-ring by Proposition 2.9.

(2). Factor rings of a left APP-ring need not be left APP. The ring $\mathbb{Z}$ of integers is an APP-ring whereas its homomorphic image $\mathbb{Z}/4\mathbb{Z}$ is not. The following is another example of such rings. Let A, B be as in (1). Suppose that $a, b, c, d \in F$ are such that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} A \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = 0$$

but $\left(\begin{smallmatrix} c & d \\ 0 & c \end{smallmatrix}\right) \neq 0$. If $c \neq 0$, then clearly $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) = 0$. If $c = 0$, then $d \neq 0$. From

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} = 0$$

it follows that $bd = 0$ and $ad = 0$, which imply that $a = b = 0$. Thus, $l_B\big(A\left(\begin{smallmatrix} c & d \\ 0 & c \end{smallmatrix}\right)\big) = 0$, and so, by Proposition 3.4, $R = R(A, B)$ is a left APP-ring since A is a left APP-ring. But the factor ring $R/(\oplus_{i=1}^{\infty} A_i)$, which is isomorphic to B by [**23**, Example 15.7(2)], is not left APP. This example also shows that if $R(A, B)$ is a left APP-ring, then B need not be a left APP-ring in general.

Note that the ring $R = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) | a, b \in F\right\}$, where F is a given field, is a generalized right PP-ring by [**12**, Proposition 3]. So Example 3.9(1) shows that generalized right PP-rings need not be left APP. On the other hand, let $R = M_2(\mathbb{Z}[x])$. Then R is both left and right APP by Proposition 3.5 and Corollary 3.12. But R is not a generalized right PP-ring by [**12**, Example 4]. Thus left APP-rings need not be generalized right PP.

Recall that a monoid S is called a *u.p.-monoid* (unique product monoid) if for any two nonempty finite subsets $A, B \subseteq S$ there exists an element $g \in S$ uniquely presented in the form ab where $a \in A$ and $b \in B$. The class of u.p.-monoids is quite large and important (see [**7**], [**16**] and [**18**]). For example, this class includes the right or left ordered monoids, submonoids of a free group, and torsion-free nilpotent groups. Every u.p.-monoid S has no non-unity element of finite order.

Let R be a ring and S a u.p.-momoid. Assume that there is a monoid homomorphism $\alpha : S \longrightarrow \text{Aut}(R)$. For any $s \in S$, we denote the image of s under α by α_s. Then we can form a *skew monoid ring* $R * S$ (induced by the monoid homomorphism α) by taking its elements to be finite formal combinations $\sum_{s \in S} a_s s$, with multiplication

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

induced by:

$$(a_s s)(b_t t) = a_s \alpha_s(b_t)(st).$$

A monoid homomorphism $\alpha : S \longrightarrow \mathrm{Aut}(R)$ is said to satisfy condition (*) if for every $a \in R$, the left ideal $\sum_{s \in S} R\alpha_s(a)$ is finitely generated. In [**11**, Theorem 3.9], it was shown that a ring R is left APP if and only if $R[x]$ is left APP. For skew monoid rings we have the following result.

THEOREM 3.10. *Let R be a left APP-ring and S a u.p.-monoid. If $\alpha : S \longrightarrow \mathrm{Aut}(R)$ satisfies the condition $(*)$, then the skew monoid ring $R * S$ (induced by the monoid homomorphism α) is a left APP-ring.*

Proof. Suppose that $f = a_1s_1 + a_2s_2 + \cdots + a_ns_n$, $g = b_1t_1 + b_2t_2 + \cdots + b_mt_m \in R * S$ are such that $f \in l_{R*S}((R * S)g)$. Then $f(R * S)g = 0$. Thus for every $s \in S$ and every $r \in R$, $f(rs)g = 0$. Suppose that $c_1, c_2, \ldots, c_n \in R$ are such that $a_i = \alpha_{s_i}(c_i)$ for $i = 1, 2, \ldots, n$. We will show that $c_i \in l_R(R\alpha_s(b_j))$ for every $s \in S$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ by induction on m.

If $m = 1$, then $g = b_1t_1$. Thus $0 = (a_1s_1 + a_2s_2 + \cdots + a_ns_n)(rs)(b_1t_1) = a_1\alpha_{s_1}(r\alpha_s(b_1))s_1st_1 + a_2\alpha_{s_2}(r\alpha_s(b_1))s_2st_1 + \cdots + a_n\alpha_{s_n}(r\alpha_s(b_1))s_nst_1$ for every $r \in R$. By [**7**, Lemma 1.1], S is a cancellative monoid. Thus $s_ist_1 \neq s_jst_1$ for $s_i \neq s_j$. Hence $a_i\alpha_{s_i}(r\alpha_s(b_1)) = 0$, which implies that $c_i \in l_R(R\alpha_s(b_1))$ since α_{s_i} is an automorphism, $i = 1, 2, \ldots, n$.

Now suppose that $m \geq 2$. Since S is a u.p.-monoid, there exist p, q with $1 \leq p \leq n$ and $1 \leq q \leq m$ such that s_pst_q is uniquely presented by considering two subsets $\{s_1, s_2, \ldots, s_n\}$ and $\{st_1, st_2, \ldots, st_m\}$ of S. Thus from $f(rs)g = 0$ it follows that $a_p\alpha_{s_p}(r\alpha_s(b_q))s_pst_q = 0$ and so $a_p\alpha_{s_p}(r\alpha_s(b_q)) = 0$. Thus $\alpha_{s_p}(c_pr\alpha_s(b_q)) = 0$, which implies that $c_pr\alpha_s(b_q) = 0$ for every $r \in R$ since α_{s_p} is an automorphism. Hence $c_p \in l_R(R\alpha_s(b_q))$. Since $l_R(R\alpha_s(b_q))$ is pure as a left ideal of R, there exists an element $e_q \in l_R(R\alpha_s(b_q))$ such that $c_p = c_pe_q$. Thus for every $r \in R$, we have

$$\begin{aligned}
0 = f(e_qrs)g &= (a_1s_1 + a_2s_2 + \cdots + a_ns_n)(e_qrs) \\
&\quad \cdot (b_1t_1 + b_2t_2 + \cdots + b_{q-1}t_{q-1} + b_{q+1}t_{q+1} + \cdots + b_mt_m) \\
&\quad + (a_1s_1 + a_2s_2 + \cdots + a_ns_n)((e_qr\alpha_s(b_q))st_q) \\
&= (a_1\alpha_{s_1}(e_q)s_1 + a_2\alpha_{s_2}(e_q)s_2 + \cdots + a_n\alpha_{s_n}(e_q)s_n)(rs) \\
&\quad \cdot (b_1t_1 + b_2t_2 + \cdots + b_{q-1}t_{q-1} + b_{q+1}t_{q+1} + \cdots + b_mt_m).
\end{aligned}$$

Since $a_i\alpha_{s_i}(e_q) = \alpha_{s_i}(c_ie_q)$, by induction, it follows that $c_ie_q \in l_R(R\alpha_s(b_j))$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, q-1, q+1, \ldots, m$. Therefore

$$c_p = c_pe_q \in \cap_{j=1}^{m} l_R(R\alpha_s(b_j)).$$

Now $a_p\alpha_{s_p}(R\alpha_s(b_j)) = \alpha_{s_p}(c_pR\alpha_s(b_j)) = 0$ for any $j = 1, 2, \ldots, m$. Thus from $f(rs)g = 0$ it follows that

$$\begin{aligned}
0 = &(a_1s_1 + a_2s_2 + \cdots + a_{p-1}s_{p-1} + a_{p+1}s_{p+1} + \cdots + a_ns_n) \\
&\cdot (rs)(b_1t_1 + b_2t_2 + \cdots + b_mt_m).
\end{aligned}$$

By using the previous method, there exists $k \in \{1, 2, \ldots, p-1, p+1, \ldots, n\}$ such that $c_k \in \cap_{j=1}^{m} l_R(R\alpha_s(b_j))$. Thus $a_k\alpha_{s_k}(R\alpha_s(b_j)) = \alpha_{s_k}(c_kR\alpha_s(b_j)) = 0$ for any $j = 1, 2, \ldots, m$.

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

Hence $(a_1s_1 + a_2s_2 + \cdots + a_{p-1}s_{p-1} + a_{p+1}s_{p+1} + \cdots + a_{k-1}s_{k-1} + a_{k+1}s_{k+1} + \cdots + a_ns_n)$ $(rs)(b_1t_1 + b_2t_2 + \cdots + b_mt_m) = 0$. Continuing this procedure yields $c_1, c_2, \ldots, c_n \in \cap_{j=1}^m l_R(R\alpha_s(b_j))$ for every $s \in S$.

Set

$$L = \sum_{j=1}^{m} \sum_{s \in S} R\alpha_s(b_j).$$

Then $c_1, c_2, \ldots, c_n \in l_R(L)$. Since α satisfies the condition (*), it is easy to see that L is finitely generated. From Proposition 2.6, $l_R(L)$ is pure as a left ideal of R. Thus there exists $d \in l_R(L)$ such that $c_i = c_id$, $i = 1, 2, \ldots, n$. Denote by η the identity of the monoid S. Then $f(d\eta) = \sum_{i=1}^n a_i\alpha_{s_i}(d)s_i = \sum_{i=1}^n \alpha_{s_i}(c_id)s_i = \sum_{i=1}^n \alpha_{s_i}(c_i)s_i = \sum_{i=1}^n a_is_i = f$. For every $r \in R$ and every $s \in S$, $r\alpha_s(b_j) \in L$ and, so $(d\eta)(rs)g = \sum_{j=1}^m dr\alpha_s(b_j)(st_j) = 0$. Thus $d\eta \in l_{R*S}((R*S)g)$. This shows that $R*S$ is a left APP-ring. □

REMARK 3.11. It is natural to ask for examples of monoid homomorphisms $\alpha : S \longrightarrow Aut(R)$ which satisfy the condition (*).

1. If $\alpha(s) = 1$ for every $s \in S$, then α satisfies the condition (*).

2. Let T be a ring and $R = T \oplus T$. Let $\gamma : R \longrightarrow R$ be an automorphism defined by $\gamma((a, b)) = (b, a)$. Let $S = \mathbb{Z}$ (or $S = \mathbb{N} \cup \{0\}$). Define $\alpha : S \longrightarrow \mathrm{Aut}(R)$ via $\alpha_0 = 1$ and $\alpha_n = \gamma^n$ for every $0 \neq n \in \mathbb{Z}$. Then $\sum_{n\in\mathbb{Z}} R\alpha_n((a, b)) = R(a, b) + R(b, a)$ for every $(a, b) \in R$. Thus α is a monoid homomorphism satisfying the condition (*).

3. Let T be a ring and $R = M_2(T)$. Let $\gamma : R \longrightarrow R$ be an automorphism defined by

$$\gamma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Let $S = \mathbb{Z}$ (or $S = \mathbb{N} \cup \{0\}$). Define $\alpha : S \longrightarrow \mathrm{Aut}(R)$ via $\alpha_0 = 1$ and $\alpha_n = \gamma^n$ for every $0 \neq n \in \mathbb{Z}$. Then

$$\sum_{n\in\mathbb{Z}} R\alpha_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = R\begin{pmatrix} a & b \\ c & d \end{pmatrix} + R\begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Thus α satisfies the condition (*).

4. Let

$$R = \left\{\begin{pmatrix} a & q \\ 0 & a \end{pmatrix} | a \in \mathbb{Z}, q \in \mathbb{Q}\right\}.$$

Define $\gamma : R \longrightarrow R$ via

$$\gamma\left(\begin{pmatrix} a & q \\ 0 & a \end{pmatrix}\right) = \begin{pmatrix} a & 2q \\ 0 & a \end{pmatrix}.$$

Then it is easy to see that $\sum_{n=0}^{\infty} R\gamma^n(\left(\begin{smallmatrix} a & q \\ 0 & a \end{smallmatrix}\right)) = R\left(\begin{smallmatrix} a & q \\ 0 & a \end{smallmatrix}\right)$. Thus $\alpha : S = \mathbb{N} \cup \{0\} \longrightarrow \mathrm{Aut}(R)$, defined by $\alpha_0 = 1$ and $\alpha_n = \gamma^n$ for every $n \in \mathbb{N}$, satisfies the condition (*).

Armendariz showed that polynomial rings over right PP-rings need not be right PP in the example in [**2**]. From [**5**, Theorem 2.1], a ring R is a left p.q.-Baer ring if and only if $R[x]$ is a left p.q.-Baer ring. It was shown in [**7**, Corollary 1.4] that R is left

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

p.q.-Baer if and only if $R[x, x^{-1}]$ is left p.q.-Baer. For monoid rings, it was shown that the monoid ring $R[S]$ of a u.p.-monoid S over a ring R is left p.q.-Baer if and only if R is left p.q.-Baer (see [**7**, Theorem 1.2]) and $R[S]$ is a reduced PP-ring if and only if R is a reduced PP-ring (see [**7**, Corollary 1.3]). For left APP-rings we have the following result.

COROLLARY 3.12. *Let S be a u.p.-monoid and X a nonempty set of not necessarily commuting indeterminates. Then the following conditions are equivalent.*

(1) *R is left APP.*
(2) *$R[X]$ is left APP.*
(3) *$R[x, x^{-1}]$ is left APP.*
(4) *$R[S]$ is left APP.*

Proof. The implication (1) ⇒ (4) follows from Theorem 3.10. (4) ⇒ (1). Let $a, b \in R$ be such that $a \in l_R(Rb)$. Then $a \in l_{R[S]}(R[S]b)$. Thus there exists $\sum_{i=0}^{n} a_i s_i \in l_{R[S]}(R[S]b)$ with $s_0 = \eta$, the identity of S, such that $a = a(\sum_{i=0}^{n} a_i s_i)$. Now it is easy to see that $a = aa_0$ and $a_0 Rb = 0$. Thus R is a left APP-ring.

(1) ⇔ (2) ⇔ (3) follow from (1) ⇔ (4), noting that the monoid generated by X is a u.p.-monoid and $R[x, x^{-1}] \cong R[\mathbb{Z}]$, the monoid ring of the u.p.-monoid $\mathbb{Z}$ over R. □

COROLLARY 3.13. *Let R be a left APP-ring and α a ring automorphism of R such that $\sum_{i=0}^{\infty} R\alpha^i(b)$ is finitely generated for every $b \in R$. Then the skew polynomial ring $R[x; \alpha]$ is a left APP-ring.*

There exists a commutative von Neumann regular ring R (hence left APP), but the ring $R[[x]]$ is not APP. For example, let R be the ring S defined in Example 2.4. Then R is a commutative von Neumann regular ring. By Example 2.4, $R[[x]]$ is not an APP-ring.

Some additional conditions were given in [**10**], [**14**] and [**15**] for right PP-rings (or left p.q.-Baer rings) under which the formal power series ring $R[[x]]$ over R is a right PP-ring (or left p.q.-Baer ring, respectively). For left APP-rings we have the following results.

PROPOSITION 3.14. *Let R be a ring satisfying descending chain condition on left and right annihilators. If R is a left APP-ring, then so is $R[[x]]$.*

Proof. Suppose that $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ and $g(x) = b_0 + b_1x + b_2x^2 + \cdots \in R[[x]]$ are such that $f(x) \in l_{R[[x]]}(R[[x]]g(x))$. Then $f(x)R[[x]]g(x) = 0$. Thus $f(x)Rg(x) = 0$. It follows that

$$\sum_{i+j=k} a_i r b_j = 0, \qquad k = 0, 1, 2, \ldots,$$

where r is an arbitrary element of R. Thus, since $a_0 r b_0 = 0$, one has $a_0 \in l_R(Rb_0)$. So there exists $w_0 \in l_R(Rb_0)$ such that $a_0 = a_0 w_0$. Let $r' \in R$ and take $r = w_0 r'$ in $a_1 r b_0 + a_0 r b_1 = 0$. Then $a_1 w_0 r' b_0 + a_0 w_0 r' b_1 = 0$. But $a_1 w_0 r' b_0 = 0$. So $a_0 w_0 r' b_1 = 0$. Since $a_0 = a_0 w_0$, we have $a_0 r' b_1 = 0$, which implies that $a_0 \in l_R(Rb_1)$. Also $a_1 r b_0 = 0$ for any $r \in R$. This means that $a_1 \in l_R(Rb_0)$.

Now assume that

$$a_i \in l_R(Rb_j), \qquad i + j = 0, 1, 2, \ldots, k - 1.$$

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

Then, since R is a left APP-ring, there exists $w_0 \in l_R(Rb_0)$ such that $a_i = a_i w_0$, $i = 0, 1, \ldots, k-1$. Let $r' \in R$ and take $r = w_0 r'$ in $\sum_{i+j=k} a_i r b_j = 0$. Then, since $a_k w_0 r' b_0 = 0$, we have

$$a_0 w_0 r' b_k + a_1 w_0 r' b_{k-1} + \cdots + a_{k-1} w_0 r' b_1 = a_0 r' b_k + a_1 r' b_{k-1} + \cdots + a_{k-1} r' b_1 = 0.$$

From $a_0, a_1, \ldots, a_{k-2} \in l_R(Rb_1)$ it follows that there exists $w_1 \in l_R(Rb_1)$ such that $a_i = a_i w_1$, $i = 0, 1, \ldots, k-2$. Let $y \in R$ and take $r' = w_1 y$. Then, since $a_{k-1} w_1 y b_1 = 0$, we have

$$a_0 w_1 y b_k + a_1 w_1 y b_{k-1} + \cdots + a_{k-2} w_1 y b_2 = a_0 y b_k + a_1 y b_{k-1} + \cdots + a_{k-2} y b_2 = 0.$$

Continuing in this manner, we have $a_0 c b_k = 0$, where c is an arbitrary element of R. This implies that $a_1 c b_{k-1} = 0, \ldots, a_{k-1} c b_1 = 0, a_k c b_0 = 0$. Thus

$$a_0 \in l_R(Rb_k), \quad a_1 \in l_R(Rb_{k-1}), \ldots, a_k \in l_R(Rb_0).$$

Therefore, by the induction principle, we have shown that $a_i \in l_R(Rb_j)$, $i, j = 0, 1, \ldots$.

Consider the descending chain as following:

$$l_R(Rb_0) \supseteq l_R(Rb_0 + Rb_1) \supseteq l_R(Rb_0 + Rb_1 + Rb_2) \supseteq \cdots.$$

Then there exists m such that $l_R(Rb_0 + Rb_1 + \cdots + Rb_m) = l_R(Rb_0 + Rb_1 + \cdots + Rb_m + Rb_{m+1}) = \cdots$. On the other hand, by considering the descending chain as following:

$$r_R(a_0) \supseteq r_R(a_0, a_1) \supseteq r_R(a_0, a_1, a_2) \supseteq \cdots,$$

there exists n such that $r_R(a_0, a_1, \ldots, a_n) = r_R(a_0, a_1, \ldots, a_n, a_{n+1}) = \cdots$. Since $a_0, a_1, \ldots, a_n \in l_R(Rb_0 + Rb_1 + \cdots + Rb_m)$, by Proposition 2.6, there exists $c \in l_R(Rb_0 + Rb_1 + \cdots + Rb_m)$ such that $a_i = a_i c$ for $i = 0, 1, \ldots, n$. Thus $1 - c \in r_R(a_0, a_1, \ldots, a_n)$. So $1 - c \in r_R(a_0, a_1, \ldots, a_n, \ldots, a_k)$ for any $k \geq n$, which implies that $a_k = a_k c$ for any $k \geq n$. Now it is easy to see that $f(x) = f(x)c$ and $c \in l_{R[[x]]}(R[[x]]g(x))$. This shows that $R[[x]]$ is a left APP-ring. □

A ring R is said to be $\aleph_0$-*self-injective* if each R-homomorphism from a countably generated left ideal L of R into R is induced by multiplication by an element of R. R is said to be *left duo* if every left ideal of R is two-sided.

Proposition 3.15. *Let R be a reduced left duo ring which is $\aleph_0$-self-injective. If R is a left APP-ring, then so is* $R[[x]]$.

Proof. Suppose that $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ and $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots \in R[[x]]$ are such that $f(x) \in l_{R[[x]]}(R[[x]]g(x))$. Then $f(x)R[[x]]g(x) = 0$. Thus $f(x)Rg(x) = 0$. By analogy with the proof of Proposition 3.14, we have $a_i R b_j = 0$ for all i and j. Let $I = \sum_{i=0}^{\infty} Ra_i$, $J = \sum_{i=0}^{\infty} Rb_i$. Then $IJ = 0$. Hence $I \bigcap J = 0$ since R is reduced. Therefore, the projection map $\alpha : I \bigoplus J \longrightarrow R$ via $\alpha(a + b) = a$, $a \in I, b \in J$, is well-defined and by hypothesis, α is given by multiplication by an element $c \in R$. Now for every $a \in I$, $a = \alpha(a + b) = (a + b)c = ac + bc$. Since R is a left duo ring, we have $a = ac$ for every $a \in I$ and $bc = 0$ for every $b \in J$. Since R is reduced, it follows that $cRb_i = 0$, $i = 0, 1, 2, \ldots$. Thus $f(x) = f(x)c$ and $c \in l_{R[[x]]}(R[[x]]g(x))$. This shows that $R[[x]]$ is a left APP-ring. □

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press

ACKNOWLEDGMENT. The authors wish to express their sincere thanks to the referee for valuable suggestions and especially for improvement of Proposition 2.9 and Corollary 2.10.

REFERENCES

1. D. D. Anderson and V. Camillo, Armendariz rings and Gaussian rings, *Comm. Algebra* **26** (1998), 2265–2272.

2. E. P. Armendariz, A note on extensions of Baer and p.p.-rings, *J. Austral. Math. Soc.* **18** (1974), 470–473.

3. G. F. Birkenmeier, J. Y. Kim and J. K. Park, A sheaf representation of quasi-Baer rings, *J. Pure Appl. Algebra* **146** (2000), 209–223.

4. G. F. Birkenmeier, J. Y. Kim and J. K. Park, On quasi-Baer rings, *Contemp. Math.* **259** (2000), 67–92.

5. G. F., Birkenmeier, J. Y. Kim and J. K. Park, On polynomial extensions of principally quasi-Baer rings, *Kyungpook Math. J.* **40** (2000), 247–254.

6. G. F. Birkenmeier, J. Y. Kim and J. K. Park, Principally quasi-Baer rings, *Comm. Algebra* **29** (2001), 639–660.

7. G. F. Birkenmeier and J. K. Park, Triangular matrix representations of ring extensions, *J. Algebra* **265** (2003), 457–477.

8. J. W. Brewer, *Power series over commutative rings* (Marcel Dekker, 1981).

9. W. E. Clark, Twisted matrix units semigroup algebras, *Duke Math. J.* **34** (1967), 417–423.

10. J. A. Fraser and W. K. Nicholson, Reduced PP-rings, *Math. Japonica* **34** (1989), 715–725.

11. Y. Hirano, On annihilator ideals of a polynomial ring over a noncommutative ring, *J. Pure Appl. Algebra* **168** (2002), 45–52.

12. C. Huh, H. K. Kim and Y. Lee, p.p. rings and generalized p.p. rings, *J. Pure Appl. Algebra* **167** (2002), 37–52.

13. I. Kaplansky, *Rings of Operators* (W. A. Benjamin, 1968).

14. Z. K. Liu and J. Ahsan, PP-rings of generalized power series, *Acta Math. Sinica, English Series* **16** (2000), 573–578.

15. Z. K. Liu, A note on principally quasi-Baer rings, *Comm. Algebra* **30** (2002), 3885–3890.

16. J. Okninski, *Semigroup algebras* (Marcel Dekker, 1991).

17. L. Oyonarte and B. Torrecillas, $\aleph$-products of injective objects in Grothendieck categories, *Comm. Algebra* **25** (1997), 923–934.

18. D. S. Passman, *The algebraic structure of group rings* (Wiley, 1977).

19. B. Stenstrom, *Rings of quotients* (Springer-Verlag, 1975).

20. M. L. Teply, Large subdirect products, *Proc. Inter. Conf. Ring Theory, Granada, Spain*, Lecture Notes in Mathematics **1328** (Springer, 1986), 283–304.

21. M. L. Teply, *Semicocritical modules* (University of Murcia Press, 1987).

22. H. Tominaga, On *s*-unital rings, *Math. J. Okayama Univ.* **18** (1976), 117–134.

23. A. Tuganbaev, *Rings close to regular* (Kluwer, 2002).

https://doi.org/10.1017/S0017089506003016 Published online by Cambridge University Press