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TRACE-CLASS OPERATORS IN *CSL* ALGEBRAS

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ABSTRACT. In this note we show that if $\mathcal{L}$ is a commutative subspace lattice, then every trace-class operator in Alg $\mathcal{L}$ lies in the norm-closure of the span of rank-one operators in Alg $\mathcal{L}$. We also give an elementary proof of a recent result of Davidson and Pitts that if $\mathcal{L}$ is a CSL generated by completely distributive lattice and finitely many commuting chains, then $\mathcal{L}$ is compact in the strong operator topology if and only if $\mathcal{L}$ is completely distributive.

1. **Introduction.** By a CSL (commutative subspace lattice) we mean a lattice of commuting self-adjoint projections on a separable Hilbert space containing O and I and closed in the strong operator topology. For a CSL $\mathcal{L}$, Alg $\mathcal{L}$, or the corresponding CSL algebra, is the algebra of all operators leaving each projection (or subspace) in $\mathcal{L}$ invariant. In other words, CSL algebras are reflexive algebras containing maximal abelian self-adjoint algebra (m.a.s.a.).

For the study of CSL and their respective algebras it is sometimes important to know what can be said about the ideal of compact operators in Alg $\mathcal{L}$. For example, if $\mathcal{L}$ is a chain, then the linear span of rank-one operators in Alg $\mathcal{L}$ (denoted by $\mathcal{R}_1(\mathcal{L})$) is weak* dense in Alg $\mathcal{L}$. On the other hand, if $\mathcal{L}$ is a non-atomic Boolean algebra, then Alg $\mathcal{L}$ does not contain non-zero compact operators. It is still an open question (see Davidson [2]) whether weak* density of compact operators in Alg $\mathcal{L}$ implies that $\mathcal{R}_1(\mathcal{L})$ is also weak* dense or, at least, non-zero. However, according to Laurie [9] $\mathcal{R}_1(\mathcal{L})$ is weak* dense if and only if Hilbert-Schmidt operators are so. Trent provided an example of a CSL algebra containing non-zero Hilbert-Schmidt operator but no rank-one one (see [7]). Froelich [6] showed that for every $p > 2$ there exists a CSL algebra which has non-zero intersection with Schatten class $\mathcal{C}_p$ but fails to meet $\mathcal{C}_q$ for $q < p$. Nevertheless, Hopenwasser and Moore [8] proved that existence of a finite-rank operator in Alg $\mathcal{L}$ implies that $\mathcal{R}_1(\mathcal{L})$ is non-zero. Davidson [3, Theorem 23.16] has generalized this result showing that the norm-closure of $\mathcal{R}_1(\mathcal{L})$ contains all finite-rank operators in Alg $\mathcal{L}$.

The main purpose of this paper is to extend this Davidson's theorem to the trace-class operators. That is, we will show (Theorem 1.5) that the norm-closure of $\mathcal{R}_1(\mathcal{L})$ contains all trace-class operators in Alg $\mathcal{L}$. As a corollary, we easily get the above-mentioned result of Laurie [9].

Recently, Davidson and Pitts [4] proved that if $\mathcal{L}$ is a CSL generated by a completely distributive lattice and a CSL of finite width, then $\mathcal{L}$ is compact in the strong operator

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topology if and only if $\mathcal{L}$ is completely distributive. While their proof is rather lattice-theoretic, we will give a very short algebraic proof of this result.

In the sequel, H denotes separable Hilbert space, $B(H)$ the algebra of bounded linear operators on H and $K(H)$ its ideal of compact operators.

2. **Trace-class operators in** CSL **algebra.** Let $\mathcal{L}$ be a CSL and $\mathcal{M}$-a m.a.s.a. which contains $\mathcal{L}$. $\mathcal{L}$ is called *synthetic* if $\mathcal{A} = \operatorname{Alg} \mathcal{L}$ is the unique weak* closed algebra that contains $\mathcal{M}$ and satisfies $\operatorname{Lat} \mathcal{A} = \mathcal{L}$ (see Arveson [1, Theorem 2.18 and Definition 2.2.1]).

A linear supspace $\mathcal{J}$ of $B(H)$ is called a $\mathcal{M}$-bimodule if $\mathcal{M}\mathcal{J}\mathcal{M} \subset \mathcal{J}$. Let us call a linear subspace $\mathcal{J} \subseteq B(H)$ *transitive* if, for every non-zero $f \in H$, $\mathcal{J}f$ is dense in H. The following Lemma is known [3, Theorem 15.9]; however, we give here a new proof of it. The idea of this proof is due to Victor Shul'man.

LEMMA 1.1. *The only transitive weak* closed m.a.s.a.-bimodule is $B(H)$.*

PROOF. Let $\mathcal{M}$ be a m.a.s.a. on H and $\mathcal{J}$ a transitive weak* closed $\mathcal{M}$-bimodule. Let $\mathcal{A}$ denote the subalgebra of $B(H \oplus H)$ which consists of all operators of the form

$$\begin{pmatrix} A & T \\ 0 & B \end{pmatrix}, \text{ where } A, B \in \mathcal{M}, T \in \mathcal{J}.$$

A routine check shows that $\mathcal{A}$ is a weak* closed algebra that contains a m.a.s.a. on $H^{(2)} = H \oplus H$ (*i.e.*, $\mathcal{M} \oplus \mathcal{M}$). Moreover, the transitivity of $\mathcal{J}$ implies that

$$\operatorname{Lat} \mathcal{A} = \left\{ L \oplus 0,\ H \oplus N : L, N \in \operatorname{Lat} \mathcal{M} \right\}.$$

Since $\operatorname{Lat} \mathcal{M}$ is a Boolean algebra (the one of all projections in $\mathcal{M}$), there exists a chain C such that $\operatorname{Lat} \mathcal{M} = C \vee C^{\perp}$.

It follows that

$$\operatorname{Lat} \mathcal{A} = (C \oplus 0) \vee (C^{\perp} \oplus 0) \vee (H \oplus C) \vee (H \oplus C^{\perp})$$

and therefore $\mathcal{L} = \operatorname{Lat} \mathcal{A}$ is a CSL of finite width. According to Arveson [1, Theorem 2.2.3], every CSL of finite width is synthetic. Hence

$$\mathcal{A} = \operatorname{Alg} \operatorname{Lat} A = \left\{ \begin{pmatrix} A & T \\ 0 & B \end{pmatrix} : A, B \in M,\ T \in B(H) \right\},$$

that is, $\mathcal{J} = B(\mathcal{H})$, as required.

LEMMA 1.2. *Let $\mathcal{J} \subseteq C_1$ be a non-zero m.a.s.a.-bimodule closed in the topology induced by the trace-norm. Then $\mathcal{J}$ contains a rank-one operator.*

PROOF. Consider the annihilator of $\mathcal{J}$, *i.e.*,

$$\mathcal{J}^{\perp} = \left\{ T \in B(H) : \operatorname{tr}(TX) = 0 \text{ for every } X \in \mathcal{J} \right\}.$$

Clearly $\mathcal{J}^{\perp} \subseteq B(H)$ is weak* closed m.a.s.a.-bimodule. Moreover, since $\mathcal{J} \neq 0$, $\mathcal{J}^{\perp}$ is strictly contained in $B(H)$. By the previous lemma, $\mathcal{J}^{\perp}$ is not transitive. Hence there exists a rank-one $R = x \otimes y$ such that $(Tx, y) = \operatorname{tr}(TR) = 0$ for each $T \in \mathcal{J}$. By a standard duality argument, $R \in \mathcal{J}$, as asserted.

We will need the following elementary lemma which shows that every trace-class operator may be approximated by trace-class pseudo-integral operators. For the definition of a pseudo-integral operator see [1, Section I.5].

LEMMA 1.3. *Let (X, m) be finite Borel measure space. Let K be a trace-class operator on $L^2(X, m)$ and $\mathcal{M}$ multiplication algebra on $L^2(X, m)$. Then there exist two sequences of projections in M, $\{P_n\}_{n=1}^{\infty}$ and $\{Q_n\}_{n=1}^{\infty}$ strongly converging to I such that P_nKQ_n is a pseudo-integral operator for $n = 1, 2, \ldots$.*

PROOF. Write K as a sum

$$K = \sum_{n=1}^{\infty} f_n \otimes g_n, \quad \text{where } f_n, g_n \in L^2(X, m)$$

satisfy $\sum_{n=1}^{\infty} \|f_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|g_n\|^2 < \infty$. Let

$$k(x, y) = \sum_{n=1}^{\infty} f_n(x)\, \overline{g_n(y)}.$$

Note that $k(x, y)$ is defined a.e., for

$$\sum_{n=1}^{\infty} |f_n(x)|^2 < \infty \text{ and } \sum_{n=1}^{\infty} |g_n(y)|^2 < \infty \text{ a.e.}$$

It follows easily from the dominated convergence theorem that $k(x, y)$ is the kernel of K, *i.e.*,

$$Kf(x) = \int k(x, y) f(y)\, dm(y) \text{ for every } f \in L^2(X, m)$$

and $k(x, y) \in L^1(X \times X, m \times m)$.

Arveson remarked [1, p. 493] that every Hilbert-Schmidt operator whose kernel function $k(x, y)$ satisfies

$$h(y) = \int_X |k(x, y)|\, dm(x) \leqq M < \infty \text{ for all } y \in X$$

and

$$g(y) = \int_X |k(x, y)|\, dm(y) \leqq M < \infty \text{ for all } x \in X$$

is a pseudo-integral operator.

In the case when $k(x, y) \in L^1(X \times X, m \times m)$ one has, by Fubini's theorem, that $h(y) < \infty$ a.e., $g(x) < \infty$ a.e. and

$$\int_X h(y)\, dm(y) = \int_X g(x)\, dm(x) < \infty.$$

Let $E_n = \{x \in X : |g(x)| \leqq n\}$ and $F_n = \{y \in X : |h(y)| \leqq n\}$. Let P_n and Q_n be projections corresponding to multiplication by χ_{E_n} and χ_{F_n}, respectively. Clearly $P_n \longrightarrow I$, $Q_n \longrightarrow I$ (strongly) and

$$P_nKQ_nf(x) = \chi_{E_n}(x) \int_X k(x,y)f(y)\,\chi_{F_n}(y)\,dm(y)$$
$$= \int_X \chi_{E_n}(x)\,\chi_{F_n}(y)k(x,y)f(y)\,dm(y),$$

which means that P_nKQ_n is an integral operator with kernel $k_n(x,y) = \chi_{E_n\times F_n}k(x,y)$. However, for all $y \in X$

$$\int_X |k_n(x,y)|\,dm(x) = \int_X \chi_{E_n}(x)\chi_{F_n}(y)|k(x,y)|\,dm(x)$$
$$= \chi_{F_n}(y)\int_X \chi_{E_n}(x)|k(x,y)|\,dm(x)$$
$$\leqq \chi_{F_n}(y)\int_X |k(x,y)|\,dm(x)$$
$$= \chi_{F_n}(y)h(y) \leqq n.$$

Similarly, for all $x \in X$,

$$\int_X |k_n(x,y)|\,dm(y) \leqq n.$$

By the above Arveson's remark, P_nKQ_n is pseudo-integral, which finishes the proof.

LEMMA 1.4. *Let $\mathcal{A}$ be a weak* closed algebra which contains m.a.s.a. $\mathcal{M}$. Then every trace-class operator in $\mathcal{A}$ belongs to weak* closure of the linear span of rank-one operators in $\mathcal{A}$.*

PROOF. Let X be a trace-class operator in $\mathcal{A}$, and let $\mathcal{J}$ denote the weak* closure of the linear span of rank-one operators in $\mathcal{A}$. We claim that if P and Q are projections in $\mathcal{M}$ such that $P\mathcal{J}Q = 0$, then $PXQ = 0$.

Let us assume the converse, *i.e.*, $P\mathcal{J}Q = 0$ but $PXQ \neq 0$. Let

$$\mathcal{S} = \{T \in \mathcal{A} : PTQ = T\}.$$

Clearly, $\mathcal{S}$ is weak* closed $\mathcal{M}$-bimodule. Since $PXQ \in \mathcal{S}$, $\mathcal{S}$ contains a non-zero trace-class operator. By Lemma 1.2 applied to $\mathcal{S} \cap \mathcal{C}_1$, $\mathcal{S}$ contains a rank-one operator R. It follows that $R \in \mathcal{J}$ and $R = PRQ \neq 0$ thus contradicting the equality $P\mathcal{J}Q = 0$.

Now let us prove that for every $f \in H$, $Xf \in \overline{\mathcal{J}f}$. Let E be the projection onto $\overline{\mathcal{J}f}$ and F the projection onto $\overline{\mathcal{M}f}$. Since $\mathcal{J}$ is M-bimodule, one has $E \in \mathcal{M}$. Also

$$\mathcal{J}F(H) \subseteq E(H),$$

and therefore $E^{\perp}\mathcal{J}F = 0$. Finally, $E^{\perp}Xf = E^{\perp}XFf = 0$, so that $Xf \in E(H)$, as asserted.

Now let $\{P_n\}_{n=1}^{\infty}$, $\{Q_n\}_{n=1}^{\infty}$ be two sequences of projections provided by the previous Lemma. By the above, for each n and for every $f \in H$ P_nXQ_nf is the closure of $\mathcal{J}f$. Since P_nXQ_n is pseudo-integral, and $\mathcal{J}$ is weak* closed, it follows, by Arveson [1, Lemma, p. 499] that $P_nXQ_n \in \mathcal{J}$. But $\|P_nXQ_n - X\| \longrightarrow 0$, so that $X \in \mathcal{J}$. This completes the proof.

Now a very little needs to be done in order to get our main result.

THEOREM 1.5. *Let $\mathcal{A}$ be a weak* closed algebra containing a m.a.s.a. $\mathcal{M}$. Then every trace-class operator in $\mathcal{A}$ lies in the norm-closure of the linear span of rank-one operators in $\mathcal{A}$.*

PROOF. It suffices to notice that if $\mathcal{S}$ is a norm-closed linear subspace of $K(H)$ and $A \in K(H)$, then A belongs to a weak* closure of S if and only if $A \in \mathcal{S}$.

COROLLARY 1.6 (DAVIDSON [3, THEOREM 23.16]). *Let $\mathcal{L}$ be a* CSL. *The norm closure of $\mathcal{R}_1(\mathcal{L})$ contains all finite-rank operators in* $\operatorname{Alg}\mathcal{L}$.

COROLLARY 1.7 (LAURIE [9]). *Let $\mathcal{L}$ be a* CSL. *The Hilbert-Schmidt operators are weak* dense in* $\operatorname{Alg}\mathcal{L}$ *if and only if $\mathcal{R}_1(\mathcal{L})$ is weak* dense in* $\operatorname{Alg}\mathcal{L}$.

PROOF. It is enough to show that weak* density of Hilbert-Schmidt operators implies the weak* density of trace-class operators. This, in turn, is equivalent to the requirement that for every semi-invariant projection E of $\operatorname{Alg}\mathcal{L}$ there exists a trace-class operator $C \in \operatorname{Alg}\mathcal{L}$ such that $ECE \neq 0$ [9, Lemma 4.1]. Suppose the converse. Since the product of two Hilbert-Schmidt operators is a trace-class one, it would follow that $EXYE = 0$ for any X, Y in $\operatorname{Alg}\mathcal{L} \cap \mathcal{C}_2$. Now, the density of the latter algebra implies that the closed linear span of

$$\{YE(H) : Y \in \operatorname{Alg}\mathcal{L} \cap \mathcal{C}_2\}$$

contains $E(H)$. Hence $EXE = 0$ for each $X \in \operatorname{Alg}\mathcal{L} \cap \mathcal{C}_2$, thus contradicting the density assumption.

The following corollary is immediately obtained from Theorem 1.5.

COROLLARY 1.8. *Let $\mathcal{L}$ be a* CSL. $\operatorname{Alg}\mathcal{L}$ *is rank-one operator free if and only if its pre-annihilator (i.e. the space of all trace-class X satisfying* $\operatorname{tr}(AX) = 0$ *for all* $A \in \operatorname{Alg}\mathcal{L}$*) is norm-dense in $K(H)$.*

Hopenwasser and Moore gave an example of a CSL algebra $\mathcal{A}$ containing a rank-two operator which can not be expressed as sum of two rank-one operators in $\mathcal{A}$ [8]. Nevertheless, in the case of rank 2, we can strengthen Davidson's Theorem as follows:

PROPOSITION 1.9. *Let* $\mathcal{A} = \operatorname{Alg}\mathcal{L}$ *be a* CSL *algebra. Then every operator of rank 2 belongs to the trace-norm closure of $\mathcal{R}_1(\mathcal{L})$.*

PROOF. Let $B \in \mathcal{A}$ be an operator of rank 2. It is enough to show that for every $T \in B(H)$ satisfying $\operatorname{tr}(TR) = 0$ for all $R \in \mathcal{R}_1(\mathcal{L})$, one has $\operatorname{tr}(TB) = 0$.

Let $\mathcal{S} = \{A \in B(H) : \operatorname{tr}(AF) = 0 \text{ for any finite-rank } F \in \mathcal{A}\}$. Let

$$\tilde{\mathcal{S}} = \left\{\begin{pmatrix} C & A \\ 0 & D \end{pmatrix} : C, D \in \mathcal{M}, A \in \mathcal{S}\right\}.$$

Fix an operator $T \in \mathcal{R}_1(\mathcal{L})^\perp$ and let

$$\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

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Clearly $\tilde{\mathcal{S}}$ is a weakly closed m.a.s.a.-containing algebra. Moreover, since $T \in \mathcal{R}_1(\mathcal{L})^{\perp}$, one has

$$\operatorname{Lat} \tilde{T} \supseteq \operatorname{Lat} \tilde{\mathcal{S}} \ [1, \text{p. } 499] .$$

By Feintuch [5] and Shul'man [10], this yields the inclusion

$$\operatorname{Lat} \tilde{T}^{(2)} \supseteq \operatorname{Lat} \tilde{\mathcal{S}}^{(2)},$$

i.e., every operator of rank 2 that annihilates $\tilde{S}$ annihilates $\tilde{T}$ as well.

Now

$$\tilde{B} = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}$$

is an operator of rank 2 which annihilates $\tilde{S}$ and therefore

$$\operatorname{tr}(\tilde{T}\tilde{B}) = \operatorname{tr}(TB) = 0.$$

This completes the proof.

Applying the same argument to Arveson's example of a non-reflexive weak* closed algebra containing a m.a.s.a., [1, p. 507] we get the following

PROPOSITION 1.10. *There exists a* CSL *algebra* $\mathcal{A} = \operatorname{Alg} \mathcal{L}$ *containing an operator of rank 10 which is not in the trace-norm closure of* $\mathcal{R}_1(\mathcal{L})$.

3. CSL **generated by a completely distributive** CSL **and finitely many chains.** In this section we will give an easy proof of a recent result of Davidson and Pitts [4].

LEMMA 2.1. *Let* $\mathcal{L} = \mathcal{D} \vee \mathcal{C}$, *where* $\mathcal{D}$ *is completely distributive and* $\mathcal{C}$ *is a chain. If* $\mathcal{L}$ *is compact in the strong operator topology, then* Alg $\mathcal{L}$ *contains an operator of rank 1.*

PROOF. Remark that without loss of generality we can assume that $\mathcal{C}$ is continuous chain. Indeed, otherwise there exist such P and Q in $\mathcal{C}$ that P is the immediate predecessor to Q. Let $E = Q - P$. Complete distributivity of $\mathcal{D}$ means that $\mathcal{R}_1(\mathcal{D})$ is weak* dense in Alg $\mathcal{D}$ [7]. Hence one can find a rank-one $R \in \mathcal{R}_1(\mathcal{D})$ such that $ERE \neq 0$. If so, ERE is a rank-one operator which leaves both $\mathcal{D}$ and $\mathcal{C}$ invariant. So let us assume henceforth that $\mathcal{C}$ is continuous.

Arguing contrapositively, suppose that Alg $\mathcal{L}$ is rank-one operator free. Since, for every $R \in \operatorname{Alg} \mathcal{D}$ and every $P \in \mathcal{C}$ an operator $PRP^{\perp}$ is in Alg $\mathcal{L}$, it follows that $PRP^{\perp} = 0$ whenever R is a rank-one operator in $\mathcal{A} = \operatorname{Alg}(\mathcal{D})$. Again the density of $\mathcal{R}_1(\mathcal{D})$ implies that

$$P\mathcal{A}P^{\perp} = \left(P^{\perp}\right)^{\perp} \mathcal{A} P^{\perp} = 0.$$

Since CSL are reflexive [1, Theorem 1.6.3], $P^{\perp} \in \mathcal{D}$. Therefore $\mathcal{C}^{\perp} \subseteq \mathcal{D}$, so that

$$\mathcal{C} \vee \mathcal{C}^{\perp} \subseteq \mathcal{L}.$$

We see that $\mathcal{L}$ contains a non-atomic Boolean algebra which contradicts the assumption about compactness of $\mathcal{L}$.

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THEOREM 2.2. *Every strongly compact* CSL *generated by a completely distributive* CSL *and finitely many chains is completely distributive.*

PROOF. It is enough to prove the theorem in the case of a completely distributive lattice and one chain. The general result will follow then by induction. Let us show that $\mathcal{L} = \mathcal{D} \vee \mathcal{C}$ is completely distributive, or equivalently, that $\mathcal{R}_1(\mathcal{L})$ is weak* dense in Alg $\mathcal{L}$. Suppose the converse. Then, as mentioned above, there is a non-zero semi-invariant projection E for Alg $\mathcal{L}$ such that $ERE = 0$ for every $R \in \mathcal{R}_1(\mathcal{L})$. On the other hand, it is clear that the lattice $\mathcal{L}_E = \{PE, P \in \mathcal{L}\}$ is compact. To complete the proof, apply the previous Lemma to $\mathcal{L}_E = \mathcal{D}_E \vee \mathcal{C}_E$.

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ADDED IN PROOF. It has become known to us that Shul'man generalized our Theorem 1.5 by showing that every trace-class operator in a CSL algebra belongs to the Hilbert-Schmidt norm closure of its rank one ideal. Moreover, Katsoulis and Moore (*On compact operators in certain reflexive operator algebras*, to appear in J.O.T.) have found a proof of Theorem 2.2 very similar to ours.

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