Appendix A
The Legendre functions

The representation functions of orbital angular momentum are (Schiff 1968, Edmonds 1960, Rose 1967) the spherical harmonics

$$Y_{lm}(\theta,\phi) = (-1)^m \left[\frac{(2l+1)(l-m)!}{4\pi(l+m)!}\right]^{\frac{1}{2}} P_l^m(z)\, e^{im\phi} \qquad (A.1)$$

where

$$z \equiv \cos\theta \qquad (A.2)$$

and where the $P_l^m(z)$ are the associated Legendre functions. Their properties are discussed in great detail in Erdelyi *et al.* (1953, vol. 1), which we shall refer to below as E followed by the appropriate page number.

Scattering problems for spinless particles are symmetrical about the beam direction, which is conventionally taken to be the z axis. This eliminates the ϕ dependence, so we are only concerned with

$$Y_{l0}(\theta,\phi) = \left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}} P_l(z) \qquad (A.3)$$

These Legendre functions are eigenfunctions of the operator for the square of the angular momentum, L^2, i.e.

$$L^2 P_l(z) = l(l+1)\, P_l(z), \quad l = 0, 1, 2, \ldots \qquad (A.4)$$

which in the co-ordinate representation becomes

$$\frac{d}{dz}\left[(1-z^2)\frac{dP_l}{dz}\right] + l(l+1)\, P_l(z) = 0 \qquad (A.5)$$

which is Legendre's equation (E, p. 121). For integer l these Legendre functions are polynomials in z, regular in the finite z plane, the first few being

$$P_0(z) = 1, \quad P_1(z) = z, \quad P_2(z) = \tfrac{1}{2}(3z^2-1), \quad P_3(z) = \tfrac{1}{2}(5z^3-3z) \quad (A.6)$$

However, $(A.5)$ also has solutions for $l \neq$ integer which (E, p. 148) may be expressed in terms of the hypergeometric function

$$P_l(z) = F(-l, l+1;\, 1;\, (1-z)/2) \qquad (A.7)$$

which is singular at $z = -1$ and ∞. These are called Legendre functions of the first kind.

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There are also solutions of (A.5) singular at $z = \pm 1$ and ∞ called Legendre functions of the second kind (E, p. 122)

$$Q_l(z) = \pi^{\frac{1}{2}} \frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})} (2z)^{-l-1} F(\tfrac{1}{2}l+1, \tfrac{1}{2}l+\tfrac{1}{2}; l+\tfrac{3}{2}; z^{-2}) \tag{A.8}$$

For integer l the first few are (E, p. 152)

$$\left.\begin{aligned} Q_0(z) &= \tfrac{1}{2}\log\left(\frac{z+1}{z-1}\right), \quad Q_1(z) = \tfrac{1}{2}z\log\left(\frac{z+1}{z-1}\right)-1, \\ Q_2(z) &= \tfrac{1}{2}P_2(z)\log\left(\frac{z+1}{z-1}\right)-\tfrac{3}{2}z. \end{aligned}\right\} \tag{A.9}$$

These functions satisfy *inter alia* the following relations which we need in this book.

The reflection relation (E, p. 140) gives

$$P_l(-z) = e^{-i\pi l}P_l(z) - \frac{2}{\pi}\sin \pi l Q_l(z) \tag{A.10}$$

$$= (-1)^l P_l(z), \quad l = \text{integer} \tag{A.11}$$

The equation (A.5) is invariant under the substitution $l \to -l-1$, so (E, p. 140)

$$P_l(z) = P_{-l-1}(z) \tag{A.12}$$

Also (E, p. 143) for real l

$$\begin{aligned} \operatorname{Im}\{P_l(z)\} &= -P_l(-z)\sin \pi l \qquad z < -1 \\ &= 0 \qquad\qquad\qquad\qquad z \geqslant 1 \end{aligned} \tag{A.13}$$

The two types of solution are related by the Neumann relation (E, p. 154) for integer l

$$Q_l(z) = -\frac{1}{2}\int_{-1}^{1} \frac{dz'}{z'-z} P_l(z'), \quad l = 0, 1, 2, \ldots \tag{A.14}$$

a 'dispersion relation' for $Q_l(z)$, from which it is obvious that (E, p. 143)

$$\begin{aligned} \operatorname{Im}\{Q(z)\} &= 0, \quad |z| > 1, \quad l = 0, 1, 2, \ldots \\ &= -\frac{\pi}{2}P_l(z), \quad -1 < z < 1, \end{aligned} \tag{A.15}$$

For $l \neq$ integer

$$\begin{aligned} \operatorname{Im}\{Q_l(z)\} &= \sin \pi l Q_l(-z), \quad -\infty < z < -1 \\ &= -\frac{\pi}{2}P_l(z), \qquad -1 < z < 1 \end{aligned} \tag{A.16}$$

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The reflection relation for the second-type functions is (E, p. 140)

$$Q_l(-z) = -\mathrm{e}^{-\mathrm{i}\pi l} Q_l(z)$$
$$= (-1)^{l+1} Q_l(z), \quad l = \text{integer} \qquad (A.17)$$

Other useful results are (E, p. 140)

$$\frac{P_l(z)}{\sin \pi l} - \frac{1}{\pi}\frac{Q_l(z)}{\cos \pi l} = -\frac{1}{\pi}\frac{Q_{-l-1}(z)}{\cos \pi l} \qquad (A.18)$$

and $$Q_l(z) = Q_{-l-1}(z), \quad l = \text{half-odd-integer} \qquad (A.19)$$

The orthogonality relation for Legendre polynomials is (E, p. 170)

$$\int_{-1}^{1} P_{l'}(z) P_l(z)\,\mathrm{d}z = \frac{2}{2l+1}\delta_{ll'}, \quad l, l' \text{ integers} \qquad (A.20)$$

and some other integral relations are (E, p. 170)

$$\int_{-1}^{1} P_\alpha(-z) P_l(z)\,\mathrm{d}z = \frac{1}{\pi}\frac{2\sin \pi\alpha}{(\alpha - l)(\alpha + l + 1)}, \quad l \text{ integer, } \alpha \text{ anything} \qquad (A.21)$$

$$\int_{1}^{\infty} P_\alpha(z) Q_l(z)\,\mathrm{d}z = \frac{1}{(l-\alpha)(l+\alpha+1)}, \quad l, \alpha \text{ anything} \qquad (A.22)$$

$$P_\alpha(-z) = -\frac{\sin \pi\alpha}{\pi}\int_1^\infty \frac{\mathrm{d}z' P_\alpha(z')}{z'-z} \qquad (A.23)$$

The asymptotic behaviour as $z \to \infty$ for fixed l may be obtained by rewriting (A.7) as (E, p. 127)

$$P_l(z) = \pi^{-\frac{1}{2}}\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)}(2z)^l F(-\tfrac{1}{2}l, -\tfrac{1}{2}l+\tfrac{1}{2}; -l+\tfrac{1}{2}; z^{-2})$$
$$+\pi^{-\frac{1}{2}}\frac{\Gamma(-l-\frac{1}{2})}{\Gamma(-l)}(2z)^{-l-1} F(\tfrac{1}{2}l+\tfrac{1}{2}, \tfrac{1}{2}l+1; l+\tfrac{3}{2}; z^{-2}) \qquad (A.24)$$

Then since $F \to 1$ as $z \to \infty$ we have (E, p. 164)

$$P_l(z) \xrightarrow[z\to\infty]{} \pi^{-\frac{1}{2}}\frac{\Gamma(l+\frac{1}{2})}{\Gamma(l+1)}(2z)^l, \quad \mathrm{Re}\{l\} \geqslant -\tfrac{1}{2} \qquad (A.25)$$

$$\xrightarrow[z\to\infty]{} \pi^{-\frac{1}{2}}\frac{\Gamma(-l-\frac{1}{2})}{\Gamma(-l)}(2z)^{-l-1}, \quad \mathrm{Re}\{l\} \leqslant -\tfrac{1}{2} \qquad (A.26)$$

Similarly, from (A.8),

$$Q_l(z) \xrightarrow[z\to\infty]{} \pi^{\frac{1}{2}}\frac{\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}(2z)^{-l-1} \qquad (A.27)$$

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The asymptotic behaviour as $l \to \infty$ for fixed z is rather more difficult (E, pp. 142, 162; Newton 1964):

$$P_l(z) \underset{l\to\infty}{\longrightarrow} (2\pi l)^{-\frac{1}{2}} (z^2-1)^{-\frac{1}{4}} \mathrm{e}^{\xi}, \quad \mathrm{Re}\{l\} \geqslant 0 \tag{A.28}$$

where $$\xi \equiv 2(\mathrm{Re}\{l\}+1)\log\left[\left(\frac{z+1}{2}\right)^{\frac{1}{2}} + \left(\frac{z-1}{2}\right)^{\frac{1}{2}}\right], \quad z > 1$$

$$\equiv 2|\mathrm{Im}\{l\}|\tan^{-1}\left(\frac{1-z}{1+z}\right)^{\frac{1}{2}} \quad z^2 < 1$$

so $$|P_l(z)| \underset{l\to\infty}{<} l^{-\frac{1}{2}} \mathrm{e}^{|\mathrm{Im}\{l\}\,\mathrm{Re}\{\theta\}+\mathrm{Re}\{l\}\,\mathrm{Im}\{\theta\}|} f(z) \tag{A.29}$$

$$\left|\frac{P_l(z)}{\sin \pi l}\right| \underset{l\to\infty}{<} l^{-\frac{1}{2}} \mathrm{e}^{|\mathrm{Im}\{l\}\,\mathrm{Re}\{\theta\}+\mathrm{Re}\{l\}\,\mathrm{Im}\{\theta\}|-\pi|\mathrm{Im}\{l\}|} f(z) \tag{A.30}$$

Also $$Q_l(z) \underset{|l|\to\infty}{\longrightarrow} l^{-\frac{1}{2}} \mathrm{e}^{-(l+\frac{1}{2})\zeta(z)} \tag{A.31}$$

where $\zeta(z) \equiv \log[z+(z^2-1)^{\frac{1}{2}}]$.

From (A.7) we see that $P_l(z)$ is an entire function of l, while from (A.8) it is clear that $Q_l(z)$ has poles in l at negative integer values due to the Γ-function in the numerator, and from (A.18)

$$Q_l(z) \approx \pi \frac{\cos \pi l}{\sin \pi l} P_{-l-1}(z), \quad l = -1, -2, -3, \ldots \tag{A.32}$$

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