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GENERATING SYSTEMS OF SUBGROUPS IN PSL$(2, \Gamma_N)$

XIANTAO WANG[1] AND WEIQI YANG[2]

[1] *Department of Applied Mathematics, Hunan University, Changsha, Hunan 410082, People's Republic of China* (xtwang@mail.hunu.edu.cn)
[2] *Department of Applied Mathematics, Beijing Institute of Technology, PO Box 327, Beijing 100081, People's Republic of China*

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Abstract It is proved in this paper that for any non-elementary subgroup G of PSL$(2, \Gamma_n)$, which has no elliptic element, to be not strict, there is a minimal generating system of G consisting of loxodromic elements, and that if G is a non-elementary subgroup of PSL$(2, \Gamma_n)$ of which each loxodromic element is hyperbolic, then G is conjugate to a subgroup of PSL$(2, \mathbb{R})$.

Keywords: Möbius transformation; Clifford algebra; generating system

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1. Introduction

Doyle and James proved in [**5**] that every non-elementary subgroup G of $SL(2, \mathbb{R})$ has a generating system consisting only of hyperbolic elements. Rosenberger proved further in [**11**] that such a system of generators can be chosen to be minimal. Isachenko [**8**] and Rosenberger [**12**] generalized some results in [**11**] and [**10**] to the case of PSL$(2, \boldsymbol{C})$.

In this paper we study the corresponding problem for the case of PSL$(2, \Gamma_n)$. The main result to be proved in this paper is that if a non-elementary subgroup G of PSL$(2, \Gamma_n)$ has no elliptic element which is not strict, then G has a minimal generating system consisting of loxodromic elements (Theorem 3.9). And it is proved that if G is a non-elementary subgroup of PSL$(2, \Gamma_n)$ of which each loxodromic element is hyperbolic, then G is conjugate to a subgroup of PSL$(2, \mathbb{R})$ (Theorem 4.1).

2. Preliminary material

We need the following preliminary material (see [**1**, **2**] for the details).

Let A_n denote the associative algebra over the real numbers generated by $1, e_1, e_2, \dots, e_{n-1}$ subject to the relations

$$e_i^2 = -1, \quad e_ie_j = -e_je_i \quad (i \neq j), \quad i, j = 1, 2, \dots, n-1. \tag{2.1}$$

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For all $a \in A_n$ there is a unique representation of the form

$$a = a_0 + \sum a_v E_v, \tag{2.2}$$

where a_0 and a_v are real, the summation is over all multi-indices $v = (v_1, v_2, \dots, v_p)$ with $0 < v_1 < v_2 < \cdots < v_p \leqslant n-1$, and $E_v = e_{v_1} e_{v_2} \dots e_{v_p}$. a_0 is said to be the real part of a denoted by $a_0 = \mathrm{Re}(a)$. The modulus of a is defined by

$$|a| = \left(a_0^2 + \sum a_v^2\right)^{1/2}. \tag{2.3}$$

Let a' be the element obtained from a by replacing every e_i in (2.2) by $-e_i$, a^* be the element obtained from a by reversing the order of the factors in each $E_v = e_{v_1} e_{v_2} \dots e_{v_p}$, and $\bar{a} = (a')^* = (a^*)'$. Obviously, $(a+b)' = a' + b'$, $(ab)' = a'b'$, and $(ab)^* = b^* a^*$.

All the elements $x = x_0 + x_1 e_1 + \cdots + x_{n-1} e_{n-1}$ ($x_k \in \mathbb{R}$, $k = 0, 1, \dots, n-1$) are said to be the *vectorial* elements in A_n, denoted by $x \in \mathbb{R}^n$. Let Γ_n be the set of all elements in A_n which can be expressed as a finite product of non-zero vectors of A_n. It is said to be *the n-dimensional Clifford group*.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is said to be *an n-dimensional Clifford matrix* if

(i) $a, b, c, d \in \Gamma_n \cup \{0\}$;

(ii) $\det(A) = ad^* - bc^* = 1$; and

(iii) $ab^*, b^*d, a^*c, cd^* \in \mathbb{R}^n$.

Let $\mathrm{SL}(2, \Gamma_n)$ denote the group of all n-dimensional Clifford matrices with the matrix product operation. Set

$$\mathrm{PSL}(2, \Gamma_n) = \mathrm{SL}(2, \Gamma_n)/\{\pm I\},$$

where I is the unit matrix.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \Gamma_n)$$

correspond to the mapping in $\bar{\mathbb{R}}^n$

$$x \mapsto Ax = (ax + b)(cx + d)^{-1}. \tag{2.4}$$

This is an isomorphic correspondence between $\mathrm{PSL}(2, \Gamma_n)$ and $M(\bar{\mathbb{R}}^n)$ (the full sense preserving Möbius group acting in $\bar{\mathbb{R}}^n$), and which are not distinguished.

Let $\tilde{f}$ denote the Poincaré extension of f (see [**4**]). Write

$$\mathrm{fix}(f) = \{x \in \bar{\mathbb{R}}^n : f(x) = x\},$$
$$\mathrm{fix}(\tilde{f}) = \{z = x + te_n \in \boldsymbol{H}^{n+1} : \tilde{f}(z) = z\}.$$

For a non-trivial element $f \in M(\bar{\mathbb{R}}^n)$, we say that

(i) f is *parabolic* if $\mathrm{card}(\mathrm{fix}(f)) = 1$ and $\mathrm{card}(\mathrm{fix}(\tilde{f})) = 0$;

(ii) f is *loxodromic* if $\mathrm{card}(\mathrm{fix}(f)) = 2$ and $\mathrm{card}(\mathrm{fix}(\tilde{f})) = 0$; and

(iii) f is *elliptic* if $\mathrm{card}(\mathrm{fix}(\tilde{f})) > 0$,

where $\mathrm{card}(M)$ is the number of the elements of the set M.

The following corollary results.

Corollary 2.1. *Let*

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \Gamma_n).$$

Then

(i) *f is loxodromic if f is conjugate to*

$$\begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix},$$

where $r > 0$, $r \neq 1$, $\lambda \in \Gamma_n$ and $|\lambda| = 1$, in particular we say that f is hyperbolic if $\lambda = \pm 1$;

(ii) *f is parabolic if f is conjugate to*

$$\begin{pmatrix} \lambda & u \\ 0 & \lambda' \end{pmatrix},$$

where $\lambda, u \in \Gamma_n$, $|\lambda| = 1$, $u \neq 0$, and $\lambda u = u\lambda'$, in particular we say that f is strictly parabolic if $\lambda = \pm 1$; and

(iii) *f is elliptic if f is conjugate to*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix},$$

where $\lambda \in \Gamma_{n+1}$, $|\lambda| = 1$ and $\lambda \neq \pm 1$, in particular we say that f is strictly elliptic if $\lambda \in \Gamma_2$.

For a non-trivial element

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \Gamma_n),$$

we say that f is vectorial if $b, c \in \mathbb{R}^n$ and $a + d^* \in \mathbb{R}$. We then have the following corollary (see [**1**]).

Corollary 2.2.

(i) *f is hyperbolic if and only if f is vectorial and $(a + d^*)^2 > 4$;*

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(ii) *f is strictly parabolic if and only if f is vectorial and $(a+d^*)^2 = 4$; and*

(iii) *f is strictly elliptic if and only if f is vectorial and $(a+d^*)^2 < 4$.*

For

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \Gamma_n)$$

with $\infty \notin \mathrm{fix}(f)$, we define the isometric sphere as follows:

$$S(f) = \{x \in \bar{\mathbb{R}}^n : |x + c^{-1}d| = |c|^{-1}\}.$$

We then have the following corollary (see [**1**] or [**13**]).

Corollary 2.3. *If $S(f) \cap S(f^{-1}) = \emptyset$, then f is loxodromic.*

3. Generating systems of subgroups of PSL$(2, \Gamma_n)$

Let G be a subgroup of $\mathrm{PSL}(2, \Gamma_n)$. G is said to be elementary if there is a finite G-orbit in $\mathbb{H}^{n+1} \cup \bar{\mathbb{R}}^n$. Otherwise, G is said to be non-elementary. The following lemma is well known (see [**14**]).

Lemma 3.1. *If G is non-elementary, then there exist loxodromic elements in G.*

The cardinal number $r(G)$ is the rank of the group G if G can be generated by a system of generators of cardinality $r(G)$, but not by a system of smaller cardinality. A system of generators of G which has cardinality $r(G)$ is said to be a minimal generating system of G.

From [**14**], we have the following lemma.

Lemma 3.2. *Any non-elementary subgroup G of $\mathrm{PSL}(2, \Gamma_n)$ has a generating system consisting of loxodromic elements.*

In order to prove our main result we need to prove the following lemmas.

Lemma 3.3. *Let f be loxodromic. For $g \in \mathrm{PSL}(2, \Gamma_n)$, if g does not interchange the two fixed points of f, then there is $n_0 \in \mathbb{N}$ such that $f^m g$ or $f^m g^{-1}$ are loxodromic for all $m \geqslant n_0$.*

Proof. We may assume that

$$f = \begin{pmatrix} r\lambda & 0 \\ 0 & r^{-1}\lambda' \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $r > 1$, $\lambda \in \Gamma_n$ and $|\lambda| = 1$.

If the group $\langle f, g \rangle$ generated by f and g is elementary, then $ad \neq 0$. Obviously, $f^m g$ is loxodromic for large enough m.

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If $\langle f, g\rangle$ is non-elementary, then $bc \neq 0$ and $\max\{|a|, |d|\} > 0$. To replace g by g^{-1} if needed we may assume that $a \neq 0$. Thus, we obtain that

$$S(f^m g) = \{x \in \bar{\mathbb{R}}^n : |x + c^{-1}d| = r^m |c|^{-1}\},$$
$$S(g^{-1} f^{-m}) = \{x \in \bar{\mathbb{R}}^n : |x - r^{2m}\lambda^m a c^{-1}(\lambda^*)^m| = r^m |c|^{-1}\},$$

and then $S(f^m g) \cap S(g^{-1} f^{-m}) = \emptyset$ for large m. It follows from Corollary 2.3 that $f^m g$ are loxodromic for all $m \geqslant n_0$. □

Lemma 3.4. *Suppose that f, g and fg are strictly elliptic, and*

$$f = \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}, \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $u \in \Gamma_2$, $|u| = 1$ and $bc \neq 0$. Then there is $t \in \mathbb{R}$ such that $c = tb' \in \Gamma_2$ and $d = a'$.

Proof. The conclusion follows from

$$b, c \in \mathbb{R}^n, \qquad ub, u'c \in \mathbb{R}^n,$$
$$a + d^* \in \mathbb{R}, \qquad ua + (u'd)^* \in \mathbb{R},$$

and $ab^* \in \mathbb{R}^n$. □

Lemma 3.5. *Under the assumptions of Lemma 3.4, if $\langle f, g\rangle$ is non-elementary, then $\langle f, g\rangle$ is conjugate to a non-elementary subgroup of $\mathrm{PSL}(2, \Gamma_2)$ that is generated by two elliptic elements.*

Proof. Since $\langle f, g\rangle$ is non-elementary, by Lemma 3.4, there is $q_1 \in \mathrm{PSL}(2, \mathbb{R})$ such that $f_1 = q_1 f q_1^{-1}$ and

$$g_1 = q_1 g q_1^{-1} = \begin{pmatrix} a & b \\ b' & a' \end{pmatrix}.$$

This implies that there is $q_2 \in \mathrm{PSL}(2, \Gamma_n)$ such that $f_2 = q_2 f_1 q_2^{-1}$, $g_2 = q_2 g_1 q_2^{-1} \in \mathrm{PSL}(2, \Gamma_{n-1})$ and $\langle f_2, g_2\rangle$ is non-elementary.

Observe that f_2, g_2 and $f_2 g_2$ are strictly elliptic. By repeating the above argument a finite number of times, our result follows. □

Lemma 3.6. *Suppose that f and g are strictly elliptic, and that $\langle f, g\rangle$ is non-elementary without elliptic elements that are non-strict. Then there are two loxodromic elements f_1 and g_1 such that $\langle f, g\rangle = \langle f_1, g_1\rangle$.*

Proof. Let $h = fg$. Then $\langle f, h\rangle = \langle f, g\rangle$ is non-elementary.

(i) If h is loxodromic, then $f_1 = h^m f$ or $h^m f^{-1}$ is loxodromic for some large m by Lemma 3.3 and $\langle f_1, h\rangle = \langle f, h\rangle = \langle f, g\rangle$. It follows that $g_1 = f_1^k h$ or $f_1^k h^{-1}$ is loxodromic for large enough k and $\langle f_1, g_1\rangle = \langle f, g\rangle$.

(ii) If h is elliptic, then h is strictly elliptic. We may assume that

$$f = \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} \quad \text{with } u \in \Gamma_2, \quad |u| = 1 \quad \text{and} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $\langle f, g\rangle$ is non-elementary, it follows from Lemma 3.5 that $\langle f, g\rangle$ is conjugate to a non-elementary subgroup G_1 in $\mathrm{PSL}(2, \Gamma_2)$ which is generated by two elliptic elements. The proof follows from [**8**, **12**] or [**15**].

(iii) If h is parabolic, then $f_1 = h^m f$ is loxodromic for some large m. Take $g_1 = f_1^k h$. Then g_1 is loxodromic for large enough k and $\langle f_1, g_1\rangle = \langle f, g\rangle$.

□

The following lemma results from Lemma 3.6 and its proof.

Lemma 3.7. *If a non-elementary two-generator subgroup G in $\mathrm{PSL}(2, \Gamma_n)$ has no elliptic element which is not strict, then G can be generated by two loxodromic elements.*

Lemma 3.8. *Let G be a non-elementary subgroup of $\mathrm{PSL}(2, \Gamma_n)$. If G has no elliptic element which is not strict, then G has a minimal generating system Y which contains two elements f, g such that $\langle f, g\rangle$ is non-elementary.*

Proof. Let X be a minimal generating system of G.
The case of $r(G) = 2$ is obvious. Hence, in the following, we suppose $r(G) \geqslant 3$.

(1) If X contains a non-elliptic element f or f, g such that fg is non-elliptic or f, g, h such that fgh is non-elliptic, then there is a minimal generating system Y of G, which contains two elements f_1, f_2 such that $\langle f_1, f_2\rangle$ is non-elementary.

(2) Suppose that all elements in X, including the compositions of any two and any three elements of X, are strictly elliptic. Let $f \in X$ and

$$f = \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix} \quad (u \in \Gamma_2,\ |u| = 1).$$

(A) If X contains g such that

$$g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \quad (b \neq 0),$$

then $d = a'$ and $b \in \Gamma_2$, since fg are strictly elliptic. We know from $ab \in \mathbb{R}^n$ that a has the following form

$$a = a_0 + \sum_{i=2}^{n-1} a_i e_i \quad (a_0, a_i \in \Gamma_2).$$

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Since G is non-elementary, we know that there exists $h \in X$ such that

$$h = \begin{pmatrix} m & k \\ p & q \end{pmatrix},$$

where $p \neq 0$.

Then $k, p \in \Gamma_2$, $m = m_0 + \sum_{i=2}^{n-1} m_i e_i$ $(m_0, m_i \in \Gamma_2)$ and $q = m' + r$ for some $r \in \mathbb{R}$.

Since all fh, gh and fgh are strictly elliptic, we know that $r = 0$ and $a, m \in \Gamma_2$.

These imply that $f, g, h \in \mathrm{PSL}(2, \Gamma_2)$. Hence, for every $w \in X$, $w \in \mathrm{PSL}(2, \Gamma_2)$. This shows that G is conjugate to a subgroup of $\mathrm{PSL}(2, \Gamma_2)$. The proof follows from [**8**, **12**] or [**15**].

(B) In the following, we need only consider the case: for any $g \in X$, either $0, \infty \in \mathrm{fix}(g)$ or $\mathrm{fix}(g) \cap \{0, \infty\} = \emptyset$.

From the assumptions, by passing to a new minimal generating system if needed, there exists

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X$$

with $abc \neq 0$. Lemma 3.4 implies that $b, c \in \Gamma_2$ and $d = a'$. Under conjugation, we assume that

$$f = \begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}, \quad g = \begin{pmatrix} a & b \\ \epsilon b' & a' \end{pmatrix} \quad (\epsilon = \pm 1,\ b \neq 0).$$

For any

$$h = \begin{pmatrix} m & k \\ p & q \end{pmatrix} \in X,$$

we know that $q = m'$ and $p = tk' \in \Gamma_2 (t \in \mathbb{R})$.

It follows from gh being strictly elliptic that $m \in \Gamma_2$ if and only if $a \in \Gamma_2$, and $t = \epsilon$ if and only if $a \notin \Gamma_2$.

If $a \in \Gamma_2$, then, for every $w \in X$, $w \in \mathrm{PSL}(2, \Gamma_2)$. Our result follows from [**8**, **12**] or [**15**].

If $a \notin \Gamma_2$, then $t = \epsilon$. Since h is arbitrary and G is non-elementary, $\epsilon = 1$. So G is conjugate to a subgroup of $\mathrm{PSL}(2, \Gamma_{n-1})$.

By repeating the above steps a finite number of times and by [**8**, **12**] or [**15**], the proof follows. □

Theorem 3.9. *Let G be a non-elementary subgroup of* $\mathrm{PSL}(2, \Gamma_n)$*. If G has no elliptic element which is not strict, then there is a minimal generating system of G consisting of loxodromic elements.*

Proof. By Lemma 3.8, G has a minimal generating system X which contains two elements f, g such that $\langle f, g \rangle$ is non-elementary. By Lemma 3.7, we can suppose that f and g are loxodromic.

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For any $h \in X - \{f, g\}$, if h is not loxodromic, then $f^m h^\epsilon$ or $f^m g h^\epsilon$ is loxodromic for large m (here $\epsilon = 1$ or -1). We replace f, h by f and $f^m h^\epsilon$ or f, g, h by f, g and $f^m g h^\epsilon$ in X.

By the arbitrariness of h, this shows that there is a minimal generating system of G consisting of loxodromic elements. □

4. A class of subgroups of PSL$(2, \Gamma_n)$

In [**7**], Greenberg proved that if a subgroup G of $\mathrm{PSL}(2, \Gamma_n)$ is a hyperbolic group (i.e. each non-trivial element is hyperbolic), then G has an invariant circle in $\bar{\mathbb{R}}^n$ that contains all fixed points of elements in G. Apanasov proved further in [**3**] that if G is a non-elementary subgroup of $\mathrm{PSL}(2, \Gamma_n)$ of which each non-trivial element is either hyperbolic, strictly elliptic or strictly parabolic, then G is conjugate to a subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

We will prove the following theorem.

Theorem 4.1. *Let G be a non-elementary subgroup of* $\mathrm{PSL}(2, \Gamma_n)$. *If each loxodromic element of G is hyperbolic, then G is conjugate to a subgroup of* $\mathrm{PSL}(2, \mathbb{R})$.

Proof. By Lemma 3.1, we may assume that there is a loxodromic element f in G of the following form:

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \quad (r > 1).$$

By Lemmas 3.2 and 3.3, there is a hyperbolic element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

such that $\langle f, g \rangle$ is non-elementary. By Lemma 3.3, $f^m g$ or $f^m g^{-1}$ is hyperbolic for large enough m. Then $a, d \in \mathbb{R}$. Observe that $\langle f, g^2 \rangle$ is also non-elementary. Then there is $t \in \mathbb{R}$ $(t \neq 0)$ such that $c = tb' \in \mathbb{R}^n$. Hence there is

$$h = \begin{pmatrix} q & 0 \\ 0 & q' \end{pmatrix} \quad (q \in \mathbb{R}^n,\ |q| = 1)$$

such that

$$hfh^{-1} = f, \quad hgh^{-1} = \begin{pmatrix} a & |b| \\ t|b| & d \end{pmatrix}.$$

Therefore, we may assume that $f, g \in \mathrm{PSL}(2, \mathbb{R})$.

For any

$$p = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$$

non-trivial, we claim that $\alpha, \delta \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{R}^n$.

We will prove our claim in two cases.

(1) $|\alpha|^2+|\delta|^2>0$.

If $\mathrm{fix}(f)\cap\mathrm{fix}(p)\neq\emptyset$, then $f^m p$ is hyperbolic for large enough m. Hence $\alpha,\delta\in\mathbb{R}$ and $\beta,\gamma\in\mathbb{R}^n$.

If $\mathrm{fix}(f)\cap\mathrm{fix}(p)=\emptyset$, then $\langle f,p\rangle$ is non-elementary. Similar argument as in the beginning of the proof implies that $\alpha,\delta\in\mathbb{R}$ and $\beta,\gamma\in\mathbb{R}^n$.

(2) $|\alpha|^2+|\delta|^2=0$.

By replacing p by gp, our claim follows case (1).

Since $gp\in G$ and

$$gp=\begin{pmatrix} a\alpha+b\gamma & a\beta+b\delta \\ c\alpha+d\gamma & c\beta+d\delta \end{pmatrix},$$

it follows from our claim that $p\in\mathrm{PSL}(2,\mathbb{R})$. The proof is completed.

□

From Theorem 4.1 and [**6**, **9**, **15**] we obtain the following corollary.

Corollary 4.2. *Let G be a non-elementary subgroup of $\mathrm{PSL}(2,\Gamma_n)$. If each loxodromic element of G is hyperbolic, then G is discrete if and only if each one-generator subgroup of G is discrete if and only if each non-elementary subgroup generated by two loxodromic elements of G is discrete.*

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