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THE STRONG CLOSURE OF BOOLEAN ALGEBRAS OF PROJECTIONS IN BANACH SPACES

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Abstract

This note improves two previous results of the second author. They turn out to be special cases of our main theorem which states: A Banach space X has the property that the strong closure of every abstractly σ-complete Boolean algebra of projections in X is Bade complete if and only if X does not contain a copy of the sequence space ℓ^∞.

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1. Statement of results

Let X be a Banach space and $\mathscr{B}$ be a Boolean algebra (briefly, B.a.) of continuous projections in X; the partial order is range inclusion, that is, $B_1 \le B_2$ means $B_1X \subseteq B_2X$, and the unit is the identity operator I in X. Recall that $\mathscr{B}$ is called *Bade complete* (respectively *Bade σ-complete*) if $\mathscr{B}$ is complete (respectively σ-complete) as an abstract B.a. and, for each family (respectively countable family) $\{B_\alpha\} \subseteq \mathscr{B}$, we have

$$(\vee_\alpha B_\alpha)X = \overline{\operatorname{span}\left\{\bigcup_\alpha B_\alpha X\right\}} \quad \text{and} \quad (\wedge_\alpha B_\alpha)X = \cap_\alpha B_\alpha X;$$

see, for example, [1, Chapter XVII]. The space of all continuous linear operators of X into itself is denoted by $\mathscr{L}(X)$; it is equipped with the strong operator topology. The dual Banach space of X is denoted by X^*.

The aim of this short note is to extend the two main results of [7]; they both turn out to be special cases of the following single result.

THEOREM. *A Banach space X has the property that the strong closure (that is, in $\mathscr{L}(X)$) of every abstractly σ-complete B.a. of projections in X is Bade complete if and only if X does not contain a copy of ℓ^∞.*

Theorem 2 of [7] states that if a Banach space X is *weakly compactly generated* (briefly, WCG), then the strong closure of any abstractly σ-complete B.a. of projections in X is Bade complete. It is known that WCG spaces cannot contain a copy of ℓ^∞, [7, page 283]. Moreover, there exist Banach spaces X which do not contain a copy of ℓ^∞, but fail to be WCG, [7, Remarks 1 (i) and 3 (i)]. So, the above theorem is a genuine extension of [7, Theorem 2].

Theorem 3 of [7] states that a Banach space X has the property that the strong closure of every abstractly *complete* B.a. of projections in X is Bade complete if, and only if, X does not contain a copy of ℓ^∞. Our main theorem is also an extension of this result; it relaxes the requirement of abstract completeness to abstract σ-completeness. Again the extension is genuine. For instance, let $X := \ell^p([0, 1])$ for any $1 \le p < \infty$ and define $\mathscr{B} := \{P(E) : E$ a Borel subset of $[0, 1]\}$ where, for each such Borel set E, the projection $P(E) \in \mathscr{L}(X)$ is defined by $P(E)\varphi = \chi_E\varphi$ (pointwise product on $[0, 1]$) and each $\varphi \in X$ is considered as a $\mathbb{C}$-valued function on $[0, 1]$. Then $\mathscr{B}$ is an abstractly σ-complete B.a. in $\mathscr{L}(X)$ which is not abstractly complete.

Further related results, due to Gillespie, can be found in [3, 2].

The extension of the above mentioned results in [7] is possible because of the following fact (answering Question 1 in [7]). Recall that a compact, totally disconnected Hausdorff space Ω is called *σ-Stonian* (or *basically disconnected*) if the closure of the union of any countable family of *clopen* sets (that is, simultaneously closed and open) is an open set. The space $C(\Omega)$, consisting of all $\mathbb{C}$-valued continuous functions on Ω, is equipped with the sup-norm.

PROPOSITION A. *Let Ω be a σ-Stonian space and X be a Banach space not containing a copy of ℓ^∞. Then every continuous linear operator from $C(\Omega)$ into X is necessarily weakly compact.*

Let us accept this result for the moment.

PROOF OF THEOREM. Suppose that X does not contain a copy of ℓ^∞. A careful examination of the proof of [7, Theorem 2] reveals that it also carries over to the current setting, provided that we now replace the use of [7, Proposition 1] in that proof with Proposition A above.

Conversely, suppose that X does contain a copy of ℓ^∞. The same example constructed in the proof of [7, Theorem 3] also applies here (since every abstractly complete B.a. is also abstractly σ-complete) to show that there necessarily exists an abstractly σ-complete, strongly closed B.a. of projections in X which fails to be Bade complete. □

REMARK. An abstractly σ-complete B.a. of projections in a Banach space not containing a copy of ℓ^∞ need not itself be Bade complete or even Bade σ-complete, [7, Remark 2].

So, back to Proposition A which is a reformulation of the following result, due to Rosenthal, [8, Theorem 3.7]; see also [6, Theorem 5.3.17 and Corollaries 3.4.5 and 5.3.14] in the setting of Banach lattices. Recall that a continuous linear operator $T : X \to Y$, with X and Y Banach spaces, is called an *isomorphism (of X into Y)* if it is injective and its range TX is a closed subspace of Y. We also say that Y contains a copy of X.

PROPOSITION B. *Let Ω be a σ-Stonian space and X be a Banach space. Let $T : C(\Omega) \to X$ be a continuous linear operator which fails to be weakly compact. Then there exists a closed subspace X_0 of $C(\Omega)$ which is isometrically isomorphic to ℓ^∞ and such that the restriction $T|_{X_0} : X_0 \to X$ is an isomorphism of X_0 into X.*

The proof of this result given in [8] is not entirely clear, especially the reference made to [4] (of our references) in the proof of [8, Proposition 3.6], which is then used in the proof of the main result, [8, Theorem 3.7]. Since we know of no other reference to Proposition B, for the sake of completeness we include a (perhaps) more transparent and self-contained proof of it. Some preliminaries will be required.

LEMMA 1 ([8, Lemma 1.1 (a)]). *Let Ω be a σ-Stonian space and $\{\mu_n\}_{n=1}^\infty$ be a bounded sequence in $C(\Omega)^*$. Suppose that $\{E_n\}_{n=1}^\infty$ is a sequence of pairwise disjoint clopen subsets of Ω and let $\varepsilon > 0$ be given. Then there exists an infinite subset $M \subseteq \mathbb{N}$ such that*

$$|\mu_m|\left(\overline{\bigcup_{k\neq m} E_k}\right) < \varepsilon, \quad m \in M.$$

Another ingredient needed for the proof of Proposition B is the following result of Grothendieck.

LEMMA 2 ([5, Theoreme 2, page 146]). *Let Ω be a compact Hausdorff space and $K \subseteq C(S)^*$ be a bounded set which is not relatively weakly compact. Then there exists $\delta > 0$, a sequence $\{\mu_n\}_{n=1}^\infty \subseteq K$ and a sequence $\{O_n\}_{n=1}^\infty$ of pairwise disjoint open subsets of Ω such that $|\mu_n|(O_n) > \delta$, for all $n \in \mathbb{N}$.*

We now formulate the main fact needed for proving Proposition B; it is the σ-Stonian version of [8, Proposition 3.6], with 'another proof'.

LEMMA 3. *Let Ω be a σ-Stonian space, X be a Banach space, $T : C(\Omega) \to X$ be a continuous linear operator, and $0 < \varepsilon < \delta$ be given. Suppose that there exists a sequence $\{x_n^*\}_{n=1}^\infty$ in the closed unit ball of X^* and a sequence $\{O_n\}_{n=1}^\infty$ of pairwise disjoint open subsets of Ω such that*

$$|x_n^* T|(O_n) > \delta \quad \text{and} \quad |x_n^* T|\left(\overline{\bigcup_{k\neq n} O_k}\right) < \varepsilon,$$

for every $n \in \mathbb{N}$. Then there exists a closed subspace X_0 of $C(\Omega)$ such that X_0 is isometrically isomorphic to ℓ^∞ and the restriction $T|_{X_0}$ is an isomorphism.

PROOF. Let $\mu_n := |x_n^* T|$, for $n \in \mathbb{N}$, where $x_n^* T$ denotes the measure representing the element $x_n^* \circ T$ of $C(\Omega)^*$. Using the regularity of μ_n and a compactness argument, we can find a clopen set $P_n \subseteq O_n$ such that $\mu_n(P_n) > \delta$, in which case also $\mu_n\left(\overline{\bigcup_{k\neq n} P_k}\right) < \varepsilon$. So, we can (and do) assume that each set O_n, for $n \in \mathbb{N}$, in the statement of the lemma is actually *clopen*.

Let $U := \bigcup_{n=1}^\infty O_n$ and put $F := \overline{U}$. Then F is clopen in Ω and F is itself σ-Stonian (for the relative topology). Actually, $F \simeq \beta(U)$ is the Čech-Stone compactification of the locally compact space U. To see this, let $f : U \to \mathbb{R}$ be any bounded continuous function. For any finite set $A \subseteq \mathbb{N}$, the function $f_A := f\chi_{O(A)}$ belongs to $C(\Omega)$, where $O(A) := \bigcup_{n\in A} O_n$. There are countably many such functions f_A and, since Ω is σ-Stonian, the lattice supremum $g := \vee_A f_A \in C(\Omega)$ exists, [4, page 52]. Clearly, $f = g|_U$.

Choose any $\delta' \in (\varepsilon, \delta)$. For each $n \in \mathbb{N}$, choose $\varphi_n \in C(\Omega)$ with support in O_n and satisfying $\|\varphi_n\|_\infty = 1$ and $\int_{O_n} \varphi_n \, d\mu_n \geq \delta'$.

Let X_0 be the collection of all elements $f \in C(\Omega)$ such that, on O_n, the function f is a constant multiple of φ_n, for each $n \in \mathbb{N}$. Since $F \simeq \beta(U)$ and, for each $f \in X_0$ each restriction $f|_{O_n}$ is a constant multiple of φ_n (for every $n \in \mathbb{N}$), it is clear that X_0 is isometrically isomorphic to ℓ^∞. In particular, X_0 is a closed subspace of $C(\Omega)$.

To show that $T|_{X_0}$ is an isomorphism, let $f \in X_0$ and $n \in \mathbb{N}$ be fixed. Noting that F is the disjoint union of O_n and $\overline{U \setminus O_n}$, we have

$$\begin{aligned} |(x_n^* T)(f)| &= \left|\int_F f \, d\mu_n\right| = \left|\int_{O_n} f \, d\mu_n + \int_{\overline{U\setminus O_n}} f \, d\mu_n\right| \\ &\geq \left|\int_{O_n} f \, d\mu_n\right| - \left|\int_{\overline{U\setminus O_n}} f \, d\mu_n\right| \geq \delta' \left\|f|_{O_n}\right\|_\infty - \varepsilon \|f\|_\infty. \end{aligned}$$

Since $\|f\|_\infty = \sup_n \left\|f|_{O_n}\right\|_\infty$ we conclude that

$$\|Tf\| \geq \sup_n |(x_n^* T)(f)| \geq \sup_n \left(\delta' \left\|f|_{O_n}\right\|_\infty - \varepsilon \|f\|_\infty\right) = (\delta' - \varepsilon)\|f\|_\infty.$$

This is valid for every $f \in X_0$, from which it follows that $T|_{X_0}$ is injective and has closed range. □

PROOF OF PROPOSITION B. Since T is not weakly compact, Lemma 2 ensures the existence of a sequence $\{x_n^*\}_{n=1}^{\infty}$ in the closed unit ball of X^*, a $\delta > 0$ and a sequence $\{O_n\}_{n=1}^{\infty}$ of pairwise disjoint open sets in Ω so that, with $\mu_n := |x_n^* T|$, we have $\mu_n(O_n) > \delta$ for each $n \in \mathbb{N}$. Arguing as in the proof of Lemma 3, the σ-Stonian nature of Ω lets us assume that each O_n, for $n \in \mathbb{N}$, is actually clopen. By Lemma 1 there is an infinite subset $M \subseteq \mathbb{N}$ so that $\mu_m\left(\overline{\bigcup_{k \neq m} O_k}\right) < \delta/2$ for each $m \in M$ and, of course, also $\mu_m(O_m) > \delta$ for each $m \in M$. Put $\varepsilon := \delta/2$. Then Lemma 3 gives the desired conclusion. □

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