LATTICE ISOMORPHIC SOLVABLE LIE ALGEBRAS

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Introduction

Let L be a Lie algebra over a field k of any characteristic, and consider the lattice $\mathscr{L}(L)$ of all subalgebras of L. In this paper we prove that if L and M are lattice isomorphic Lie algebras, over a field of any characteristic, and L' and M' are nilpotent, then the difference between the orders of solvability of L and M differs by at most one.

1. Full intervals

DEFINITION. An $(n+1)$-dimensional $(n \geqq 1)$ Lie algebra is called *almost abelian* if it has a basis $e_0, e_1, \cdots, e_n$ such that $e_0 e_i = e_i$ for $i \geqq 1$ and $e_i e_j = 0$ for $i, j \geqq 1$ (cf. [3] p. 150).

Let L be a Lie algebra and A and B subalgebras of L such that $A \subseteqq B$. We shall denote the lattice of all subalgebras C of L such that $A \subseteqq C \subseteqq B$ by $\mathscr{L}(B \div A)$.

DEFINITION. We call a lattice $\mathscr{L}(L)$ *projective* if it is isomorphic to the lattice of all subspaces of a projective geometry.

DEFINITION. An interval $\mathscr{L}(B \div A)$ of a Lie algebra L is called *full* if every subspace U of L, $A \subseteqq U \subseteqq B$, is a subalgebra.

Clearly, if L is a Lie algebra, then $\mathscr{L}(L)$ is projective if and only if $\mathscr{L}(L \div O)$ is full.

In this paper we denote the derived algebra of a Lie algebra L by L' and the derived algebra of $L^{(r-1)}$ by $L^{(r)}$. We use the symbol $\cup$ to denote the join in the lattice of subalgebras. Also, $\langle S \rangle$ is the subspace spanned by the set S and $\langle U, V \rangle$ is the subspace spanned by the subsets U and V.

PROPOSITION 1. *For a Lie algebra L, $\mathscr{L}(L)$ is projective if and only if L is abelian or almost abelian.*

PROOF. If L is abelian or almost abelian, then clearly $\mathscr{L}(L \div O)$ is full.

[1] The authors wish to thank the referee for his suggestions, which resulted in shortening several of the proofs.

Conversely, suppose that $\mathscr{L}(L \div O)$ is full and that L is not abelian. Then there exists a two dimensional non-abelian subalgebra of L. Hence, there exist $e, x \in L$ such that $ex = x \neq o$. Now suppose that e, x and y are linearly independent. Then $ey = \lambda e + \mu y$ for some λ, μ in the field, and

$$e(x+y) = x + \lambda e + \mu y \in \langle e, x+y \rangle.$$

It then follows that $\mu = 1$ and that $e(ey) = ey$. Thus, $L = \langle e, eL \rangle$. Since eL is a subalgebra we conclude that $L' = eL$.

Now $(e+x)x = x \neq o$, and so by the above $(e+x)L = L'$ and $(e+x)y = y$ for all $y \in L'$. But $ey = y$ for $y \in L'$, and thus $xy = 0$. It then follows that L is almost abelian. This completes the proof.

It is well known that in a nilpotent Lie algebra L, $L' = \Phi(L)$, the Frattini subalgebra. If $\mathscr{L}(L \div A)$ is full, then A is an intersection of maximal subalgebras, and hence $A \supseteqq \Phi(L) = L'$. Therefore, a nilpotent Lie algebra L is abelian if and only if $\mathscr{L}(L)$ is projective. Also, if L is a nilpotent Lie algebra with subalgebras A and B, $A \subseteqq B$, and if $\mathscr{L}(B \div A)$ is full then $B' \subseteqq A$.

Lemma 1. *Let L and M be solvable Lie algebras and let $\varphi : \mathscr{L}(L) \to \mathscr{L}(M)$ be a lattice isomorphism. If A and B are subalgebras of L such that $A \subseteqq B$ and $\mathscr{L}(B \div A)$ is full then $\mathscr{L}(\varphi(B) \div \varphi(A))$ is full.*

Proof. Let V be a subspace of M such that $\varphi(A) \subseteqq V \subseteqq \varphi(B)$. Let $x, y \in V$, we show that $xy \in V$. Since $\langle x \rangle$, $\langle y \rangle$ are subalgebras of M, there exist $x_0, y_0 \in L$ such that $\varphi(\langle x_0 \rangle) = \langle x \rangle$ and $\varphi(\langle y_0 \rangle) = \langle y \rangle$. Let $U = \langle x_0, y_0, A \rangle$. Then $A \subseteqq U \subseteqq B$ and so by assumption U is a subalgebra of L. Thus, $U = \langle x_0 \rangle \cup \langle y_0 \rangle \cup A$. Since L and M are solvable, φ preserves dimensions. From $\dim A = \dim \varphi(A)$ it follows that

$$\dim \langle x_0, y_0, A \rangle = \dim \langle x, y, \varphi(A) \rangle.$$

But $\dim U = \dim \varphi(U)$ and therefore $\varphi(U) = \langle x, y, \varphi(A) \rangle \subseteqq V$. Thus, $xy \in V$.

2. Order of solvability

Theorem 1. *If L and M are lattice isomorphic nilpotent Lie algebras, then L and M have the same order of solvability.*

Proof. Since L/L' is abelian, we have that $\mathscr{L}(L/L')$ is projective, which implies that $\mathscr{L}(L \div L')$ is full. If φ is the lattice isomorphism between $\mathscr{L}(L)$ and $\mathscr{L}(M)$ we then have that $\mathscr{L}(M \div \varphi(L'))$ is full and hence $\varphi(L') \supseteqq M'$. Similarly, $\varphi^{-1}(M') \supseteqq L'$. Thus, $M' = \varphi(L')$. By induction, $M^{(k)} = \varphi(L^{(k)})$, which implies that L and M have the same order of solvability.

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REMARK. We also note that Theorem 1 follows from Corollaries 1′ and 2′ on pages 458 and 459 of [2].

THEOREM 2. *Let L and M be lattice isomorphic Lie algebras, with L′ and M′ nilpotent. Then the orders of solvability of L and M differ by at most one.*

PROOF. Let φ be the lattice isomorphism between $\mathscr{L}(L)$ and $\mathscr{L}(M)$. Now $\varphi(L')/\varphi(L') \cap M'$ is abelian for it is isomorphic to $\varphi(L') \cup M'/M'$. Therefore, $\mathscr{L}(\varphi(L') \div \varphi(L') \cap M')$ is full. By Lemma 1,

$$\mathscr{L}(L' \div L' \cap \varphi^{-1}(M'))$$

is full. Since L' is nilpotent,

$$L'' \subseteqq L' \cap \varphi^{-1}(M') \subseteqq L'.$$

Similarly,

$$M'' \subseteqq M' \cap \varphi(L') \subseteqq M'.$$

Now $L' \cap \varphi^{-1}(M')$ and $\varphi(L') \cap M'$ are lattice isomorphic. By Theorem 1 they have the same order of solvability, say r. We then have

$$L^{(r)} = (L')^{(r-1)} \supseteqq (L' \cap \varphi^{-1}(M'))^{(r-1)} \neq O,$$

and

$$L^{(r+2)} \subseteqq (L' \cap \varphi^{-1}(M'))^{(r)} = O.$$

Thus, the order of solvability of L is either $r+1$ or $r+2$. Similarly, we find that $M^{(r)} \neq O$ and $M^{(r+2)} = O$, which implies that the order of solvability of M is either $r+1$ or $r+2$. This completes the proof.

COROLLARY 1. *If L and M are lattice isomorphic solvable Lie algebras over a field of characteristic zero, then the orders of solvability of L and M differ by at most one.*

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