COMMUTATORS OF MATRICES WITH PRESCRIBED DETERMINANT

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1. Introduction. Let K be a commutative field, let $\mathrm{GL}(n, K)$ be the multiplicative group of all non-singular $n \times n$ matrices with elements from K, and let $\mathrm{SL}(n, K)$ be the subgroup of $\mathrm{GL}(n, K)$ consisting of all matrices in $\mathrm{GL}(n, K)$ with determinant one. We denote the determinant of matrix A by $|A|$, the identity matrix by I_n, the companion matrix of polynomial $p(\lambda)$ by $C(p(\lambda))$, and the transpose of A by A^T. The multiplicative group of non-zero elements in K is denoted by K^*. We let $\mathrm{GF}(p^n)$ denote the finite field having p^n elements.

The goal of this paper is to prove Theorem 1.

THEOREM 1. *Let $x, y \in K^*$ and let $A \in \mathrm{SL}(n, K)$. Then $X, Y \in \mathrm{GL}(n, K)$ exist such that*

$$A = XYX^{-1}Y^{-1} \tag{1}$$

with

$$|X| = x, \qquad |Y| = y, \tag{2}$$

unless: (i) $n = 2$, *K is* $\mathrm{GF}(2)$, *or* $\mathrm{GF}(3)$, $x = y = 1$, *and A is similar within* $\mathrm{GL}(2, K)$ *to* $C((\lambda \pm 1)^2)$; *or* (ii) $A = fI_n$ *where f has order n in K^* and K has infinitely many elements.*

The cases (i) and (ii) are genuinely exceptional. In case (i) the matrices $C((\lambda \pm 1)^2)$ do not lie in the commutator group of $\mathrm{SL}(2, K)$. Whether (1) and (2) possess a solution $X, Y \in \mathrm{GL}(n, K)$ in case (ii) depends very much on the field K. In Theorems 2 and 3 and their corollaries we produce criteria that can be used to determine the solvability of (1) and (2) in case (ii).

Theorem 1, in the case $x = y = 1$, was the result obtained in **(1; 2; 3)**. The methods used to prove Theorem 1 are extensions of the methods of **(1; 2)**. It does not, however, appear to be the case that Theorem 1 follows from the results of **(1; 2; 3)**. Without further explanation we use notation, terminology, and results from **(1; 2; 3)**.

2. The scalar case. Let $f \in K^*$ have order n and let $A = fI_n$. Suppose (1) and (2) hold. Let $x_1 = (-1)^{n-1}x$, $y_1 = (-1)^{n-1}y$. From (1) we get

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$fY = XYX^{-1}$. Let ξ be an eigenvalue of Y in some extension field of K. Then $f\xi$ is an eigenvalue of fY, hence of Y also as fY is similar to Y. Thus $\xi, f\xi, \ldots, f^{n-1}\xi$ are all eigenvalues of Y. Since f has order n and $\xi \neq 0$, these eigenvalues are distinct and therefore are all the eigenvalues of Y. Thus Y must be non-derogatory and is therefore similar within $\mathrm{GL}(n, K)$ to the companion matrix of its characteristic polynomial. Note that, if n is even, $f^{n(n-1)/2} = (f^{n/2})^{n-1} = (-1)^{n-1}$; and if n is odd, $f^{n(n-1)/2} = (f^n)^{(n-1)/2} = 1 = (-1)^{n-1}$. Multiplying together the eigenvalues of Y we find that

$$|Y| = (-1)^{n-1}y_1 = (-1)^{n-1}\xi^n.$$

Thus $\xi^n = y_1$ and so each eigenvalue of Y is a zero of the polynomial $\lambda^n - y_1$. Consequently $\lambda^n - y_1$ is the characteristic polynomial of Y, and so

$$SYS^{-1} = C(\lambda^n - y_1)$$

for some $S \in \mathrm{GL}(n, K)$. Now from (1) we get

$$fI_n = (SXS^{-1})(SYS^{-1})(SXS^{-1})^{-1}(SYS^{-1})^{-1},$$

and so, after a change of notation, we may assume that $Y = C(\lambda^n - y_1)$. Let

$$\Delta = (f^{n-1}) \dotplus (f^{n-2}) \dotplus \ldots \dotplus (f) \dotplus (1). \tag{3}$$

Then $|\Delta| = (-1)^{n-1}$ and $fY = \Delta Y \Delta^{-1}$. So (1) becomes $\Delta Y \Delta^{-1} = XYX^{-1}$ and thus $Z = \Delta^{-1}X$ commutes with Y. Conversely, if for any non-singular Z commuting with Y we put $X = \Delta Z$, then $fI_n = XYX^{-1}Y^{-1}$. Since Y is non-derogatory, the only matrices commuting with Y are polynomials in Y. This completes the proof of Theorem 2.

THEOREM 2. *Let $f \in K^*$ have order n. Let $y \in K^*$, and put $y_1 = (-1)^{n-1}y$. Then all solutions of*

$$fI_n = XYX^{-1}Y^{-1} \tag{4}$$

with $|Y| = y$ are given by

$$\begin{aligned} Y &= SC(\lambda^n - y_1)S^{-1}, \\ X &= S\Delta\left(\sum_{i=0}^{n-1} z_i\, C(\lambda^n - y_1)^i\right)S^{-1}, \end{aligned} \tag{5}$$

where $z_0, z_1, \ldots, z_{n-1}$ are arbitrary elements of K (such that X is non-singular), S is an arbitrary element of $\mathrm{GL}(n, K)$, and Δ is defined by (3).

COROLLARY 1. *Let $x_1 \in K^*$. The necessary and sufficient condition that (4) have a solution $X, Y \in \mathrm{GL}(n, K)$ with $|X| = (-1)^{n-1}x_1$, $|Y| = (-1)^{n-1}y_1$ is that the polynomial equation*

$$\left|\sum_{i=0}^{n-1} z_i\, C(\lambda^n - y_1)^i\right| = x_1 \tag{6}$$

have a solution $z_0, z_1, \ldots, z_{n-1} \in K$.

COROLLARY 2. (i) *Let y be fixed in K^*. The set of all $x_1 \in K^*$ such that* (4) *has a solution $X, Y \in \mathrm{GL}(n, K)$ with $|X| = (-1)^{n-1}x_1$, $|Y| = y$ forms a multiplicative group G_y in K^* containing y and z^n for each $z \in K^*$.* (ii) *Let x be fixed in K^*. The set of all $y_1 \in K^*$ such that* (4) *has a solution $X, Y \in \mathrm{GL}(n, K)$ with $|X| = x$, $|Y| = (-1)^{n-1}y_1$ forms a multiplicative group H_x in K^* containing x and z^n for each $z \in K^*$.*

Proofs. Corollary 1 is clear from (5). Corollary 2(i) is also immediate since all X are given by $X = S\Delta ZS^{-1}$ where Z runs over the multiplicative group of matrices in $\mathrm{GL}(n, K)$ commuting with $C(\lambda^n - y_1)$, with $y_1 = (-1)^{n-1}y$. If we put $Z = C(\lambda^n - y_1)$ or $Z = zI_n$ we find that $|Z| = y$ or $|Z| = z^n$. This proves (i). We may deduce (ii) from (i) by noting that (4) holds if and only if $fI_n = Y^{-1}XYX^{-1}$ holds, since from (4), we get

$$fI_n = Y^{-1}(fI_n)Y = Y^{-1}(XYX^{-1}Y^{-1})Y = Y^{-1}XYX^{-1}.$$

THEOREM 3. *Let $f \in K^*$ have order n. Let $m|n$ and let $x, \gamma \in K^*$. Then* (4) *has a solution $X, Y \in \mathrm{GL}(n, K)$ with*

$$(7) \qquad |X| = (-1)^{n-1}x, \qquad |Y| = (-1)^{n-1}\gamma^{n/m}$$

if and only if

$$(8) \qquad f^{n/m}I_m = XYX^{-1}Y^{-1}$$

has a solution $X, Y \in \mathrm{GL}(m, K)$ with

$$(9) \qquad |X| = (-1)^{m-1}x, \qquad |Y| = (-1)^{m-1}\gamma.$$

Proof. Let $\xi, f\xi, \ldots, f^{n-1}\xi$ be the eigenevalues of $C(\lambda^n - \gamma^{n/m})$, where we choose ξ so that $\xi^m = \gamma$. Then, by Corollary 1, (4) has a solution $X, Y \in GL(n, K)$ satisfying (7) if and only if

$$\sum_{j=1}^{n} z_{j-1}\, C(\lambda^n - \gamma^{n/m})^{j-1}$$

has determinant equal to x. Since the eigenvalues of $C(\lambda^n - \gamma^{n/m})$ are $f^{i-1}\xi$, $1 \leqslant i \leqslant n$, this condition is equivalent to

$$(10) \qquad x = \prod_{i=1}^{n} \left(\sum_{j=1}^{n} z_{j-1} f^{(i-1)(j-1)} \xi^{j-1} \right).$$

Put $j - 1 = t - 1 + m(\sigma - 1)$ and $i - 1 = \rho - 1 + (\mu - 1)n/m$, where $1 \leqslant \rho, \sigma \leqslant n/m$, $1 \leqslant t, \mu \leqslant m$. Then, upon setting $\zeta = f^{n/m}$ and using $\xi^m = \gamma$ and $f^n = 1$, (10) becomes

$$(11) \qquad x = \prod_{\mu=1}^{m} \prod_{\rho=1}^{n/m} \left(\sum_{t=1}^{m} \zeta^{(\mu-1)(t-1)} \xi^{t-1} \sum_{\sigma=1}^{n/m} f^{(\rho-1)(t-1+m(\sigma-1))} \gamma^{\sigma-1} z_{t-1+m(\sigma-1)} \right).$$

Introduce new variables $w_{t\rho}$ by setting, for each fixed t, $1 \leqslant t \leqslant m$, and variable ρ, $1 \leqslant \rho \leqslant n/m$:

$$\sum_{\sigma=1}^{n/m} f^{(\rho-1)(t-1+m(\sigma-1))}\gamma^{\sigma-1}z_{t-1+m(\sigma-1)} = w_{t\rho}. \tag{12}$$

If we view (12) as a linear system linking variables $\gamma^{\sigma-1}z_{t-1+m(\sigma-1)}$, $1 \leqslant \sigma \leqslant n/m$, with variables $w_{t\rho}$, $1 \leqslant \rho \leqslant n/m$, then the coefficient matrix

$$\left(f^{(\rho-1)(t-1+m(\sigma-1))}\right)_{1\leqslant\rho,\sigma\leqslant n/m}$$

is non-singular since, upon removing $f^{(\rho-1)(t-1)}$ from row ρ, the resulting matrix $(f^{(\rho-1)(\sigma-1)m})_{1\leqslant\rho,\sigma\leqslant n/m}$ is a Vandermonde matrix and is non-singular because f^m is a primitive root of unity of order n/m. Thus for fixed t, as the $z_{t-1+m(\sigma-1)}$, $1 \leqslant \sigma \leqslant n/m$, run freely over K, so also do the $w_{t\rho}$, $1 \leqslant \rho \leqslant n/m$. We now write (11) as

$$x = \prod_{\rho=1}^{n/m}\left\{\prod_{\mu=1}^{m}\left(\sum_{t=1}^{m}\zeta^{(\mu-1)(t-1)}w_{t\rho}\,\xi^{t-1}\right)\right\}. \tag{13}$$

The expression in braces in (13) is the determinant of the matrix

$$\sum_{t=1}^{m} w_{t\rho}\, C(\lambda^m - \gamma)^{t-1}. \tag{14}$$

So we may rewrite (13) as

$$x = \prod_{\rho=1}^{n/m}\left|\sum_{t=1}^{m} w_{t\rho}\, C(\lambda^m - \gamma)^{t-1}\right|. \tag{15}$$

Since any polynomial in $C(\lambda^m - \gamma)$ has the form

$$\sum_{t=1}^{m} W_{t-1}\, C(\lambda^m - \gamma)^{t-1},$$

it follows that if (4), (7) hold, then

$$x = \left|\sum_{t=1}^{m} W_{t-1}\, C(\lambda^m - \gamma)^{t-1}\right| \tag{16}$$

has a solution $W_0, W_1, \ldots, W_{m-1} \in K$. By Corollary 1, this implies that (8) has a solution $X, Y \in \mathrm{GL}(m, K)$ satisfying (9). Conversely, if X, Y exist in $\mathrm{GL}(m, K)$ satisfying (8) and (9), then Corollary 1 implies that (16) has a solution $W_0, W_1, \ldots, W_{m-1} \in K$. From this solution we construct a solution of (15): simply put $w_{t1} = W_{t-1}$ for $1 \leqslant t \leqslant m$, $w_{1\rho} = 1$ for all $\rho > 1$, $w_{t\rho} = 0$ for t and $\rho > 1$. With this choice of the $w_{t\rho}$, (16) coincides with (15). We may then use (12) to find values for the $z_{t-1+m(\sigma-1)}$, $1 \leqslant \sigma \leqslant n/m$, $1 \leqslant t \leqslant m$, for which (10) holds. Corollary 1 and (10) then imply that X, Y exist in $\mathrm{GL}(n, K)$ satisfying (4) and (7).

COROLLARY 3. *Let* $n \equiv 0 \pmod 2$, *let* f *have order* n *in* K^*, *and let* $x, \gamma \in K^*$. *Then* $X, Y \in \mathrm{GL}(n, K)$ *exist satisfying* (4) *and* $|X| = -x$, $|Y| = -\gamma^{n/2}$ *if and only if*

$$xu^2 + \gamma v^2 = 1 \tag{17}$$

has a solution $u, v \in K$.

Proof. By Theorem 3, with $m = 2$, (4) holds with $|X| = -x$, $|Y| = -\gamma^{n/2}$ if and only if $-I_2 = XYX^{-1}Y^{-1}$ has a solution $\in \mathrm{GL}(2, K)$ with $|X| = -x$, $|Y| = -\gamma$. By Corollary 1 this latter event will hold if and only if

$$\begin{vmatrix} z_0 & z_1 \\ \gamma z_1 & z_0 \end{vmatrix} = x$$

has a solution in K. This equation in turn is $z_0^2 - \gamma z_1^2 = x$ and is easily shown to have a solution $z_0, z_1 \in K$ if and only if (17) has a solution $u, v \in K$. (Consider separately the cases $z_0 = 0$, $z_0 \neq 0$; $u = 0$, $u \neq 0$.)

If we put $x = \gamma = -1$, then we find from Corollary 3 that *for* $n \equiv 2 \pmod 4$, fI_n *is a commutator of* $\mathrm{SL}(n, K)$ *if and only if* -1 is a sum of two squares in K. This is part of Theorem 1 of **(1)**.

COROLLARY 4. *Let* $n \equiv 0 \pmod 2$, *let* $x\gamma \in K^*$, *and let* f *have order* n *in* K^*. *Suppose that* (17) *does not have a solution* $u, v \in K$. *Then* (4) *does not have a solution* $X, Y \in \mathrm{GL}(n, K)$ *with* $|X| = -x$, $|Y| = -\gamma$.

Proof. Suppose $\gamma^{n/2} \in H_{-x}$. Then (4) would have a solution $X, Y \in \mathrm{GL}(n, K)$ with $|X| = -x$, $|Y| = -\gamma^{n/2}$. By Corollary 3 this implies that (17) has a solution $u, v \in K$. This contradiction implies $\gamma^{n/2} \notin H_{-x}$. If $\gamma \in H_{-x}$, then, as H_{-x} is a group, $\gamma^{n/2} \in H_{+x}$. Hence $\gamma \notin H_{-x}$.

COROLLARY 5. *Let* Q *be the rational number field Let* $x, \gamma \in Q^*$. *Let* f *be a primitive root of unity of order* $n \equiv 0 \pmod 2$ *in the complex number field. If there exists a prime* $p \equiv 1 \pmod n$ *such that* (17) *does not have a solution in the* p*-adic number field, then* (4) *has no solution* $X, Y \in \mathrm{GL}(n, Q(f))$ *with* $|X| = -x$, $|Y| = -\gamma$.

Proof. Let Q_p be the p-adic number field. It is known that if $n \mid (p-1)$ then Q_p^* contains an element of order n. So we may assume that $f \in Q_p^*$, and hence $Q(f) \subset Q_p$. If (17) does not have a solution in Q_p, then surely it has no solution in $Q(f)$.

We remark that well-known techniques are available for determining the solvability of (17) in Q_p. These techniques and Corollary 5 suffice to show that many combinations of determinants cannot be reached in $Q(f)$ to satisfy (1) and (2).

THEOREM 4. *Let* $K = \mathrm{GF}(p^k)$ *be a finite field. Let* $n \mid (p^k - 1)$. *Let* $x, y \in K^*$. *Then* (4) *has a solution* $X, Y \in \mathrm{GL}(n, K)$ *with* $|X| = x$, $|Y| = y$.

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Note that $GF(p^k)^*$ contains an element f of order n if and only if $n \mid (p^k - 1)$.

Proof. Let Ψ be a generator of the cyclic multiplicative group K^*. First note by Corollary 2 that $\Psi \in H_\Psi$. Since H_Ψ is a group, this implies that $H_\Psi = K^*$. Thus $(-1)^{n-1}y \in H_\Psi$. Hence there exist elements $X, Y \in GL(n, K)$ such that (4) holds with $|X| = \Psi, |Y| = y$. In turn this says that $(-1)^{n-1}\Psi \in G_y$. Consequently G_y contains the cyclic group generated by $(-1)^{n-1}\Psi$. If K has characteristic 2 or if n is odd, we have $(-1)^{n-1} = 1$ and hence $G_y = K^*$. Thus $(-1)^{n-1}x \in G_y$ so that a solution of (4) with $|X| = x$, $|Y| = y$ exists in $GL(n, K)$. Now let n be even and p be odd. Let $m = p^k - 1$ be the order of K^*. Then $\Psi^m = 1$ and $-1 = \Psi^{m/2}$. The order of $(-1)^{n-1}\Psi = -\Psi = \Psi^{1+m/2}$ is $m/(m, 1 + m/2) = m$ if $4 \mid m$, $= m/2$ if $2 \| m$. Thus if $m \equiv 0 \pmod 4$, $(-1)^{n-1}\Psi$ is also a generator of K^*; consequently $G_y = K^*$ and hence again we may solve (4) within $GL(n, K)$ with $|X| = x$, $|Y| = y$. Now let $m \equiv 2 \pmod 4$. In this case G_y contains the cyclic group of order $m/2$ generated by $-\Psi$; therefore $[K^*: G_y] \leqslant 2$. Since the order of Ψ^2 is $m/(m, 2) = m/2$, the cyclic group generated by $-\Psi$ is also generated by Ψ^2, hence consists of exactly the even powers of Ψ. We now find an odd power of Ψ in G_y. Note the following chain of equivalences: $fI_n = XYX^{-1}Y^{-1}$ with $|X| = -\Psi^{n/2}$, $|Y| = y \Leftrightarrow fI_n = Y^{-1}XYX^{-1}$ with $|Y^{-1}| = y^{-1}$,

$$|X| = -\Psi^{n/2} \Leftrightarrow -y^{-1}u^2 + \Psi v^2 = 1$$

has a solution in $K \Leftrightarrow -yu^2 + \Psi v^2 = 1$ has a solution in K. That this equation always has a solution in a finite field is well known and can be seen as follows: the map $u \to u^2$ is 2: 1 in K^* and $0 \to 0$. Thus $-yu^2$ assumes $1 + m/2$ values as u runs over K. So also does $1 - \Psi v^2$ as v runs over K. If these two sets of values were disjoint, K would have $m + 2$ elements. Since K has only $m + 1$ elements, $-yu^2 = 1 - \Psi v^2$ has a solution in K. Consequently we know that $\Psi^{n/2} \in G_y$. But $m \equiv 2 \pmod 4$ and n even implies $n \equiv 2 \pmod 4$; thus $\Psi^{n/2}$ is an odd power of Ψ. Hence G_y is properly larger than the subgroup of index two in K^*, and hence $G_y = K^*$.

Now let K be again an arbitrary field.

THEOREM 5. *Let f have order n in K^*, let $m > 1$, and let $x, y \in K^*$. Then $fI_{mn} = XYX^{-1}Y^{-1}$ has a solution $X, Y \in GL(mn, K)$ with $|X| = x$, $|Y| = y$.*

Proof. Since for any α, $1 \in H_\alpha$ and for any β, $1 \in G_\beta$, it follows that we can find $X_1, Y_1, X_2, Y_2, X_3, Y_3 \in GL(n, K)$ such that

$$fI_n = X_1 Y_1 X_1^{-1} Y_1^{-1} = X_2 Y_2 X_2^{-1} Y_2^{-1} = X_3 Y_3 X_3^{-1} Y_3^{-1}$$

with $|X_1| = (-1)^{n-1}$, $|Y_1| = (-1)^{n-1}y$, $|X_2| = (-1)^{(m-1)(n-1)}x$, $|Y_2| = (-1)^{n-1}$, $|X_3| = (-1)^{n-1}$, $|Y_3| = 1$. Put

$$X = X_1 \dotplus X_2 \dotplus X_3 \dotplus X_3 \dotplus \ldots \dotplus X_3,$$
$$Y = Y_1 \dotplus Y_2 \dotplus Y_3 \dotplus Y_3 \dotplus \ldots \dotplus Y_3,$$

(X_3, Y_3 each appear $m-2$ times.) Then $fI_{mn} = XYX^{-1}Y^{-1}$ and $|X| = x$, $|Y| = y$.

3. The non-scalar case. Let $A \in \mathrm{SL}(n, K)$. To avoid conflict with notation used in **(1; 2)** we now let $\phi, \tau \in K^*$ and we attempt to find matrices S, D such that

$$A = SDS^{-1}D^{-1}, \qquad S, D \in \mathrm{GL}(n, K), \tag{18}$$

$$|S| = \phi, \qquad |D| = \tau. \tag{19}$$

Following the discussion in **(1**, §4**)** we let $A = A_1 \dotplus \ldots \dotplus A_m$ where $A_i \in \mathrm{GL}(j(i), K)$ is a companion matrix, $1 \leqslant i \leqslant m$, and

$$j(1) \leqslant j(2) \leqslant \ldots \leqslant j(m).$$

If we can locate a matrix $D \in \mathrm{GL}(n, K)$ with $|D| = \tau$, possessing a linear elementary divisor $\lambda - \alpha$ where $\alpha \in K$, and such that AD is similar to D, then **(1**, Lemma 6**)** can be used to find $S \in \mathrm{GL}(n, K)$ with $|S| = \phi$ such that $AD = SDS^{-1}$. Hence (18) and (19) will hold. The rest of this paper is devoted to locating matrix D.

Note that, in several places in **(1; 2)**, the determinant of a direct sum $A_1 \dotplus \ldots \dotplus A_m$ is written as $|A_1 \ldots A_m|$ when a better notation would be $|A_1| \ldots |A_m|$ or $|A_1 \dotplus \ldots \dotplus A_m|$.

Case 1. $m = 1$. If $\tau \neq 1$ construct, by **(1**, Lemma 4**)** a standard matrix $D \in \mathrm{GL}(n, K)$ such that both D and AD have elementary divisors $\lambda - \tau$, $(\lambda - 1)^{n-1}$. This finishes the case $m = 1$, $\tau \neq 1$ for any field. This argument also works if $\tau = 1$ and $n = 1$. If $n > 1$ and $\tau = 1$ choose $\rho \in K^*$ such that $\rho^2 \neq 1$. This is possible if $K \neq \mathrm{GF}(2)$ or $\mathrm{GF}(3)$. By **(1**, Lemma 4**)** construct $D \in \mathrm{SL}(n, K)$ such that both D and AD have elementary divisors $\lambda - \rho$, $\lambda - \rho^{-1}$, $(\lambda - 1)^{n-2}$. This finishes the case $m = 1$, $\tau = 1$, $n > 1$ over any field except $\mathrm{GF}(3)$ or $\mathrm{GF}(2)$.

Case 2. $m > 1$, $j(m) \geqslant 3$ or $j(m) = j(m-1) = 2$, K has more than six elements. These cases go almost exactly the same as cases 2 and 3 of **(1)**. We need only make the following small changes: replace equations (13) and (15) of **(1)** by (13′) and (15′) below:

$$\left(\prod_{i=1}^{m} \delta_i\right)\left(\prod_{i=1}^{m-1} \gamma_i^{\,j(i)-1}\right) \gamma_m^{\,j(m)-2}\gamma''' = \tau, \tag{13′}$$

$$\left(\prod_{i=1}^{m} \delta_i\right)\left(\prod_{i=1}^{m-1} \gamma_i^{\,j(i)-1}\right) \gamma_m^{\,j(m)-3}\gamma'\gamma'' = \tau. \tag{15′}$$

Construct matrices $D_1, \ldots, D_m$ as in **(1**, Cases 2 and 3**)**. Put

$$D = D_1 \dotplus \ldots \dotplus D_m.$$

Then D and DA are similar. D has a linear elementary divisor, and $|D| = \tau$. This finishes Case 2.

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Case 3. As in the discussion at the end of Case 3 of **(1)**, it only remains to consider the case $A = fI_{n-2} \dotplus C((\lambda - f)(\lambda - g))$ where $f, g \in K$ and $f^{n-1}g = 1$, $n \geqslant 3$. The following proof is valid if $K \neq \mathrm{GF}(2)$ or $\mathrm{GF}(3)$. First let $f \neq 1$. Let a_2 be any element of K^* such that $a_2 \neq 2 + 2(-1)^n\tau - n$. Put $a_3 = 1 + (-1)^n\tau - a_2$ and let $p(\lambda) = \lambda^n - a_3\lambda^2 - a_2\lambda + (-1)^n\tau$. Then $p(1) = 0$ and $dp(\lambda)/d\lambda \neq 0$ when $\lambda = 1$. Thus $\lambda = 1$ is a simple zero of $p(\lambda)$. Set $x = a_2(1 - f^{-1})f^{2-n}$, $y = a_3(1 - f^{-2})f^{3-n}$. Then $x \neq 0$, Let $B = (b_{ij}) \in \mathrm{SL}(n, K)$ where $b_{ii} = f$ for $1 \leqslant i \leqslant n - 1$, $b_{nn} = g$, $b_{n1} = x$, $b_{n2} = y$; and all other $b_{ij} = 0$. Then B is similar to A. This is most easily seen by reducing $\lambda I_n - B$ to Smith canonical form. In $\lambda I^n - B$ add $x^{-1}(\lambda - f)$ times row n and yx^{-1} times row 2 to row 1. Then add $x^{-1}(\lambda - g)$ times column 1 to column n. Finally multiply column 1 by $-x^{-1}$ and row 1 by x. Interchange rows 1 and n and add y times column one to column two to display invariant factors $\lambda - f, \ldots, \lambda - f$ ($n - 2$ times) together with the invariant factor $(\lambda - f)(\lambda - g)$. Thus B is similar to A. Now $\Delta^{-1}BC(p(\lambda))\Delta = C(p(\lambda))$. Thus $BC(p(\lambda))$ is similar to $C(p(\lambda))$ and has $\lambda - 1$ as an elementary divisor. Moreover, $|C(p(\lambda))| = \tau$. Thus B, and hence and A, is a commutator of the required type. The case $f = 1$ follows from Lemma 1 below.

LEMMA 1. *Let $K \neq \mathrm{GF}(2)$ or $\mathrm{GF}(3)$, let $\phi, \tau \in K^*$, and let $A \in \mathrm{SL}(n, K)$ have $\lambda - 1$ as an elementary divisor. Then S, D may be found to satisfy* (18) *and* (19).

Proof. Within $\mathrm{GL}(n, K)$, A is similar to a matrix of the form $W \dotplus I_1$ where W is a direct sum of companion matrices: $W = W_1 \dotplus W_2 \dotplus \ldots \dotplus W_k$, where $W_i \in \mathrm{GL}(w(i), K)$, say, $1 \leqslant i \leqslant k$. Select $\delta_1 \in K^*$, define $\delta_{i+1} = |W_i|\delta_i$ for $1 \leqslant i \leqslant k - 1$, select $\gamma_i \in K^*$ such that $\gamma_i \neq \delta_i, \delta_{i+k}$. Construct, by **(1**, Lemma 4**)**, a standard matrix $D_i \in \mathrm{GL}(w(i), K)$ such that D_i has elementary divisors $\lambda - \delta_i$, $(\lambda - \gamma_i)^{w(i)-1}$ and such that $W_i D_i$ has elementary divisors $\lambda - |W_i| \delta_i$, $(\lambda - \gamma_i)^{w(i)-1}$, for $1 \leqslant i \leqslant k$. Put $E = D_1 \dotplus \ldots \dotplus D_k$. Then WE is similar to E, so that $WE = TET^{-1}$ for some $T \in \mathrm{GL}(n - 1, K)$. Put $S = T \dotplus (|T|^{-1}\phi)$, $D = E \dotplus (|E|^{-1}\tau)$. Then $W \dotplus I_1 = SDS^{-1}D^{-1}$ and $|S| = \phi$, $|D| = \tau$, as required.

Theorem 1 is now completely established, except when K has five or fewer elements. The rest of this paper is devoted to finishing the proof of Theorem 1 when K is one of the exceptional fields $\mathrm{GF}(3)$, $\mathrm{GF}(2^2)$, $\mathrm{GF}(5)$. The case $K = \mathrm{GF}(2)$ was treated completely in **(3)**.

4. The case $K = \mathrm{GF}(3)$. We use the notation of **(2)**.

LEMMA 2. *Let $K = \mathrm{GF}(3)$ and let $A \in \mathrm{SL}(n, K)$ be a companion matrix. Then matrices S, D satisfying* (18), (19) *exist where* $(\phi, \tau) = (1, -1), (-1, 1), (-1, -1)$, *as demanded. If $A \neq C((\lambda \pm 1)^2)$, then we may also have* $(\phi, \tau) = (1, 1)$.

Proof. By § 3, case 1 above with $\tau = -1$ we have (18), (19) with $(\phi, \tau) = (1, -1)$ or $(-1, -1)$, at will. If we apply this result to A^T (which is similar to A) we achieve (18), (19) with $(\phi, \tau) = (-1, 1)$. By **(2**, Lemma 5**)** we obtain (18), (19) with $\phi = \tau = 1$, if $n \neq 2$. Finally

$$C(\lambda^2 + 1) = SDS^{-1}D^{-1}$$

where

$$S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

LEMMA 3. *Let* $K = \mathrm{GF}(3)$. *Let* $A = A_1 \dotplus A_2 \in \mathrm{SL}(n, K)$ *where* A_i *is similar to the companion matrix of a power of a polynomial irreducible over* K *and* $|A_i| = -1$, $i = 1, 2$. *Then* (18), (19) *hold where, as demanded, we have* $(\phi, \tau) = (1, 1), (1, -1), (-1, 1), (-1, -1)$.

Proof. That we can achieve $(\phi, \tau) = (1, 1)$ is the result of **(2**, Lemma 7**)**. The proof of **(2**, Lemma 8**)** shows how to construct a matrix $D \in \mathrm{GL}(n, K)$ such that D and AD have elementary divisors $\lambda + 1$, $(\lambda - 1)^{n-1}$. This shows we can achieve $\phi = \pm 1$, $\tau = -1$. Applying this result to A^T, we achieve $\phi = -1$, $\tau = 1$.

LEMMA 4. *Let* $K = \mathrm{GF}(3)$. *Let* $A = C((\lambda \pm 1)^2) \dotplus C((\lambda \pm 1)^2)$, *where either sign may appear in each direct summand. Then* (18), (19) *hold, where, at will,* $(\phi, \tau) = (1, 1), (1, -1), (-1, 1), (-1, -1)$.

Proof. Let C_1, C_2 each be either $C((\lambda + 1)^2)$ or $C((\lambda - 1)^2)$. By Lemma 2, $C_1 = S_1 D_1 S_1^{-1} D_1^{-1}$ where $(|S_1|, |D_1|) = (-\phi, -1)$, and $C_2 = S_2 D_2 S_2^{-1} D_2^{-1}$ where $(|S_2|, |D_2|) = (-1, -\tau)$. Put $S = S_1 \dotplus S_2$, $D = D_1 \dotplus D_2$. Then $A = SDS^{-1}D^{-1}$ and $(|S|, |D|) = (\phi, \tau)$.

We now prove Theorem 1 in the case $K = \mathrm{GF}(3)$ and A not scalar. Let $A = A_1 \dotplus \ldots \dotplus A_m$ where, here, either A_i is the companion matrix of a power of a polynomial irreducible over GF(3) and $|A_i| = 1$, or else $A_i = A_{i1} \dotplus A_{i2}$ where A_{i1} and A_{i2} are each companion matrices of powers of polynomials irreducible over GF(3) and $|A_{i1}| = |A_{i2}| = -1$. If an A_i appears which is not $C((\lambda \pm 1)^2)$, choose the notation so that A_m is not $C((\lambda \pm 1)^2)$. If each A_i is $C((\lambda \pm 1)^2)$, then $m \geqslant 2$. (Since, if $m = 1$, the result follows from Lemma 2 above.) In this event change notation so that $A_m = C((\lambda \pm 1)^2) \dotplus C((\lambda \pm 1)^2)$. By Lemmas 2, 3 we may find S_i, D_i with elements in GF(3) so that $A_i = S_i D_i S_i^{-1} D_i^{-1}$, $1 \leqslant i \leqslant m - 1$. By Lemmas 2, 3, 4 we may express $A_m = S_m D_m S_m^{-1} D_m^{-1}$, where

$$|S_m| = |S_1| \ldots |S_{m-1}|\phi, \qquad |D_m| = |D_1| \ldots |D_{m-1}|\tau.$$

Put $S = S_1 \dotplus \ldots \dotplus S_m$, $D = D_1 \dotplus \ldots \dotplus D_m$. Then (18), (19) are satisfied. This proves Theorem 1 when $K = \mathrm{GF}(3)$.

4. Some lemmas. To handle the cases $K = \mathrm{GF}(4)$ and GF(5) we require the following rather complicated lemmas. The proofs of these lemmas are extensions of the method used to prove Lemmas 7 and 8 of **(2)**. For the moment K will still be an arbitrary field. Let $e_i = (0, 0, \ldots, 0, 1)$ have i components, of which all but the last are zero.

LEMMA 5. *Let $t \geqslant 2$. Suppose matrices $A_i, U_i, \Delta_i \in \mathrm{GL}(j(i), K)$ and polynomials $p_i(\lambda)$ over K are given satisfying $U_i A_i \Delta_i U_i^{-1} = C(p_i(\lambda))$, such that A_i is a companion matrix, the last column of U_i is $e_{j(i)}{}^T$, and Δ_i is upper triangular, $1 \leqslant i \leqslant t$. Suppose also that vectors v_i with $j(i+1)$ components from K are given, such that whenever $j(i) = 1$, $v_i = |A_i|^{-1}\rho_{i+1}$, ρ_{i+1} being the first row of U_{i+1}, $1 \leqslant i \leqslant t-1$. Let D be a triangular matrix, presented in partitioned form as*

$$D = \begin{bmatrix} \Delta_1 & D_{12} & D_{13} & \dots & D_{1t} \\ 0 & \Delta_2 & D_{23} & \dots & D_{2t} \\ 0 & 0 & \Delta_3 & \dots & D_{3t} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & \Delta_t \end{bmatrix}.$$

We suppose that v_i is the last row of $D_{i,i+1}$, $1 \leqslant i \leqslant t-1$. Let $A = A_1 \dotplus \dots \dotplus A_t$. Then it is possible to select the as yet unspecified elements in $D_{12}, D_{13}, \dots, D_{t-1,t}$ from K in such a manner that AD is non-derogatory and has $p_1(\lambda) \dots p_t(\lambda)$ as its characteristic polynomial.

Proof. Let $v_i = (v_{i1}, v_{i2}, \dots, v_{i,j(i+1)})$. Let α be fixed, $\alpha < t$. We first specify the elements of $D_{\alpha,\alpha+1}$. If $j(\alpha) = 1$, this has already been done by the hypotheses. Let $j(\alpha) > 1$ and for this fixed α let $R_1, R_2, \dots, R_{j(\alpha+1)}$ denote the rows of $A_{\alpha+1}\Delta_{\alpha+1}$, and let

$$D_{\alpha,\alpha+1} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1,j(\alpha+1)} \\ d_{21} & d_{22} & \dots & d_{2,j(\alpha+1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_{j(\alpha)-1,1} & d_{j(\alpha)-1,2} & \dots & d_{j(\alpha)-1,j(\alpha+1)} \\ v_{\alpha 1} & v_{\alpha 2} & \dots & v_{\alpha,j(\alpha+1)} \end{bmatrix}.$$

Let δ_α denote the bottom right corner element of Δ_α. As Δ_α is triangular and non-singular, $\delta_\alpha \neq 0$. Let C_α be the last column of $A_\alpha \Delta_\alpha$. Because A_α is a companion matrix, the next to bottom element of C_α is δ_α. Now $A_\alpha D_{\alpha,\alpha+1}$ has the form

$$A_\alpha D_{\alpha,\alpha+1} = \begin{bmatrix} d_{21} & d_{22} & \dots & d_{2,j(\alpha+1)} \\ d_{31} & d_{32} & \dots & d_{3,j(\alpha+1)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ d_{j(\alpha)-1,1} & d_{j(\alpha)-1,2} & \dots & d_{j(\alpha)-1,j(\alpha+1)} \\ v_{\alpha 1} & v_{\alpha 2} & \dots & v_{\alpha,j(\alpha+1)} \\ z_1 & z_2 & \dots & z_{j(\alpha+1)} \end{bmatrix}$$

where

$$(20)\qquad z_s = (-1)^{j(\alpha)-1}\,|A_\alpha|\, d_{1s} + \text{a fixed linear combination of}$$
$$d_{2s}, d_{3s}, \ldots, d_{j(\alpha)-1,s}, v_{\alpha s}, \qquad 1 \leqslant s \leqslant j(\alpha+1).$$

We now impose the following condition upon the elements of $D_{\alpha,\alpha+1}$:

$$(21)\qquad \begin{bmatrix} O_{j(\alpha)-1,\,j(\alpha+1)} \\ \rho_{\alpha+1} \end{bmatrix} + \delta_\alpha^{-1}(v_{\alpha 1}\, C_\alpha, v_{\alpha 2}\, C_\alpha, \ldots, v_{\alpha, j(\alpha+1)}\, C_\alpha)$$
$$- \delta_\alpha^{-1} \begin{bmatrix} O_{j(\alpha)-1,\,j(\alpha+1)} \\ \sum_{\mu=1}^{j(\alpha+1)} v_{\alpha\mu}\, R_\mu \end{bmatrix} = A_\alpha\, D_{\alpha,\alpha+1}.$$

Here, in (21), the first matrix in the left member has $j(\alpha)$ rows, the first $j(\alpha) - 1$ of which are zero vectors and the last $\rho_{\alpha+1}$; the second matrix on the left has $j(\alpha+1)$ columns, each of which is the indicated multiple of C_α; the third matrix on the left has $j(\alpha) - 1$ rows of zeros, followed by a row which is the indicated linear combination of $R_1, \ldots, R_{j(\alpha+1)}$. From an examination of the form of $A_\alpha\, D_{\alpha,\alpha+1}$, we see that (21) immediately determines all rows of $D_{\alpha,\alpha+1}$ except the first; and then (20) can be used to determine the first row of $D_{\alpha,\alpha+1}$ in such a manner that (21) is satisfied. All this can be done for $\alpha = 1, 2, \ldots, t-1$. Hence $D_{12}, D_{23}, \ldots, D_{t-1,t}$ are now constructed.

Now form AD. We find that

$$AD = \begin{bmatrix} A_1\,\Delta_1 & A_1\, D_{12} & E_{13} & E_{14} & \ldots & E_{1t} \\ 0 & A_2\,\Delta_2 & A_2\, D_{23} & E_{24} & \ldots & E_{2t} \\ 0 & 0 & A_3\,\Delta_3 & A_3\, D_{34} & \ldots & E_{3t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & A_t\,\Delta_t \end{bmatrix}.$$

Here $E_{\alpha\beta} = A_\alpha\, D_{\alpha\beta}$. Let α be fixed, $1 \leqslant \alpha \leqslant t-1$. We now perform the following similarity transformations on AD. If $j(\alpha) = 1$, we do nothing. If $j(\alpha) > 1$, we subtract $\delta_\alpha^{-1} v_{\alpha s}$ times column $j(1) + j(2) + \ldots + j(\alpha)$ of AD from column $j(1) + j(2) + \ldots + j(\alpha) + s$, then add $\delta_\alpha^{-1} v_{\alpha s}$ times row $j(1) + j(2) + \ldots + j(\alpha) + s$ to row $j(1) + \ldots + j(\alpha)$, for $s = 1, 2, \ldots, j(\alpha+1)$. Owing to (21) this results in converting the block $A_\alpha\, D_{\alpha,\alpha+1}$ into a block whose last row is $\rho_{\alpha+1}$ and whose other rows are all zero. If $j(\alpha) = 1$, it is already true that $A_\alpha\, D_{\alpha,\alpha+1}$ has $\rho_{\alpha+1}$ for its only row. In addition observe that these similarity transformations leave all diagonal blocks $A_1\Delta_1, \ldots, A_t\Delta_t$ unchanged. The only block in the block diagonal just above and parallel to the main block diagonal that changes is $A_\alpha\, D_{\alpha,\alpha+1}$. Also observe that while certain of the E matrices change, they do so only in the following way. If an E matrix, say E_{pq}, becomes altered, the only alteration is to add to the

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elements of E_{pq} certain known linear combinations of elements from matrices which lie in block row p and which are to the left of E_{pq}, or to add to some of the elements of E_{pq} certain known linear combinations of elements from matrices which lie in block column q and which are below E_{pq}.

We now perform the above similarities on AD with $\alpha = t - 1$, then on the result perform the above similarities with $\alpha = t - 2$, then $t - 3, \ldots, 1$. The result of all of this is to find a non-singular W such that

$$WADW^{-1} = \begin{bmatrix} A_1\Delta_1 & F_{12} & G_{13} & G_{14} & \ldots & G_{1t} \\ 0 & A_2\Delta_2 & F_{23} & G_{24} & \ldots & G_{2t} \\ 0 & 0 & A_3\Delta_3 & F_{34} & \ldots & G_{3t} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & A_t\Delta_t \end{bmatrix}.$$

Owing to our construction of $D_{\alpha,\alpha+1}$, we have

$$F_{\alpha,\alpha+1} = \begin{bmatrix} O_{j(\alpha)-1,\, j(\alpha+1)} \\ \rho_{\alpha+1} \end{bmatrix}, \qquad 1 \leqslant \alpha \leqslant t-1.$$

Now set $G_{13} = 0$, $G_{24} = 0, \ldots, G_{t-2,t} = 0$. Because of the manner in which the G matrices arise from the E matrices, this amounts to setting

$$A_\alpha D_{\alpha,\alpha+2} = T_\alpha, \qquad 1 \leqslant \alpha \leqslant t-2,$$

where the T_α are some matrices of known elements. So we may solve for $D_{13}, D_{24}, \ldots, D_{t-2,t}$ such that $G_{13} = 0$, $G_{24} = 0, \ldots, G_{t-2,t} = 0$. Now set $G_{14} = 0$, $G_{25} = 0, \ldots, G_{t-3,t} = 0$. By the same kind of argument this amounts to putting $A_\alpha D_{\alpha,\alpha+3} = V_\alpha$, $1 \leqslant \alpha \leqslant t-3$, where the V_α are certain matrices of known elements. In this manner we construct in succession the block side diagonals of D parallel to the main block diagonal such that all the G matrices are zero. D is now completely specified.

Now, for $\alpha < t$, note that because of the special forms of U_α and $F_{\alpha,\alpha+1}$, we have $U_\alpha F_{\alpha,\alpha+1} = F_{\alpha,\alpha+1}$. And also observe that since the last row of $F_{\alpha,\alpha+1}$ is the first row of $U_{\alpha+1}$, $F_{\alpha,\alpha+1} U_{\alpha+1}^{-1} = N_\alpha$, say, is a matrix consisting entirely of zeros except for its extreme lower left corner element, which is a one. Now put $U = U_1 \dotplus U_2 \dotplus \ldots \dotplus U_t$. Then

$$UWADW^{-1}U^{-1} = \begin{bmatrix} C(p_1(\lambda)) & N_1 & 0 & 0 & \ldots & 0 \\ 0 & C(p_2(\lambda)) & N_2 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & C(p_t(\lambda)) \end{bmatrix}.$$

The proof is now complete since $UWADW^{-1}U^{-1}$ clearly has the required characteristic polynomial and is non-derogatory since the $(n-1) \times (n-1)$

subdeterminant of $\lambda I_n - UWADW^{-1}U^{-1}$ obtained by deleting column 1 and row n is a non-zero constant.

The next lemma uses notation explained in (**2**, pp. 144–145).

LEMMA 6. *Let* $A = C(p(\lambda)) \in \mathrm{GL}(n, K)$, *where* $K \neq \mathrm{GF}(2)$. *Let* $g_1, g_3 \in K^*$. *Then it is possible to choose* $g_2, g_4 \in K$ *and a vector* d *with elements in* K *so that* $U \in \mathrm{GL}(n, K)$ *exists satisfying*: (i) *the last column* of U *is* e_n^T; (ii) *if* $n \geqslant 2$, $UA\Delta_n(g_1, g_2, g_3, g_4, d)U^{-1} = C((\lambda - |A|g_1)(\lambda - g_3)(\lambda - 1)^{n-2})$; (iii) *if* $n = 1$, $UA\Delta_n(g_1, g_2, g_3, g_4, d)U^{-1} = C(\lambda - |A|g_1)$; (iv) *if* $n \geqslant 3$, $g_2, g_4 \neq 0$; (v) *if* $n = 2$ *and* $g_1 = g_3$, *then* $g_2 = 0$ *if and only if* $p(\lambda) = (\lambda - |A|)(\lambda - 1)$.

Proof. This is a specialization of Lemma 2 of (**2**). The matrix U is the matrix ST whose existence is asserted in (**2**, Lemma 2). Let $-a_2$ be the coefficient of λ in $p(\lambda)$. When $n \geqslant 3$ to get g_2, g_4 we require that the coefficient of λ in $(\lambda - |A|g_1)(\lambda - g_3)(\lambda - 1)^{n-2}$ be

$$(-1)^n|A|\,(g_2 + g_1g_4 + g_1g_3(n-3)) - a_2g_3.$$

This is a linear equation in two unknowns g_2, g_4. Set $g_2 = 1$ and solve for g_4. If $g_4 = 0$, set instead g_2 equal to any other non-zero value in K. Then solve for g_4; it must now turn out that $g_4 \neq 0$. When $n = 2$ we determine g_2 from $-|A|g_1 - g_3 = |A|g_2 - a_2g_3$. If $g_2 = 0$ and $g_1 = g_3$, we get $a_2 = |A| + 1$. This implies that $p(\lambda) = (\lambda - |A|)(\lambda - 1)$.

LEMMA 7. *Let* $A = A_1 \dotplus A_2 \in \mathrm{SL}(n, K)$ *where* A_i *is a* $j(i) \times j(i)$ *companion matrix and* $|A_i| \neq 1$ *for* $i = 1, 2$. *Let* $j(1) \geqslant 2$, $j(1) \geqslant j(2)$. *Suppose* $\lambda - 1$ *is not an elementary divisor of* A. *Let* $\delta_1, \delta_2, \delta_3 \in K^*$ *be such that one of* (i), (ii), (iii), (iv) *holds*: (i) $1 \neq \delta_2 \neq \delta_1 = \delta_3 \neq 1$; (ii) $\delta_3 = \delta_2 = 1 \neq \delta_1$; (iii) $\delta_1, \delta_2, \delta_3$, 1 *are all different*; (iv) $\delta_1 = \delta_3 = 1 \neq \delta_2$. *Then we may find* $D \in \mathrm{GL}(n, K)$ *such that* D *and* AD *are both non-derogatory with characteristic polynomials* $(\lambda - \delta_1)(\lambda - \delta_2)(\lambda - \delta_3)(\lambda - 1)^{n-3}$, $(\lambda - |A_1|\delta_1)(\lambda - |A_2|\delta_2)(\lambda - \delta_3)(\lambda - 1)^{n-3}$ *respectively*.

Proof. Use Lemma 6 to construct matrices $U_1, \Delta_{j(1)}(\delta_1, g_2, \delta_3, g_4, d) \in \mathrm{GL}(j(1), K)$ satisfying (i), (ii), (iv), (v) of Lemma 6. Note that if $j(1) = 2$ and $\delta_1 = \delta_3$, then $g_2 \neq 0$ since $g_2 = 0$ implies $A_1 = C((\lambda - |A_1|)(\lambda - 1))$, hence $\lambda - 1$ is an elementary divisor of A, contrary to hypothesis. If $j(1) = 2$, set $g_4 = 1$. Use Lemma 6 to construct matrices

$$U_2, \Delta_{j(2)}(\delta_2, h_2, 1, h_4, d') \in \mathrm{GL}(j(2), K)$$

satisfying the five conditions of Lemma 6. Note that if $j(2) = 2$ and $\delta_2 = 1$, then $h_2 \neq 0$, since $h_2 = 0$ implies $A_2 = C((\lambda - |A_2|)(\lambda - 1))$, so that $\lambda - 1$ is an elementary divisor of A, contrary to hypothesis. If $j(2) \leqslant 2$, set $h_4 = 1$, and if $j(2) = 1$, set $h_2 = 1$. Now put

$$D = \begin{bmatrix} \Delta_{j(1)}(\delta_1, g_2, \delta_3, g_4, d) & D_{12} \\ 0 & \Delta_{j(2)}(\delta_2, h_2, 1, h_4, d') \end{bmatrix}$$

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where the last row of D_{12} is $(0, 1, 0, 0, \ldots, 0)$ if $j(2) > 1$ and $\delta_2 \neq 1$; otherwise the last row of D_{12} is $(|A_1|^{-1}, 0, \ldots, 0)$. Then the conditions of Lemma 5 are satisfied and hence we may choose the other elements of D_{12} so that AD is non-derogatory and has the required characteristic polynomial. Since D is triangular, it is clear that D has the required characteristic polynomial. It is only necessary to show that D is non-derogatory. This will be accomplished by showing that the greatest common divisor of the $(n-1) \times (n-1)$ subdeterminants of $\lambda I_n - D$ is one. Let $D[\alpha|\beta]$ denote the subdeterminant of $\lambda I_n - D$ obtained by deleting row α and column β, and let $D[\alpha|\beta]_{\lambda=1}$ denote this subdeterminant evaluated when $\lambda = 1$. In case (i) consider

$$D[j(1)+1|j(1)+1] = (\lambda - \delta_1)^2(\lambda - 1)^{n-3};$$

$D[2|1] = -g_2(\lambda - \delta_2)(\lambda - 1)^{n-3} \neq 0$; $D[n|3]_{\lambda=1} = \pm(1-\delta_1)^2(1-\delta_2)h_4 \neq 0$ (when $j(1) > 2$ and $j(2) > 1$), $D[n-1|3]_{\lambda=1} = \pm(1-\delta_1)^2(1-\delta_2) \neq 0$ (when $j(1) > 2$ and $j(2) = 1$), or $D[n|n] = (\lambda-\delta_1)^2(\lambda-\delta_2)$ when $n = 4$. In case (ii) consider $D[1|1] = (\lambda - 1)^{n-1}$; and

$$D[n|2]_{\lambda=1} = \pm(1-\delta_1)|A_1|^{-1}g_4 h_2 h_4 \neq 0.$$

In case (iii) consider $D[1|1] = (\lambda-\delta_2)(\lambda-\delta_3)(\lambda-1)^{n-3}$;

$$D[2|2] = (\lambda-\delta_1)(\lambda-\delta_2)(\lambda-1)^{n-3};$$
$$D[j(1)+1|j(1)+1] = (\lambda-\delta_1)(\lambda-\delta_3)(\lambda-1)^{n-3};$$
$$D[n|3]_{\lambda=1} = \pm(1-\delta_1)(1-\delta_2)(1-\delta_3)h_4 \neq 0$$

(when $j(1) > 2, j(2) > 1$), or

$$D[n-1|3]_{\lambda=1} = \pm(1-\delta_1)(1-\delta_2)(1-\delta_3) \neq 0$$

(when $j(1) > 2$, $j(2) = 1$), or $D[4|4] = (\lambda-\delta_1)(\lambda-\delta_2)(\lambda-\delta_3)$ (when $j(1) = j(2) = 2$). In case (iv) consider $D[j(1)+1|j(1)+1] = (\lambda-1)^{n-1}$; $D[n|1]_{\lambda=1} = \pm g_2 g_4 h_4(1-\delta_2) \neq 0$ (when $j(2) > 1$) or

$$D[n-1|1]_{\lambda=1} = \pm g_2 g_4(1-\delta_2) \neq 0$$

(when $j(2) = 1$). In all four cases we have computed sufficiently many subdeterminants of $\lambda I_n - D$ to show that D is non-derogatory.

The rest of this paper is devoted to finishing the proof of Theorem 1 when $K = \mathrm{GF}(4)$ or when $K = \mathrm{GF}(5)$. We may, by Lemma 1, assume that $\lambda - 1$ is not an elementary divisor of A. We may also assume that $\tau \neq 1$ since in (**1**, §§ 5, 6) it was shown how to construct a matrix $D \in \mathrm{SL}(n, K)$ possessing a linear elementary divisor $\lambda - \alpha$ with $\alpha \in K$ such that AD is similar to D. Hence (18) and (19) can be satisfied when $\tau = 1$. If A is a companion matrix, the required proof to complete Theorem 1 is supplied by § 3, case 3 above. If $A = A_1 \dotplus A_2$ when $A_i \in \mathrm{GL}(j(i), K)$ is a companion matrix, $i = 1, 2$, we may assume that $j(1) \geqslant j(2)$, by use of the following device.

Since the inverse of a companion matrix is similar to a companion matrix, let B_1, B_2 be companion matrices similar to A_2^{-1}, A_1^{-1}, respectively. Put $B = B_1 \dot{+} B_2$. If, for A, $j(1) < j(2)$, then, for B, $j(1) > j(2)$, and moreover $(|B_1|, |B_2|) = (|A_1|, |A_2|)$ since $|A_1|\,|A_2| = 1$. Since B is similar to A^{-1}, if $B = SDS^{-1}D^{-1}$ where S, D have arbitrary prescribed determinant, the same will hold for A also. In general we may suppose $A = A_1 \dot{+} \ldots \dot{+} A_m$ where $A_i \in \mathrm{GL}(j(i), K)$ is a companion matrix. Thus when $m = 2$ we may take $j(1) \geqslant j(2)$. We shall take advantage of the simplifying assumption explained in (**1**, §§ 5, 6). By rearranging the A_i, we can order the integers $j(1), \ldots, j(m)$ in any manner that is convenient at the moment and by considering A^{-1} instead of A we can eliminate some cases. By virtue of these remarks we need consider only the following possibilities when $K = \mathrm{GF}(4)$: $\tau = \theta$ or $\tau = \theta^2$; $m = 2$, $|A_1| = \theta$, $|A_2| = \theta^2$, $j(1) \geqslant j(2)$; $m = 3$, $|A_1| = |A_2| = |A_3| = \theta$, $j(1) \geqslant j(2) \geqslant j(3)$. And when $K = \mathrm{GF}(5)$ we need consider only the following possibilities: $\tau = 2, 3, 4$; $m = 2$, $|A_1| = 2$, $|A_2| = 3$, $j(1) \geqslant j(2)$; $m = 2$, $|A_1| = |A_2| = 4$, $j(1) \geqslant j(2)$; $m = 3$, $|A_1| = 2$, $|A_2| = 2$, $|A_3| = 4$, and $j(1), j(2)$ ordered in any convenient manner;

$$m = 4,\ |A_1| = |A_2| = |A_3| = |A_4| = 2,$$

and $j(1), j(2), j(3), j(4)$ ordered in any convenient manner.

5. The case $K = \mathrm{GF}(4)$. Let $K = \mathrm{GF}(4)$ and first suppose $m = 2$, $|A_1| = \theta$, $|A_2| = \theta^2$, $j(1) \geqslant j(2)$. Let $\delta_1 = \delta_3 = \theta$, $\delta_2 = \theta^2$. Then by Lemma 7, part 1, we may find non-derogatory $D \in \mathrm{GL}(n, \mathrm{GF}(4))$ with characteristic polynomial $(\lambda - \theta)^2(\lambda - \theta^2)(\lambda - 1)^{n-3}$ such that AD is non-derogatory and has characteristic polynomial $(\lambda - \theta^2)(\lambda - \theta)^2(\lambda - 1)^{n-3}$. Thus D and AD are similar, D has a linear elementary divisor, and $|D| = \theta$, finishing the case $\tau = \theta$, $m = 2$. Now let $\delta_1 = \theta^2$, $\delta_2 = \delta_3 = 1$. Then by Lemma 7, part (ii), we may find $D \in \mathrm{GL}(n, \mathrm{GF}(4))$ such that D and AD are non-derogatory and both have $(\lambda - \theta^2)(\lambda - 1)^{n-1}$ as characteristic polynomial. Since $|D| = \theta^2$ and D has a linear elementary divisor, this completes the case $m = 2$.

Let $m = 3$, $|A_1| = |A_2| = |A_3| = \theta$, $j(1) \geqslant j(2) \geqslant j(3)$. If $j(1) = 1$, A is scalar and § 2 supplies the result. So let $j(1) \geqslant 2$. Let $\mu = 1$ or -1 (mod 3), to be specified later. Use Lemma 6 to choose

$$U_1,\ \Delta_{j(1)}(\theta^\mu, g_2, \theta^\mu, g_4, d) \in \mathrm{GL}(j(1), \mathrm{GF}(4))$$

with $g_2 \neq 0$ such that

$$U_1 A_1 \Delta_{j(1)} U_1^{-1} = C((\lambda - \theta^{1+\mu})(\lambda - \theta^\mu)(\lambda - 1)^{j(1)-2}).$$

Use Lemma 6 to choose $U_2, \Delta_{j(2)}(1, h_2, 1, h_4, d') \in \mathrm{GL}(j(2), \mathrm{GF}(4))$ with $h_2 \neq 0$, $h_4 \neq 0$ such that $U_2 A_2 \Delta_{j(2)} U_2^{-1} = C((\lambda - \theta)(\lambda - 1)^{j(2)-1})$. Use Lemma 6 to choose $U_3, \Delta_{j(3)}(\theta^{-\mu}, k_2, 1, k_4, d'') \in \mathrm{GL}(j(3), \mathrm{GF}(4))$ with $k_4 \neq 0$ such that $U_3 A_3 \Delta_{j(3)} U_3^{-1} = C((\lambda - \theta^{1-\mu})(\lambda - 1)^{j(3)-1})$. Put

$$D = \begin{bmatrix} \Delta_{j(1)} & D_{12} & D_{13} \\ 0 & \Delta_{j(2)} & D_{23} \\ 0 & 0 & \Delta_{j(3)} \end{bmatrix}. \tag{22}$$

Here the last row of D_{12} is $(\theta^2, 0, \ldots, 0)$ and the last row of D_{23} is $(0, 1, 0, 0, \ldots, 0)$ when $j(3) \geqslant 2$ and (θ^2) when $j(3) = 1$. Then by Lemma 5 we may construct the remaining elements of D such that AD is non-derogatory and has $(\lambda - \theta^{\mu})^2(\lambda - \theta^{-\mu})(\lambda - 1)^{n-3}$ as its characteristic polynomial. This is also the characteristic polynomial of D. Now

$$D[2|1] = -g_2(\lambda - \theta^{-\mu})(\lambda - 1)^{n-3} \neq 0;$$

$$D[j(1) + j(2) + 1|j(1) + j(2) + 1] = (\lambda - \theta^{\mu})^2(\lambda - 1)^{n-3};$$

and $D[n|3]$ (when $j(3) > 1$) or $D[n - 1|3]$ (when $j(3) = 1$) is a polynomial in λ not vanishing when $\lambda = 1$. Hence D is non-derogatory and has a linear elementary divisor, so that (18) holds with $|S| = \phi$, $|D| = \theta^{\mu}$. By choosing $\mu = 1$ or -1 we get $|D| = \theta$ or θ^2 as required. This completes the case $K = GF(4)$.

6. The case $K = GF(5)$. First let $|A_1| = 2$, $|A_2| = 3$, $j(1) \geqslant j(2)$. If $j(1) = j(2) = 1$, then A is similar to $C((\lambda - 2)(\lambda - 3))$ which falls into case 1 of § 3. So suppose $j(1) \geqslant 2$. Let $\delta_1 = \delta_3 = \delta$, $\delta_2 = 2\delta$, where $\delta = 1, 2$, or 4. Then by Lemma 7, part (i) or (iv), we may find $D \in GL(n, GF(5))$ such that D and AD are both non-derogatory with characteristic polynomial $(\lambda - \delta)^2(\lambda - 2\delta)(\lambda - 1)^{n-3}$. As D has the linear elementary divisor $\lambda - 2\delta$, we can satisfy (18), (19) with $|D| = 2\delta^3 = 2, 1$, or 3. (This supplies a second proof for the case $\tau = 1$, $|A_1| = 2$, $|A_2| = 3$.) Now set $\delta_1 = 2$, $\delta_2 = 4$, $\delta_3 = 3$. By Lemma 7, part (iii), we get $D \in GL(n, GF(5))$ such that both D and AD are non-derogatory and both have $(\lambda - 2)(\lambda - 3)(\lambda - 4)(\lambda - 1)^{n-3}$ as characteristic polynomial. So we can satisfy (18) with $\tau = 4$. This finishes the case $(|A_1|, |A_2|) = (2, 3)$.

Now let $|A_1| = |A_2| = 4$, $j(1) \geqslant j(2)$. Let $j(1) > 1$ and let $\delta_1 = \delta_3 = \delta$, $\delta_2 = 4\delta$ where δ is 1, 2, or 3. Then by Lemma 7, part (i) or (iv) we get $D \in GL(n, GF(5))$ with D and AD both non-derogatory and having the same characteristic polynomial $(\lambda - \delta)^2(\lambda - 4\delta)(\lambda - 1)^{n-3}$. Thus again (18), (19) are satisfied, with $\tau = 4\delta^3 = 4, 2$, or 3. This completes the case

$$|A_1| = |A_2| = 4, \qquad j(1) > 1, \qquad j(1) \geqslant j(2).$$

If $j(1) = j(2) = 1$, A is scalar and § 2 supplies the result. This finishes all $m = 2$ cases.

We now suppose $m = 3$ and $(|A_1|, |A_2|, |A_3|) = (2, 2, 4)$. Using (**1**, Lemma 4) construct a standard matrix $D_1 \in GL(j(1), GF(5))$ with elementary divisors $\lambda - 2$, $(\lambda - 1)^{j(1)-1}$ such that the elementary divisors of $A_1 D_1$ are $\lambda - 4$, $(\lambda - 1)^{j(1)-1}$. Similarly construct $D_2 \in GL(j(2), GF(5))$ with elementary divisors $\lambda - 4$, $(\lambda - 1)^{j(2)-1}$ such that $A_2 D_2$ has elementary divisors $\lambda - 3$, $(\lambda - 1)^{j(2)-1}$. Construct $D_3 \in GL(j(3), GF(5))$ with elementary divisors $\lambda - 3$, $(\lambda - 1)^{j(3)-1}$ such that $A_3 D_3$ has elementary divisors $\lambda - 2$, $(\lambda - 1)^{j(3)-1}$. Set $D = D_1 \dotplus D_2 \dotplus D_3$. Then D and AD have the same elementary divisors, including a linear elementary divisor, and $|D| = 4$. So (18), (19) can be satisfied when $\tau = 4$.

If not both $j(1) = 1$, $j(2) = 1$, we arrange A_1, A_2 so that $j(1) > 1$. Construct by **(1**, Lemma 4**)** a standard matrix $D_1 \in \mathrm{GL}(j(1), K)$ with elementary divisors $\lambda - 2, \lambda - 3, (\lambda - 1)^{j(1)-2}$ such that A_1, D_1 has elementary divisors $\lambda - 4, \lambda - 3, (\lambda - 1)^{j(1)-2}$. Similarly construct $D_2 \in \mathrm{GL}(j)(2), \mathrm{GF}(5))$ with elementary divisors $\lambda - 4, (\lambda - 1)^{j(2)-1}$ such that the elementary divisors of $A_2 D_2$ are $(\lambda - 3), (\lambda - 1)^{j(2)-1}$. Construct $D_3 \in \mathrm{GL}(j(3), \mathrm{GF}(5))$ with elementary divisors $\lambda - 3, (\lambda - 1)^{j(3)-1}$ such that $A_3 D_3$ has elementary divisors $\lambda - 2, (\lambda - 1)^{j(3)-1}$. Set $D = D_1 \dotplus D_2 \dotplus D_3$. Then D and AD are similar and $|D| = 2$. So (18), (19) are satisfied with $\tau = 2$. However, this computation fails when $j(1) = j(2) = 1$. If also $j(3) = 1$, then A is similar to $C(\lambda - 2) \dotplus C((\lambda - 2)(\lambda - 4))$, which falls under the already treated case $m = 2$. So let $j(1) = j(2) = 1 \neq j(3)$. Use Lemma 6 to construct $U_3, \Delta_{j(3)}(3, g_2, 3, g_4, d) \in \mathrm{GL}(j(3), \mathrm{GF}(5))$ with $g_2 \neq 0$, such that

$$U_3 A_3 \Delta_{j(3)} U_3^{-1} = C((\lambda - 2)(\lambda - 3)(\lambda - 1)^{j(3)-2}).$$

Set $U_1 = U_2 = I_1$, and set

$$D = \begin{bmatrix} 2 & 3 & D_{13} \\ 0 & 4 & D_{23} \\ 0 & 0 & \Delta_{j(3)} \end{bmatrix}.$$

Here $D_{23} = 3$ (top row of U_3). Use Lemma 5 to construct D_{13} such that AD is non-derogatory with $(\lambda - 2)(\lambda - 4)(\lambda - 3)^2(\lambda - 1)^{n-4}$ as characteristic polynomial. This is also the characteristic polynomial of D. Moreover D is non-derogatory since 2, 4 are simple eigenvalues of D,

$$D[4|3] = -g_2(\lambda - 2)(\lambda - 4)(\lambda - 1)^{n-4} \neq 0,$$

and $D[n|5]_{\lambda=1} \neq 0$ (when $n > 5$). Then, in the usual way, D and AD are similar and $|D| = 2$. This shows that (18), (19) can always be solved with $\tau = 2$.

If not both $j(1) = 1$, $j(2) = 1$, let $j(2) > 1$. Use Lemma 6 to construct $U_1, \Delta_{j(1)}(2, g_2, 1, g_4, d) \in \mathrm{GL}(j(1), \mathrm{GF}(5))$ with $g_4 \neq 0$ such that

$$U_1 A_1 \Delta_{j(1)} U_1^{-1} = C((\lambda - 4)(\lambda - 1)^{j(1)-1}).$$

Use Lemma 6 to construct $U_2, \Delta_{j(2)}(1, h_2, 1, h_4, d') \in \mathrm{GL}(j(2), \mathrm{GF}(5))$ with $h_2 \neq 0$, $h_4 \neq 0$ such that $U_2 A_2 \Delta_{j(2)} U_2^{-1} = C((\lambda - 2)(\lambda - 1)^{j(2)-1})$. Use Lemma 6 to construct $U_3, \Delta_{j(3)}(4, k_2, 1, k_4, d'') \in \mathrm{GL}(j(3), \mathrm{GF}(5))$ with $k_4 \neq 0$ such that $U_3 A_3 \Delta_{j(3)} U_3^{-1} = C((\lambda - 1)^{j(3)})$. Define D by (22). We let the last row of D_{12} be $(1, 0, 0, \ldots, 0)$ if $j(1) > 1$, and 3 (first row of U_2) if $j(1) = 1$. We let the last row of D_{23} be $(0, 1, 0, 0, \ldots, 0)$ if $j(3) > 1$, and (1) if $j(3) = 1$. We use Lemma 5 to construct the remaining elements of D so that AD is non-derogatory with characteristic polynomial $(\lambda-2)(\lambda-4) \times (\lambda - 1)^{n-2}$. This is also the characteristic polynomial of D. Since 2, 4 are simple eigenvalues of D, and $D[n-1|2]_{\lambda=1} \neq 0$ (if $j(3) = 1$) or $D[n|2]_{\lambda=1} \neq 0$ (if $j(3) > 1$), it follows that D is non-derogatory also. Thus AD is similar to D and as D has a linear elementary divisor and $|D| = 3$, we can satisfy (18), (19) when $\tau = 3$. However, this computation fails if $j(1) = j(2) = 1$. Assume $j(1) = j(2) = 1$ and let $A_3 = C(p(\lambda))$. If $p(2) \neq 0$, then A is similar to $C(\lambda - 2) \dotplus C((\lambda - 2)p(\lambda))$, already treated under case $m = 2$. Let $p(2) = 0$ (hence $j(3) > 1$). If $j(3) = 2$, then $A = 2I_2 \dotplus C((\lambda - 2)^2)$, which has already been handled in § 3, case 3. So let $j(3) > 2$. Use **(2**, Lemma 2**)** to construct $U_3, \Delta_{j(3)}(2, g_2, 2, g_4, d) \in GL(j(3), GF(5))$ with $g_2 \neq 0$, such that $U_3 A_3 \Delta_{j(3)} U_3^{-1} = C((\lambda - 2)^2(\lambda - 4)(\lambda - 1)^{j(3)-3})$. ($U_3$ is the matrix ST of **(2**, Lemma 2**)**.) Then set

$$D = \begin{bmatrix} 4 & 3 & D_{13} \\ 0 & 3 & D_{23} \\ 0 & 0 & \Delta_{j(3)} \end{bmatrix}.$$

Here $D_{23} = 3$ (first row of U_3). Then by Lemma 5 we may construct D_{13} so that AD is non-derogatory with $(\lambda - 2)^2(\lambda - 4)(\lambda - 3)(\lambda - 1)^{n-4}$ as characteristic polynomial. This is also the characteristic polynomial of D. Moreover, D is non-derogatory since 4, 3 are simple eigenvalues,

$$D[4|3] = -g_2(\lambda - 4)(\lambda - 3)(\lambda - 1)^{n-4} \neq 0,$$

and $D[n|5]_{\lambda=1} \neq 0$ (when $n > 5$). Since $|D| = 3$, we have now solved (18), (19) when $\tau = 3$. This completely finishes all $m = 3$ cases.

Now let $m = 4$ and $|A_1| = |A_2| = |A_3| = |A_4| = 2$. As in **(1**, § 5**)**, we need only find an element $\delta_1 \in GF(5)^*$ and integers $e(i)$ satisfying

$$0 \leqslant e(i) \leqslant j(i) - 1$$

such that

$$\delta_1^n 3^{2(1+j(1)+j(4))-e(1)+e(2)+3e(3)+e(4)} = \tau. \tag{23}$$

If some $j(i)$ is $\geqslant 4$, let $j(2) \geqslant 4$. Set $\delta_1 = 1$, $e(1) = e(3) = e(4) = 0$, $e(2) = 0, 1, 2, 3$ so as to satisfy (23). Now suppose all $j(i)$ are $\leqslant 3$. Suppose $j(2) = 3$ and $j(4) \geqslant 2$. Then put $\delta_1 = 1$, $e(1) = e(3) = 0$, $e(2) = 0, 1$, or 2, and $e(4) = 0$ or 1 so that $e(2) + e(4) = 0, 1, 2$, or 3 as necessary to satisfy (23). Now suppose $j(2) = 3$ and $j(1) = j(3) = j(4) = 1$. Hence $n = 6$ and the left member of (23) becomes $\delta_1^2 3^{2+e(2)}$. If $\tau = 1$, take $\delta_1 = 1$, $e(2) = 2$. If $\tau = 2$, take $\delta_1 = 1$, $e(2) = 1$. If $\tau = 3$, take $\delta_1 = 3$, $e(2) = 1$. If $\tau = 4$, take $\delta_1 = 1$, $e(2) = 0$. Hence we may assume each $j(i) \leqslant 2$. If there exist at least three $j(i)$ not one, let $j(2) = j(3) = j(4) = 2$. Set $\delta_1 = 1$ and take $e(2)$, $e(3)$, $e(4)$ to be 0 or 1 so that $e(2) + 3e(3) + e(4) \equiv 0, 1, 2$, or 3 (mod 4) as required to satisfy (23). If exactly two $j(i)$ are two and exactly two are one, let $j(1) = j(2) = 2$, $j(3) = j(4) = 1$. Then $n = 6$ and (23) becomes $\delta_1^2 3^{-e(1)+e(2)} = \tau$. If $\tau = 1$, take $\delta_1 = 1$, $e(1) = e(2) = 0$. If $\tau = 2$, take $\delta_1 = 1$, $e(1) = 1$, $e(2) = 0$. If $\tau = 3$, take $\delta_1 = 1$, $e(1) = 0$, $e(2) = 1$. If $\tau = 4$, take $\delta_1 = 2$, $e(1) = e(2) = 0$. Now suppose $j(1) = j(2) = j(3) = 1$,

$j(4) = 2$. Then $n = 5$ and if we put $e(4) = 0$ and $\delta_1 = \tau$, (23) will be satisfied. Finally if $j(1) = j(2) = j(3) = j(4) = 1$, then A is scalar and this case was handled in § 2. This completes the $m = 4$ case and the proof of Theorem 1.

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