Glasgow Math. J. **41** (1999) 45–57. © Glasgow Mathematical Journal Trust 1999. Printed in the United Kingdom

GREEN'S RELATIONS AND REGULARITY IN CENTRALIZERS OF PERMUTATIONS

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(Received 18 February, 1997)

Abstract. For any permutation σ on $N = \{1,2,\ldots,n\}$: (i) Green's relations are characterized in the centralizer $C(\sigma)$ of σ (relative to the semigroup PT_n of partial transformations on N); and (ii) a criterion is given for $C(\sigma)$ to be a regular semigroup (inverse semigroup, union of groups).

1. Introduction. Let PT_n denote the semigroup of partial transformations on $N = \{1,2,\ldots,n\}$, and let S_n denote the symmetric group of permutations on N, the group of units of PT_n. For $\gamma \in PT_n$, the set

$$C(\gamma) = \{\alpha \in PT_n : \alpha\gamma = \gamma\alpha\}$$

is a subsemigroup of PT_n, called the *centralizer* of γ.

Centralizers of partial transformations are studied in [**3**], where the elements of $C(\gamma)$ are characterized. It is shown in [**7**] that for a permutation $\sigma \in S_n$, $C(\sigma)$ can be embedded into a wreath product of two semigroups determined by the number and length of cycles in σ. Centralizers in some subsemigroups of PT_n have also been studied. A structure of centralizers in the symmetric group S_n is presented in [**8**]. A representation and order of centralizers in the symmetric inverse semigroup I_n are given in [**4**] and [**5**]. A construction of centralizers in the full transformation semigroup T_n is presented in [**1**]. Many results from the above references are collected in [**6**].

In this paper, we study centralizers of permutations in PT_n. Section 2 introduces notation, definitions, and some preliminary results. In Section 3, Green's relations in $C(\sigma)$ (for any $\sigma \in S_n$) are determined. Section 4 characterizes the permutations $\sigma \in S_n$ whose centralizer $C(\sigma)$ is a regular semigroup (inverse semigroup, union of groups). In particular, we find that $C(\sigma)$ is an inverse semigroup if and only if it is a union of groups. As an illustration, the egg-box structure of a specific centralizer is presented (Section 5).

2. Preliminary results. For $\alpha \in PT_n$, the domain and range of α will be denoted by dom α and ran α, respectively. The *kernel* of α, denoted by ker α, is the equivalence relation on dom α defined by x (ker α) $y \iff x\alpha = y\alpha$. Denoting by dom α/ker α the partition of dom α induced by ker α, we have |dom α/ker α| = |ran α|. This common cardinality of dom α/ker α and ran α is called the *rank* of α and denoted rank α. For example, for

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 4 & - & 4 & 3 & 1 \end{pmatrix} \in PT_7,$$

dom α = {1,2,3,5,6,7}, ran α = {1,3,4}, ker α = | 1 3 5 | 2 6 | 7 | (we identify ker α with dom α/ker α), and rank α = 3.

Throughout the paper, we shall use the following characterization of the elements of $C(\sigma)$ ($\sigma \in S_n$), which is a special case of [**3**, Theorem 5].

THEOREM 2.1. *Let $\sigma \in S_n$ and $\alpha \in PT_n$. Then $\alpha \in C(\sigma)$ if and only if for every cycle $(x_0x_1\ldots x_{k-1})$ in σ such that some $x_i \in$ dom α, the following conditions are satified:*

(i) $\{x_0,x_1,\ldots,x_{k-1}\} \subseteq$ dom α;
(ii) *there is a cycle $(y_0y_1\ldots y_{m-1})$ in σ such that m divides k and for some index j,*

$$x_0\alpha = y_j, \qquad x_1\alpha = y_{j+1}, \qquad x_2\alpha = y_{j+2}, \ldots,$$

where the subscripts on ys are calculated modulo m. □

Let $\sigma \in S_n$ be a permutation with cycle decomposition $\sigma = a_1 \cdots a_t$ (1-cycles included). For $\alpha \in C(\sigma)$, define a partial transformation t_α on the set $A = \{a_1, \ldots, a_t\}$ of the cycles in σ by:

(1) dom t_α consists of all cycles $a = (x_0x_1\ldots x_{k-1}) \in A$ such that some x_i is in dom α;
(2) for each $a = (x_0x_1 \ldots x_{k-1}) \in$ dom t_α and each $b = (y_0y_1\ldots y_{m-1}) \in A$

$$at_\alpha = b \Longleftrightarrow x_i\alpha = y_j \text{ for some } x_i \text{ and some } y_j.$$

By Theorem 2.1, t_α is well defined. Speaking informally, $at_\alpha = b$ if α wraps cycle a around cycle b one or more times. As an example, consider the permutation $\sigma = abc = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9)$ in S_9 and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ - & - & 4 & 5 & 3 & 1 & 2 & 1 & 2 \end{pmatrix} \in C(\sigma)$. Then $t_\alpha = \begin{pmatrix} a & b & c \\ - & b & a \end{pmatrix}$.

For a cycle a, $\ell(a)$ will denote the length of a. For example, if $a = (1\ 2\ 3)$, then $\ell(a) = 3$.

We shall frequently use the following lemma.

LEMMA 2.2. *If $\sigma \in S_n$, $a = (x_0x_1\ldots x_{k-1})$ and $b = (y_0y_1\ldots y_{m-1})$ are cycles in σ, and α, $\beta \in C(\sigma)$, then:*

(1) $t_{\alpha\beta} = t_\alpha t_\beta$;
(2) *if $at_\alpha = b$ then $\ell(b)$ divides $\ell(a)$;*
(3) *$b \in$ ran t_α if and only if $\{y_0,y_1,\ldots,y_{m-1}\} \subseteq$ ran α;*
(4) *$at_\alpha = bt_\alpha$ if and only if $x_i\alpha = y_j\alpha$ for some x_i and some y_j.*

Proof. Immediate by the definition of t_α and Theorem 2.1. □

3. Green's relations. If S is a semigroup and $a,b \in S$, we say that $a\,\mathcal{L}\,b$ if $S^1a = S^1b$, $a\,\mathcal{R}\,b$ if $aS^1 = bS^1$, and $a\,\mathcal{J}\,b$ if $S^1aS^1 = S^1bS^1$, where S^1 is the semigroup S with an identity adjoined. We define $\mathcal{H}$ as the intersection of $\mathcal{L}$ and $\mathcal{R}$, and $\mathcal{D}$ as the join of $\mathcal{L}$ and $\mathcal{R}$, i.e., the smallest equivalence containing both $\mathcal{L}$ and $\mathcal{R}$. These five equivalences are known as *Green's relations* [**2**, p. 45]. The relations $\mathcal{L}$ and $\mathcal{R}$

commute, and consequently $\mathcal{D} = \mathcal{L}\circ\mathcal{R} = \mathcal{R}\circ\mathcal{L}$. If S is finite then $\mathcal{D} = \mathcal{J}$. For $a \in S$, we denote the equivalence classes of a with respect to $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$, and $\mathcal{D}$ by L_a, R_a, J_a, H_a, and D_a, respectively.

Green's relations in the semigroup PT_n are well known [**2**, Exercise 17, p. 63].

LEMMA 3.1. *If* $\alpha,\beta \in PT_n$, *then the following hold.*

(1) $\alpha\ \mathcal{L}\ \beta \iff \text{ran}\ \alpha = \text{ran}\ \beta$.
(2) $\alpha\ \mathcal{R}\ \beta \iff \ker \alpha = \ker \beta$.
(3) $\alpha\ \mathcal{H}\ \beta \iff \text{ran}\ \alpha = \text{ran}\ \beta$ *and* $\ker \alpha = \ker \beta$.
(4) $\alpha\ \mathcal{D}\ \beta \iff \text{rank}\ \alpha = \text{rank}\ \beta$. □

A description of Green's relations in $C(\sigma)$ ($\sigma \in S_n$) will involve t_α ($\alpha \in C(\sigma)$). The following lemma clarifies the relation between the range and kernel of α and t_α.

LEMMA 3.2. *If* $\sigma \in S_n$ *and* $\alpha,\beta \in C(\sigma)$, *then*

(1) $\text{ran}\ \alpha = \text{ran}\ \beta \iff \text{ran}\ t_\alpha = \text{ran}\ t_\beta$,
(2) $\ker \alpha = \ker \beta \implies \ker t_\alpha = \ker t_\beta$.

Proof. Statement (1) follows from (3) of Lemma 2.2 and Theorem 2.1. To show (2), suppose $\ker \alpha = \ker \beta$. Let $a = (x_0x_1\ldots x_{k-1})$ and $b = (y_0y_1\ldots y_{m-1})$ be cycles in σ. Then,

$$\begin{aligned}(a,b) \in \ker t_\alpha &\iff at_\alpha = bt_\alpha\\ &\iff x_i\alpha = y_j\alpha \ \text{ for some } x_i \text{ and some } y_j \quad \text{(by (4) of Lemma 2.2)}\\ &\iff x_i\beta = y_j\beta \ \ (\text{since } \ker\alpha = \ker\beta)\\ &\iff at_\beta = bt_\beta \quad \text{(by (4) of Lemma 2.2)}\\ &\iff (a,b) \in \ker t_\beta. \qquad \square\end{aligned}$$

The implication in (2) cannot be reversed. For example, if $\sigma = ab = (1\ 2)(3) \in S_3$, then for $\alpha = \begin{pmatrix} 1 & 2 & 3\\ 1 & 2 & - \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3\\ 3 & 3 & - \end{pmatrix}$ in $C(\sigma)$, we have $t_\alpha = \begin{pmatrix} a & b\\ a & - \end{pmatrix}$ and $t_\beta = \begin{pmatrix} a & b\\ b & - \end{pmatrix}$. Thus $\ker t_\alpha = \ker t_\beta = |\ a\ |$, but $\ker \alpha = |\ 1\ |\ 2\ |$ is different from $\ker \beta = |\ 1\ 2\ |$.

There is no corresponding result for ranks. It is possible to have $\alpha,\beta \in C(\sigma)$ with $\text{rank}\ \alpha = \text{rank}\ \beta$ but $\text{rank}\ t_\alpha \neq \text{rank}\ t_\beta$ as well as with $\text{rank}\ t_\alpha = \text{rank}\ t_\beta$ but $\text{rank}\ \alpha \neq \text{rank}\ \beta$.

For $\sigma \in S_n$, $\alpha \in C(\sigma)$, and $b \in \text{ran}\ t_\alpha$, we denote by $t_\alpha^{-1}(b)$ the set of all cycles $a \in \text{dom}\ t_\alpha$ such that $at_\alpha = b$.

The following theorem characterizes Green's $\mathcal{L}$ relation in $C(\sigma)$.

THEOREM 3.3. *Let* $\sigma \in S_n$ *and let* $\alpha,\beta \in C(\sigma)$. *Then,* $\alpha\ \mathcal{L}\ \beta$ (*in* $C(\sigma)$) *if and only if the following conditions are satisfied*:

(1) $\text{ran}\ t_\alpha = \text{ran}\ t_\beta$;
(2) *for every* $c \in \text{ran}\ t_\alpha = \text{ran}\ t_\beta$:

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(a) *if* $a \in t_\alpha^{-1}(c)$, *then there exists* $b \in t_\beta^{-1}(c)$ *such that* $\ell(b)$ *divides* $\ell(a)$;
(b) *if* $a \in t_\beta^{-1}(c)$, *then there exists* $b \in t_\alpha^{-1}(c)$ *such that* $\ell(b)$ *divides* $\ell(a)$.

Proof. Suppose $\alpha \mathcal{L} \beta$. Then $\alpha \mathcal{L} \beta$ in PT_n and so (1) holds by (1) of Lemma 3.1 and (1) of Lemma 3.2. To show (2)(a), suppose $c \in \operatorname{ran} t_\alpha = \operatorname{ran} t_\beta$ and let $a \in t_\alpha^{-1}(c)$. Since $\alpha \mathcal{L} \beta$, we have $\alpha = \gamma\beta$ for some $\gamma \in C(\sigma)$ and so, by (1) of Lemma 2.2, $t_\alpha = t_\gamma t_\beta$. Since $at_\alpha = c$, there is a cycle b in σ such that $at_\gamma = b$ and $bt_\beta = c$. Thus $\ell(b)$ divides $\ell(a)$ (by (2) of Lemma 2.2) and $b \in t_\beta^{-1}(c)$. The condition 2(b) follows by symmetry.

Conversely, suppose (1) and (2) hold. We shall construct $\gamma \in C(\sigma)$ such that $\alpha = \gamma\beta$. First, we set dom γ = dom α. To define the values of γ, let $a = (x_0x_1\ldots x_{k-1})$ be a cycle in σ with $a \in \operatorname{dom} t_\alpha$, and let $c = (y_0y_1\ldots y_{m-1}) = at_\alpha$. By Theorem 2.1, m divides k and for some index j,

$$x_0\alpha = y_j, \qquad x_1\alpha = y_{j+1}, \qquad x_2\alpha = y_{j+2}, \ldots,$$

where the subscripts on ys are calculated modulo m. By (1) and (2)(a), $c \in \operatorname{ran} t_\beta$ and there is $b = (w_0w_1\ldots w_{p-1}) \in \operatorname{dom} t_\beta$ such that $bt_\beta = c$ and p divides k. By Theorem 2.1, m divides p and for some index i,

$$w_0\beta = y_i, \qquad w_1\beta = y_{i+1}, \qquad w_2\beta = y_{i+2}, \ldots,$$

where the subscripts on ys are calculated modulo m. Let $u \in \{0,1,\ldots,p-1\}$ be an index such that $w_u\beta = y_j$. Since p divides k, we may define

$$x_0\gamma = w_u, \qquad x_1\gamma = w_{u+1}, \qquad x_2\gamma = w_{u+2}, \ldots,$$

where the subscripts on ws are calculated modulo p. By the construction of γ and Theorem 2.1, we have $\alpha = \gamma\beta$ and $\gamma \in C(\sigma)$. By symmetry, there is $\delta \in C(\sigma)$ such that $\beta = \delta\alpha$, which concludes the proof. □

To illustrate Theorem 3.3, let $\sigma = abcd = (1\ 2)(3\ 4\ 5)(6)(7) \in S_7$ and consider $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 7 & 7 & 7 & 6 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 7 & 6 & 6 & 6 & 7 & - \end{pmatrix}$ in $C(\sigma)$. Calculating $t_\alpha = \begin{pmatrix} a & b & c & d \\ c & d & c & c \end{pmatrix}$ and $t_\beta = \begin{pmatrix} a & b & c & d \\ d & c & d & - \end{pmatrix}$, we see that (1) of Theorem 3.3 holds, but (2) does not hold. Indeed, $a \in t_\alpha{}^{-1}(c)$ and $\ell(a) = 2$, but the only element of $t_\beta^{-1}(c)$ is b, for which $\ell(b) = 3$. Hence, α and β are not in the same $\mathcal{L}$-class in $C(\sigma)$. Note, however, that $\alpha \mathcal{L} \beta$ in PT_n since ran α = ran β.

For any integers i and m, $m \geq 1$, we denote by $(i)_m$ the unique integer j such that $i \equiv j \pmod m$ and $0 \leq j \leq m-1$.

Unlike the $\mathcal{L}$ relation, Green's $\mathcal{R}$ relation in $C(\sigma)$ is simply the restriction of the $\mathcal{R}$ relation in PT_n to $C(\sigma) \times C(\sigma)$.

THEOREM 3.4. *Let* $\sigma \in S_n$ *and let* $\alpha,\beta \in C(\sigma)$. *Then* $\alpha \mathcal{R} \beta$ *(in* $C(\sigma)$*) if and only if* ker α = ker β.

Proof. If $\alpha \mathcal{R} \beta$ in $C(\sigma)$, then $\alpha \mathcal{R} \beta$ in PT_n and so ker α = ker β by (2) of Lemma 3.1. Conversely, suppose ker α = ker β. We shall construct $\gamma \in C(\sigma)$ such that $\alpha\gamma = \beta$. First, we set dom γ = ran α. To define the values of γ, let

$b = (y_0y_1 \dots y_{m-1}) \in \operatorname{ran} t_\alpha$ and let $a = (x_0x_1 \dots x_{k-1})$ be a cycle in dom t_α such that $at_\alpha = b$. By Theorem 2.1, m divides k and for some index j,

$$x_0\alpha = y_j, \qquad x_1\alpha = y_{j+1}, \qquad x_2\alpha = y_{j+2}, \dots,$$

where the subscripts on ys are calculated modulo m. Since $\ker \alpha = \ker \beta$, we have $\ker t_\alpha = \ker t_\beta$ by (2) of Lemma 3.2, which implies dom t_α = dom t_β. Thus $a \in$ dom t_β and let $c = (z_0z_1 \dots z_{p-1}) = at_\beta$. By Theorem 2.1, p divides k and for some index i,

$$x_0\beta = z_i, \qquad x_1\beta = z_{i+1}, \qquad x_2\beta = z_{i+2}, \dots,$$

where the subscripts on zs are calculated modulo p. Note that $\ker \alpha = \ker \beta$ implies $m = p$. (Indeed, if, say, $m < p$, then $x_0\alpha = x_m\alpha = y_j$, implying $z_i = x_0\beta = x_m\beta = z_{(i+m)_p}$, which is a contradiction since for $m < p$, $z_i \neq z_{(i+m)_p}$.) Thus we may define

$$y_j\gamma = z_i, \qquad y_{j+1}\gamma = z_{i+1}, \qquad y_{j+2}\gamma = z_{i+2}, \dots,$$

where the subscripts on ys and on zs are calculated modulo m $(= p)$. By the construction of γ and Theorem 2.1, $\gamma \in C(\sigma)$. It remains to show that $\alpha\gamma = \beta$. Since dom γ = ran α and dom α = dom β, we have dom $(\alpha\gamma)$ = dom β. Let $w \in$ dom $(\alpha\gamma)$ = dom β. Then there is $d = (w_0w_1 \dots w_{q-1}) \in$ dom t_α such that $w = w_s$ for some index s. Let $b = (y_0y_1 \dots y_{m-1}) = dt_\alpha$, $a = (x_0x_1 \dots x_{k-1})$, and $c = (z_0z_1 \dots z_{p-1})$ be the cycles used in the construction of γ. Let $y_v = w_s\alpha$ $(v \in \{0,1,\dots,m-1\})$ and let u be the unique number in $\{0,1,\dots,m-1\}$ such that $v = (j+u)_m$. Then $w_s(\alpha\gamma) = y_v\gamma = z_{(i+u)_m}$. Note that $x_u\alpha = y_{(j+u)_m} = y_v = w_s\alpha$. This and the fact that $\ker \alpha = \ker \beta$ give $w_s\beta = x_u\beta = z_{(i+u)_m}$, which shows that $\alpha\gamma = \beta$. By a similar construction, we obtain $\delta \in C(\sigma)$ such that $\beta\delta = \alpha$, which concludes the proof. □

Corollary 3.5. *Let $\sigma \in S_n$ and let $\alpha,\beta \in C(\sigma)$. Then, $\alpha\,\mathcal{H}\,\beta$ (in $C(\sigma)$) if and only if* ran t_α = ran t_β, $\ker \alpha = \ker \beta$, *and* (2) *of Theorem 3.3 is satisfied.*

Proof. Follows from Theorem 3.3, Theorem 3.4, and the fact that $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. □

The next theorem characterizes Green's $\mathcal{D}$ relation in $C(\sigma)$.

Theorem 3.6. *Let $\sigma \in S_n$ and let $\alpha,\beta \in C(\sigma)$. Then, $\alpha\mathcal{D}\beta$ (in $C(\sigma)$) if and only if the following conditions are satisfied.*

(1) rank t_α = rank t_β.

(2) *The sets* ran t_α *and* ran t_β *can be ordered, say,*

$$\operatorname{ran} t_\alpha : c_1, c_2, \dots, c_u,$$

$$\operatorname{ran} t_\beta : d_1, d_2, \dots, d_u,$$

in such a way that for each i, $1 \le i \le u$, $\ell(c_i) = \ell(d_i)$ and:

(a) *if* $a \in t_\alpha^{-1}(c_i)$, *then there exists $b \in t_\beta^{-1}(d_i)$ such that $\ell(b)$ divides $\ell(a)$;*

(b) *if* $a \in t_\beta^{-1}(d_i)$, *then there exists $b \in t_\alpha^{-1}(c_i)$ such that $\ell(b)$ divides $\ell(a)$.*

Proof. Suppose $\alpha\mathcal{D}\beta$. Since $\mathcal{D} = \mathcal{R}\circ\mathcal{L}$, there is $\delta \in C(\sigma)$ such that $\alpha\mathcal{R}\delta$ and $\delta\mathcal{L}\beta$. Then $\ker t_\alpha = \ker t_\delta$ (by Theorem 3.4 and (2) of Lemma 3.2) and ran t_δ = ran t_β (by

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Theorem 3.3), which implies rank t_α = rank t_δ = rank t_β. Select an ordering of

$$\text{ran } t_\alpha : c_1, c_2, \ldots, c_u.$$

Since $\alpha\mathcal{R}\delta$, $\alpha\gamma = \delta$ for some $\gamma \in C(\sigma)$, which gives $t_\alpha t_\gamma = t_\delta$ by (1) of Lemma 2.2. Moreover, by the proof of Theorem 3.4, γ can be selected in such a way that dom t_γ = ran t_α, ran t_γ = ran t_δ, and for each $c_i \in$ dom t_γ = ran t_α, the cycle $c_i t_\gamma$ has the same length as c_i. Since t_γ maps ran t_α onto ran t_δ and $|\text{ran } t_\alpha| = |\text{ran } t_\delta|$, we also have that t_γ is one-one. Therefore, setting $d_i = c_i t_\gamma$ $(1 \leq i \leq u)$, we obtain the corresponding ordering of

$$\text{ran } t_\beta = \text{ran } t_\delta = \text{ran } t_\gamma : d_1, d_2, \ldots, d_u,$$

with $\ell(c_i) = \ell(d_i)$ for each i. Let $i \in \{1,\ldots,u\}$. Then, for every cycle a in σ,

$$\begin{aligned} a \in t_\alpha^{-1}(c_i) &\iff a t_\alpha = c_i \\ &\iff (a t_\alpha) t_\gamma = d_i \quad (\text{since } c_i t_\gamma = d_i \text{ and } t_\gamma \text{ is one-one}) \\ &\iff a t_\delta = d_i \quad (\text{since } t_\delta = t_\alpha t_\gamma) \\ &\iff a \in t_\delta^{-1}(d_i). \end{aligned}$$

Thus $t_\alpha^{-1}(c_i) = t_\delta^{-1}(d_i)$ and so (2) is satisfied by the fact that $\delta\mathcal{L}\beta$ and Theorem 3.3.

Conversely, suppose that (1) and (2) are satisfied. For $i \in \{1,\ldots,u\}$, let $c_i = (x_{i0}x_{i1}\ldots x_{i,r_i-1})$ and $d_i = (y_{i0}y_{i1}\ldots y_{i,r_i-1})$. Let $\gamma,\gamma' \in PT_n$ be transformations with dom γ = ran α, dom γ' = ran β, and values determined by $x_{ij}\gamma = y_{ij}$ and $y_{ij}\gamma' = x_{ij}$ $(1 \leq i \leq u, 0 \leq j \leq r_{i-1})$. Then, by Theorem 2.1, $\gamma,\gamma' \in C(\sigma)$. Setting $\delta = \alpha\gamma$, we have $\delta\gamma' = \alpha\gamma\gamma' = \alpha$, which gives $\alpha\mathcal{R}\delta$. By the definitions of γ and δ, we have that ran t_δ = $\{d_1,\ldots,d_u\}$ = ran t_β and that for each i, $1 \leq i \leq u$, $t_\alpha{}^{-1}(c_i) = t_\delta^{-1}(d_i)$. This, (2), and Theorem 3.3 imply $\delta\mathcal{L}\beta$, which, coupled with $\alpha\mathcal{R}\delta$, gives $\alpha\mathcal{D}\beta$. □

Recall the example given after Theorem 3.3: $\sigma = abcd = (1\ 2)(3\ 4\ 5)(6)(7) \in S_7$, $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 6 & 7 & 7 & 7 & 6 & 6 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 7 & 6 & 6 & 6 & 7 & - \end{pmatrix}$ in $C(\sigma)$. Calculating $t_\alpha = \begin{pmatrix} a & b & c & d \\ c & d & c & c \end{pmatrix}$ and $t_\beta = \begin{pmatrix} a & b & c & d \\ d & c & d & - \end{pmatrix}$, we have rank t_α = rank t_β = 2. Moreover, ordering ran $t_\alpha : c, d$ and ran $t_\beta : d, c$ we see that (2) of Theorem 3.6 is also satisfied. Hence $\alpha\mathcal{D}\beta$ in $C(\sigma)$.

In a finite semigroup S, the $\mathcal{D}$-classes are partially ordered by the following relation:

$$D_a \leq D_b \iff S^1 a S^1 \subseteq S^1 b S^1,$$

where $a, b \in S$. The relation $\leq$ is a partial ordering since in a finite semigroup $\mathcal{D} = \mathcal{J}$. When studying the structure of a finite semigroup, it is important to determine not only the $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, and $\mathcal{D}$-classes, but also the partial ordering of $\mathcal{D}$-classes.

The next theorem determines the partial ordering of $\mathcal{D}$-classes in $C(\sigma)$.

THEOREM 3.7. *Let $\sigma \in S_n$ and let $\alpha,\beta \in C(\sigma)$ with* ran $t_\alpha = \{c_1,c_2,\ldots,c_u\}$. *Then, $D_\alpha \leq D_\beta$ if and only if to each sequence*

$$s : a_1 \in t_\alpha^{-1}(c_1), a_2 \in t_\alpha^{-1}(c_2), \ldots, a_u \in t_\alpha^{-1}(c_u), \tag{1}$$

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we can assign a sequence of elements of ran t_β:

$$d^s : d^s_1, d^s_2, \ldots, d^s_u, \qquad (2)$$

in such a way that for all sequences s and t as in (1) *and for all* $i,j \in \{1,\ldots,u\}$:

(i) $\ell(c_i)$ *divides* $\ell(d^s_i)$;
(ii) *there is* $b_i \in t_\beta^{-1}(d^s_i)$ *such that* $\ell(b_i)$ *divides* $\ell(a_i)$;
(iii) *if* $d^s_i = d^t_j$, *then* $i = j$.

Proof. Suppose $D_\alpha \le D_\beta$, i.e., $\alpha = \delta\beta\gamma$ for some $\delta,\gamma \in C(\sigma)$. By (1) of Lemma 2.2, $t_\alpha = t_\delta t_\beta t_\gamma$. Consider a sequence s as in (1) and let $i \in \{1,\ldots,u\}$. Since $a_i t_\alpha = c_i$ and $t_\alpha = t_\delta t_\beta t_\gamma$, there are cycles b_i and d^s_i in σ such that $a_i t_\delta = b_i$, $b_i t_\beta = d^s_i$, and $d^s_i t_\gamma = c_i$. Then $b_i \in t_\beta^{-1}(d^s_i)$ and, by (2) of Lemma 2.2, $\ell(c_i)$ divides $\ell(d^s_i)$ and $\ell(b_i)$ divides $\ell(a_i)$. Thus, assigning d^s_1, d^s_2, …, d^s_u to s, we have that (i) and (ii) are satisfied. To show (iii), assume that s and t are sequences as in (1) and that $i,j \in \{1,\ldots,u\}$. Then,

$$d^s_i = d^t_j \;\Rightarrow\; d^s_i t_\gamma = d^t_j t_\gamma \;\Rightarrow\; c_i = c_j \;\Rightarrow\; i = j.$$

Conversely, suppose that to each sequence (1) we can assign a sequence (2) in such a way that the conditions (i)–(iii) are satisfied. We shall construct $\delta,\gamma \in C(\sigma)$ such that $\alpha = \delta\beta\gamma$. First, we define dom γ to be the set of all elements that occur in any cycle d in σ such that $d = d^s_v$ for some sequence s as in (1) and some $v \in \{1,\ldots,u\}$. To define the values of γ, let $d = d^s_v = (w_0 w_1 \ldots w_{q-1})$ and let $c_v = (z_0 z_1 \ldots z_{p-1})$. By (i), p divides q, and so we may define

$$w_0\gamma = z_0, \qquad w_1\gamma = z_1, \qquad w_2\gamma = z_2, \ldots,$$

where the subscripts on zs are calculated modulo p. By (iii), γ is well-defined. Next, we set dom δ = dom α. To define the values of δ, let $a = (x_0 x_1 \ldots x_{k-1}) \in$ dom t_α. Then $a \in t_\alpha^{-1}(c_v)$ for some $v \in \{1,\ldots,u\}$. Select a sequence s as in (1) with $a_v = a$, and let $d^s_v = (w_0 w_1 \ldots w_{q-1})$ and $c_v = (z_0 z_1 \ldots z_{p-1})$ be as in the construction of γ. By (ii), there is $b_v = (y_0 y_1 \ldots y_{m-1}) \in t_\beta^{-1}(d^s_v)$ such that m divides k. By Theorem 2.1, p divides q, q divides m, and for some indices $i \in \{0,1,\ldots,p-1\}$ and $j \in \{0,1,\ldots,q-1\}$,

$$x_0\alpha = z_i, x_1\alpha = z_{i+1}, x_2\alpha = z_{i+2}, \ldots, \quad \text{and} \quad y_0\beta = w_j, y_1\beta = w_{j+1}, y_2\beta = w_{j+2}, \ldots,$$

where the subscripts on zs are calculated modulo p and the subscripts on ws are calculated modulo q. Let $r \in \{0,1,\ldots,m-1\}$ be an index such that $y_r\beta = w_i$. Since m divides k, we may define

$$x_0\delta = y_r, \qquad x_1\delta = y_{r+1}, \qquad x_2\delta = y_{r+2}, \ldots,$$

where the subscripts on ys are calculated modulo m. By the constructions of γ and δ and Theorem 2.1, we have $\delta,\gamma \in C(\sigma)$ and $\alpha = \delta\beta\gamma$. This concludes the proof. □

Note that taking $s = t$ in (iii), we get that d^s_1, d^s_2,…, d^s_u are pairwise distinct. This, coupled with (i), shows that if $D_\alpha \le D_\beta$, then rank $t_\alpha \le$ rank t_β and rank $\alpha \le$ rank β.

To illustrate Theorem 3.7, consider $\sigma = abcde = (1\ 2)(3\ 4)(5\ 6\ 7)(8)(9) \in S_9$, and $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 8 & - & - & 8 & 8 & 8 & - & 9 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & - & - & 7 & 5 & 6 & - & 8 \end{pmatrix}$ in $C(\sigma)$. Since $t_\alpha =$

$\left(\begin{smallmatrix} a & b & c & d & e \\ d & - & d & - & e \end{smallmatrix}\right)$, we have ran $t_\alpha = \{d, e\}$ and two sequences of type (1): $s : a, e$ and $t : c, e$. Since $t_\beta = \left(\begin{smallmatrix} a & b & c & d & e \\ b & - & c & - & d \end{smallmatrix}\right)$ with ran $t_\beta = \{b, c, d\}$, we can construct the corresponding sequences of type (2): $d^s : b, d$ and $d^t : c, d$ that satisfy (i)–(iii). Therefore, $D_\alpha \leq D_\beta$. Note that it is impossible to construct a sequence d_1, d_2 of elements of ran t_β that would work for both s and t.

4. Regularity. An element a of a semigroup S is called *regular* if $a = axa$ for some x in S. If all elements of S are regular, we say that S is a *regular semigroup*. An element a' in S is called an *inverse* of a in S if $a = aa'a$ and $a' = a'aa'$. Since regular elements are precisely those that have inverses (if $a = axa$, then $a' = xax$ is an inverse of a), we may define a regular semigroup as a semigroup in which every element has an inverse.

If a $\mathcal{D}$-class D in S contains a regular element, then every element in D is regular, and we call D a *regular $\mathcal{D}$-class*. In a regular $\mathcal{D}$-class, every $\mathcal{L}$-class and every $\mathcal{R}$-class contains an idempotent (an element e with $e = ee$). If an $\mathcal{H}$-class H contains an idempotent, then H is a maximal subgroup of S.

If every element of a semigroup S has exactly one inverse, then S is called an *inverse semigroup*. An alternative definition is that S is an inverse semigroup if it is regular and its idempotents commute. If every element of S is in some subgroup of S, then S is called a *union of groups*. In other words, unions of groups are semigroups in which every $\mathcal{H}$-class is a group. (Unions of groups are also called completely regular semigroups [**2**, Proposition 4.1.1].) Both inverse semigroups and unions of groups are regular semigroups.

The following lemma describes regular elements in $C(\sigma)$.

LEMMA 4.1. *Let $\sigma \in S_n$. Then a transformation $\alpha \in C(\sigma)$ is regular if and only if for every $b \in$ ran t_α, there is $a \in t_\alpha^{-1}(b)$ such that $\ell(a) = \ell(b)$.*

Proof. Suppose $\alpha \in C(\sigma)$ is regular, i.e., $\alpha = \alpha\beta\alpha$ for some $\beta \in C(\sigma)$. Let $b \in$ ran t_α and select $c \in t_\alpha^{-1}(b)$. Since $t_\alpha = t_\alpha t_\beta t_\alpha$ (by (1) of Lemma 2.2) and $ct_\alpha = b$, there is a cycle a in σ such that $ct_\alpha = b$, $bt_\beta = a$, and $at_\alpha = b$. Then $a \in t_\alpha^{-1}(b)$ and, by (2) of Lemma 2.2, $\ell(c) \geq \ell(b) \geq \ell(a) \geq \ell(b)$, implying $\ell(a) = \ell(b)$.

Conversely, suppose that the given condition is satisfied. We shall define $\beta \in C(\sigma)$ such that $\alpha = \alpha\beta\alpha$. First, set dom β = ran α. To define the values of β, let $b = (y_0 y_1 \ldots y_{m-1}) \in$ ran t_α. Then, by the assumption, we can find a cycle $a = (x_0 x_1 \ldots x_{k-1})$ in dom t_α such that $at_\alpha = b$ and $k = m$. By Theorem 2.1, for some index j,

$$x_0\alpha = y_j, \qquad x_1\alpha = y_{j+1}, \qquad x_2\alpha = y_{j+2}, \ldots,$$

where the subscripts on ys are calculated modulo m. Since $k = m$, we may define

$$y_j\beta = x_0, \qquad y_{j+1}\beta = x_1, \qquad y_{j+2}\beta = x_2, \ldots,$$

where the subscripts on ys and on xs are calculated modulo m $(= k)$. By the construction of β and Theorem 2.1, we have $\beta \in C(\sigma)$ and $\alpha = \alpha\beta\alpha$. This concludes the proof. □

Using Lemma 4.1, we characterize the permutations $\sigma \in S_n$ for which $C(\sigma)$ is a regular semigroup.

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THEOREM 4.2. *Let $\sigma \in S_n$. Then $C(\sigma)$ is a regular semigroup if and only if*

$$\text{for all cycles } a, b \in C(\sigma) : \ell(b) \text{ divides } \ell(a) \Rightarrow \ell(b) = \ell(a). \tag{3}$$

Proof. Suppose $C(\sigma)$ is a regular semigroup. Let $a = (x_0x_1 \ldots x_{k-1})$ and $b = (y_0y_1 \ldots y_{m-1})$ be cycles in σ such that m divides k. Consider $\alpha \in PT_n$ with dom $\alpha = \{x_0, x_1, \ldots, x_{k-1}\}$ and with values defined by

$$x_0\alpha = y_0, \qquad x_1\alpha = y_1, \qquad x_2\alpha = y_2, \ldots,$$

where the subscripts on ys are calculated modulo m. By Theorem 2.1, $\alpha \in C(\sigma)$. Since dom $t_\alpha = \{a\}$ and ran $t_\alpha = \{b\}$, we have $m = k$ by the fact that α is regular and Lemma 4.1.

Conversely, suppose (3) holds. Let $\alpha \in C(\sigma)$ and let $b \in$ ran t_α. Select an $a \in t_\alpha^{-1}(b)$. By (2) of Lemma 2.2 and (3), we have $\ell(b) = \ell(a)$. It follows by Lemma 4.1 that α is regular. □

For example, for $\sigma = (1\ 2)(3\ 4\ 5)(6\ 7\ 8)$ and $\rho = (1\ 2)(3\ 4)(5\ 6\ 7\ 8)$ in S_8, the centralizer $C(\sigma)$ is a regular semigroup whereas $C(\rho)$ is nonregular. Note that for any permutation $\sigma \in S_n$ (other than the identity) with at least one 1-cycle, $C(\sigma)$ is nonregular.

In an inverse semigroup, only one $\mathcal{H}$-class in each $\mathcal{L}$-class ($\mathcal{R}$-class) is a group. In contrast, in a union of groups, every $\mathcal{H}$-class is a group. We note that in the class of centralizers of permutations, inverse semigroups and unions of groups coincide.

THEOREM 4.3. *For any $\sigma \in S_n$, the following conditions are equivalent*:
(a) *$C(\sigma)$ is an inverse semigroup*;
(b) *$C(\sigma)$ is a union of groups*;
(c) *for all cycles a,b in σ, if $\ell(b)$ divides $\ell(a)$ then $b = a$.*

Proof. To show (a) $\Rightarrow$ (c), suppose $C(\sigma)$ is an inverse semigroup and let $a = (x_0x_1 \ldots x_{k-1})$ and $b = (y_0y_1 \ldots y_{m-1})$ be cycles in σ such that m divides k. By Theorem 4.2, $m = k$. Suppose $a \neq b$. Define $\varepsilon, \xi \in PT_n$ by: dom $\varepsilon = \{x_0, \ldots, x_{k-1}, y_0, \ldots, y_{k-1}\}$, dom $\xi = \{y_0, \ldots, y_{k-1}\}$, $x_i\varepsilon = y_i$, $y_i\varepsilon = y_i$, and $y_i\xi = y_i$ $(0 \leq i \leq k-1)$. By the construction and Theorem 2.1, ε and ξ are idempotents in $C(\sigma)$ with $\varepsilon\xi = \varepsilon \neq \xi = \xi\varepsilon$, which is a contradiction (since idempotents commute in an inverse semigroup). Hence $b = a$.

To show (b) $\Rightarrow$ (c), suppose $C(\sigma)$ is a union of groups and let a and b be cycles in σ as above. Again, $k = m$ and suppose $a \neq b$. Define $\alpha \in PT_n$ by: dom $\alpha = \{x_0, \ldots, x_{k-1}\}$ and $x_i\alpha = y_i$ $(0 \leq i \leq k-1)$. By the construction and Theorem 2.1, $\alpha \in C(\sigma)$ and $\alpha^2 = 0$, where 0 is the zero (empty) transformation. Since H_α is a group, we have $\alpha^2 \in H_\alpha$ and so $\alpha\mathcal{H}0$. This is a contradiction (by Corollary 3.5). Hence $b = a$.

Suppose (c) holds. Then, by Theorem 2.1, for every $\alpha \in C(\sigma)$, α is a permutation on its domain and t_α fixes each element of its domain. It follows that for some integer $p \geq 1$, $\alpha^p = \varepsilon$ is an idempotent such that dom $\varepsilon =$ dom α, $x\varepsilon = x$ for each $x \in$ dom ε, and $t_\varepsilon = t_\alpha$. By Corollary 3.5, $\alpha\mathcal{H}\varepsilon$, which shows that $C(\sigma)$ is a union of groups. Further, the fact that elements of $C(\sigma)$ are permutations on their domains implies that idempotents in $C(\sigma)$ are one-one. Since one-one idempotents in PT_n commute, we have that $C(\sigma)$ is also an inverse semigroup. □

5. Example. In this section, we shall use the results of Section 3 and Section 4 to present the structure of the centralizer $C(\sigma)$ for

$$\sigma = abc = (1\ 2)(3\ 4)(5\ 6\ 7\ 8). \tag{4}$$

We shall visualize each $\mathcal{D}$-class as an egg-box diagram, with each $\mathcal{R}$-class R_α (row) labelled by ker α (see Theorem 3.4) and each $\mathcal{L}$-class L_α (column) labelled by ran t_α (see Theorem 3.3). In each $\mathcal{H}$-class H (cell), we shall place a representative α of H together with t_α, with α being an idempotent if H is a group. Idempotents will be indicated by asterisks.

To simplify notation, we shall write both $\alpha \in C(\sigma)$ and t_α as sequences of images. For example, for $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & - & - & 3 & 4 & 3 & 4 \end{pmatrix}$ and $t_\alpha = \begin{pmatrix} a & b & c \\ a & - & b \end{pmatrix}$, we shall write $\alpha = 1\ 2 - - 3\ 4\ 3\ 4$ and $t_\alpha = a - b$.

If $\alpha \in C(\sigma)$ with rank $\alpha = k$ and rank $t_\alpha = m$, we say that the $\mathcal{D}$-class D_α is of rank (k, m). This definition is justified by Theorem 3.6, which implies that if $\alpha\mathcal{D}\beta$, then rank α = rank β and rank t_α = rank t_β.

By Theorem 2.1, the possible ranks of $\mathcal{D}$-classes in $C(\sigma)$ for the permutation (4) are: (8,3), (6,2), (4,2), (4,1), (2, 1), and (0,0).

Rank (8, 3). There is one $\mathcal{D}$-class of this rank, say D_1, namely the group of units of $C(\sigma)$ (see Fig. 1). Every member of D_1 maps either a onto a, b onto b, and c onto c or a onto b, b onto a, and c onto c. We have $2{\cdot}2{\cdot}4 = 16$ possibilities for the former case and the same number for the latter, giving the total of 32 elements in D_1.

	abc
|1|2|3|4|5|6|7|8|	**1 2 3 4 5 6 7 8*** ***a b c***

Figure 1. D_1 (group of units, 32 elements).

Rank (6, 2). There is one $\mathcal{D}$-class of this rank, say D_2 (see Fig. 2). Look at the $\mathcal{H}$-class in the lower right-hand corner. Each member of this $\mathcal{H}$-class maps b onto b and c onto c. This can be done in $2{\cdot}4 = 8$ ways. Since all $\mathcal{H}$-classes in the same $\mathcal{D}$-class have the same cardinality, D_2 has $8{\cdot}8 = 64$ elements.

	ac	***bc***
|13|24|5|6|7|8|	**1 2 1 2 5 6 7 8*** ***a a c***	**3 4 3 4 5 6 7 8*** ***b b c***
|14|23|5|6|7|8|	**1 2 2 1 5 6 7 8*** ***a a c***	**4 3 3 4 5 6 7 8*** ***b b c***
|1|2|5|6|7|8|	**1 2 − − 5 6 7 8*** ***a − c***	**3 4 − − 5 6 7 8** ***b − c***
|3|4|5|6|7|8|	**− − 1 2 5 6 7 8** ***− a c***	**− − 3 4 5 6 7 8*** ***− b c***

Figure 2. D_2 (regular, 64 elements).

https://doi.org/10.1017/S0017089599970301 Published online by Cambridge University Press

Rank (4, 2). There are two $\mathcal{D}$-classes of this rank, say D_3 and D_4, one regular and one nonregular (see Figs 3 and 4). Each $\mathcal{H}$-class in D_3 has 8 elements and each $\mathcal{H}$-class in D_4 has 4 elements.

	ab
|157|268|3|4|	1 2 3 4 1 2 1 2* *a b a*
|168|257|3|4|	1 2 3 4 2 1 2 1* *a b a*
|1|2|357|468|	1 2 3 4 3 4 3 4* *a b b*
|1|2|368|457|	1 2 3 4 4 3 4 3* *a b b*
|1|2|3|4|	1 2 3 4 — — — —* *a b* —

Figure 3. D_3 (regular, 40 elements).

	ab	*ab*
|13|24|57|68|	1 2 1 2 3 4 3 4 *a a b*	3 4 3 4 1 2 1 2 *b b a*
|14|23|57|68|	1 2 2 1 4 3 4 3 *a a b*	4 3 3 4 1 2 1 2 *b b a*
|1|2|57|68|	1 2 — — 3 4 3 4 *a* — *b*	3 4 — — 1 2 1 2 *b* — *a*
|3|4|57|68|	— — 3 4 1 2 1 2 — *b a*	— — 1 2 3 4 3 4 — *a b*

Figure 4. D_4 (nonregular, 32 elements).

Rank (4, 1). There is one $\mathcal{D}$-class of this rank, say D_5, with a single $\mathcal{H}$-class (see Fig. 5).

	c
|5|6|7|8|	— — — — 5 6 7 8* — — *c*

Figure 5. D_5 (regular, 4 elements).

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	a	b
|1357|2468|	1 2 1 2 1 2 1 2* a a a	3 4 3 4 3 4 3 4* b b b
|1457|2368|	1 2 2 1 1 2 1 2* a a a	4 3 3 4 4 3 4 3* b b b
|1368|2457|	1 2 1 2 2 1 2 1* a a a	3 4 3 4 4 3 4 3* b b b
|1468|2357|	1 2 2 1 2 1 2 1* a a a	3 4 4 3 3 4 3 4* b b b
|157|268|	1 2 – – 1 2 1 2* a – a	3 4 – – 3 4 3 4 b – b
|168|257|	1 2 – – 2 1 2 1* a – a	3 4 – – 4 3 4 3 b – b
|357|468|	– – 1 2 1 2 1 2 – a a	– – 3 4 3 4 3 4* – b b
|368|457|	– – 1 2 2 1 2 1 – a a	– – 3 4 4 3 4 3* – b b
|13|24|	1 2 1 2 – – – –* a a –	3 4 3 4 – – – –* b b –
|14|23|	1 2 2 1 – – – –* a a –	4 3 3 4 – – – –* b b –
|1|2|	1 2 – – – – – –* a – –	3 4 – – – – – – b – –
|3|4|	– – 1 2 – – – – – a –	– – 3 4 – – – –* – b –

Figure 6. D_6 (regular, 48 elements).

	a	b
|57|68|	– – – – 1 2 1 2 – – a	– – – – 3 4 3 4 – – b

Figure 7. D_7 (nonregular, 4 elements).

Rank (2, 1). There are two $\mathcal{D}$-classes of this rank, say D_6 and D_7, one regular and one nonregular (see Figs 6 and 7). Each $\mathcal{H}$-class in each of these two $\mathcal{D}$-classes has 2 elements.

Rank (0, 0). There is one $\mathcal{D}$-class of this rank, containing the zero transformation as the only element.

Thus the semigroup $C(\sigma)$ has 225 elements (189 regular and 36 nonregular) and 8 $\mathcal{D}$-classes (6 regular and 2 nonregular). Using Theorem 3.7, we can determine the partial ordering of $\mathcal{D}$-classes (see Fig. 8). Regular $\mathcal{D}$-classes are marked with asterisks.

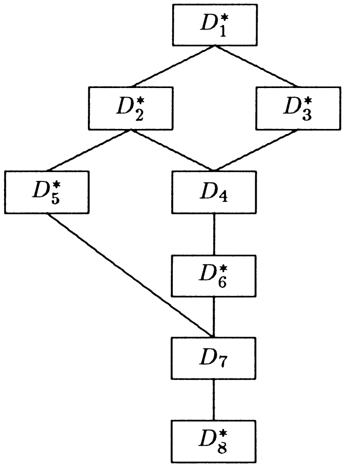

Figure 8. Global structure of $C(\sigma)$.

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