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SOME INTEGRALS INVOLVING LEGENDRE POLYNOMIALS PROVIDING COMBINATORIAL IDENTITIES

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Abstract

An integral involving a combination of Legendre polynomials, exponential and algebraic terms is solved using the generating function. Special cases of this result are compared with known expansions, and the previously known results are shown to be extendible to a particularly pleasing result as a limiting case. Comparisons provide some new combinatorial identities involving binomials. Finally, an effective numerical procedure is described which evaluates the integral to machine accuracy.

1. Introduction

In the following, we obtain analytic solutions to the definite integral

$$V_{2m}(\lambda) = \int_0^1 e^{-\lambda x}[P_{2m}(x) - P_{2m}(0)]/x\,dx, \tag{1}$$

where λ is a non-negative parameter, and P_n is the Legendre polynomial of order n. Because of the nature of the Legendre polynomials of even order, this integral is well defined for all non-negative integer m, and in particular $V_0(\lambda) \equiv 0$.

By solving in two entirely different ways, we obtain a number of identities involving gamma functions and binomial coefficients.

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2. The generating-function approach

THEOREM 1. *For non-negative real* λ,

$$V_{2m}(\lambda) = \sum_{k=1}^{m} \frac{(4k-1)!!}{(2k)!\lambda^{2k}} \gamma(2k, \lambda) \binom{-2k-1/2}{m-k}, \tag{2}$$

where $\gamma(a, x)$ *is one of the incomplete gamma functions described in Abramowitz and Stegun* [1]. *In particular, it is defined by*

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1}\, dt,$$

for $\Re a > 0$, *and the extension* $\gamma^*(a, x) = x^{-a}\gamma(a, x)/\Gamma(a)$ *is a single-valued analytic function of* a *and* x.

PROOF. Consider the straight-forward extension of the right-hand side of (1):

$$\sum_{n=0}^{\infty} V_n(\lambda) t^n = \int_0^1 e^{-\lambda x}[(1-2xt+t^2)^{-1/2} - (1+t^2)^{-1/2}]/x\, dx,$$

obtained by using the generating function for the Legendre polynomials. This can be expressed as the double integral

$$\int_0^1 \int_0^x t e^{-\lambda x}(1-2yt+t^2)^{-3/2}/x\, dy\, dx.$$

Swapping order gives

$$\int_0^1 \int_y^1 t e^{-\lambda x}(1-2yt+t^2)^{-3/2}/x\, dx\, dy.$$

From the definition of the exponential integral, [1, page 288],

$$E_1(z) = \int_z^{\infty} e^{-t}/t\, dt,$$

we have

$$\sum_{n=0}^{\infty} V_n(\lambda) t^n = t \int_0^1 \frac{E_1(\lambda y) - E_1(\lambda)}{(1-2yt+t^2)^{3/2}}\, dy, \tag{3}$$

which we can treat as a generating function for the integrals $V_n(\lambda)$.

Expanding on an integral from Gradsteyn and Ryzhik, [2, page 641], we can write

$$\int_0^1 [E_1(\lambda y) - E_1(\lambda)] y^{n-1}\, dy = \frac{1}{n\lambda^n}\gamma(n, \lambda), \quad \text{for } n \geq 1.$$

Thus, expanding the denominator in (3) in a binomial expansion in powers of y, and using this last formula, we can write

$$\sum_{n=0}^{\infty} V_n(\lambda)t^n = \sum_{k=1}^{\infty} \frac{(2k-1)!!}{k!\lambda^k}\gamma(k\lambda)[t^k(1+t^2)^{-k-1/2}].$$

Making the final binomial expansion, exchanging the order of the two sums, and equating coefficients of even powers of t we obtain the result required.

Since the generating function produces all integer orders of the Legendre polynomials, we can also equate coefficients of odd powers of t.

COROLLARY 1.

$$\int_0^1 \frac{e^{-\lambda x}}{x} P_{2n-1}(x)\,dx = \sum_{k=1}^{n} \frac{(4k-3)!!}{(2k-1)!\lambda^{2k-1}}\gamma(2k-1,\lambda)\binom{-2k+1/2}{n-k}.$$

3. Expansion approach

In the right-hand side of (1) we can expand the exponential, to give

$$V_{2m}(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n\lambda^n}{n!} J(2m, n), \tag{4}$$

where

$$J(2m, n) = \int_0^1 x^{n-1}[P_{2m}(x) - P_{2m}(0)]\,dx, \tag{5}$$

this integral being well defined for at least $n \geq 0$.

Now Gradsteyn and Ryzhik, [2, page 822], state the following identity:

$$\int_0^1 x^a P_{2m}(x)\,dx = \frac{(-1)^m\Gamma(m-a/2)\Gamma(1/2+a/2)}{2\Gamma(-a/2)\Gamma(m+3/2+a/2)},$$

as applying for all real $a > -1$. This is true, but one should note the care required in the right-hand side when a is an even integer no larger than $2m$. In fact, since the integrand on the left-hand side is then even, that integral can be written as one half the integral from -1 to 1, and this integral is identically zero.

It follows, then, that for all real $a > 0$,

$$J(2m, a) = \frac{(-1)^m(2m-1-a)(2m-3-a)\cdots(1-a)}{(2m+a)(2m-2+a)\cdots(2+a)(a)} + \frac{(-1)^{m+1}}{a2^{2m}}\binom{2m}{m}, \tag{6}$$

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where the second term comes from the explicit expression for $P_{2m}(0)$, and the first term vanishes for a odd and $\leq 2m-1$. The only outstanding required result is the value of $J(2m, 0)$.

THEOREM 2.

$$J(2m, 0) = (-1)^{m+1}\frac{1}{2^{2m}}\binom{2m}{m}\sum_{k=1}^{2m}\frac{1}{k}. \tag{7}$$

PROOF. Let us write

$$J(2m, a) = \frac{f_m(a)}{g_m(a)a} + h_m(a), \tag{8}$$

defining each of the functions f, g and h by direct comparison with (6).

Expand the leading expression on the right of (8) in partial fractions

$$\frac{f_m(a)}{g_m(a)a} = \frac{p_m(a)}{g_m(a)} + \frac{A}{a}, \tag{9}$$

where A is a constant, and is given explicitly by

$$A = f_m(0)/g_m(0) = \frac{(-1)^m}{2^{2m}}\binom{2m}{m}.$$

Thus $A/a = -h_m(a)$, and these terms cancel in (8).

All we now require to complete the proof is an expression for $p_m(0)$, since $g_m(a)$ is well defined in the limit $a \to 0$. From (9) we have,

$$ap_m(a) = f_m(a) - Ag_m(a).$$

Thus $p_m(a)$ is a polynomial in a. Differentiating, we have

$$p_m(a) = df_m/da - A\,dg_m/da - a\,dp_m/da.$$

So

$$p_m(0) = df_m(0)/da - A dg_m(0)/da. \tag{10}$$

Now

$$\frac{df_m(0)}{da} = (-1)^{m+1}(2m-1)!!\sum_{k=1}^{m} 1/(2k-1),$$

and

$$dg_m(0)/da = 2^{m-1}m!\sum_{k=1}^{m} 1/k.$$

Combining these in (10), with the previously obtained value of A, and then using $g_m(0) = 2^m m!$, we obtain the required result.

It should be pointed out that (6) and (7) are very well behaved and enable the integral (5) to be evaluated to machine accuracy for all values of m and

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n. Asymptotically in n, $J(2m, n)$ goes as $1/n$, and so (4) is well suited for evaluation of integral (1) for $\lambda \le 1$, and, with care, for values a little larger.

In this context, (2) is suitable for near machine accuracy evaluation of the integral for any value of λ, since

$$\frac{1}{(2k)!\lambda^{2k}}\gamma(2k, \lambda) = \frac{e^{-\lambda}}{2k}\sum_{n=2k}^{\infty}\lambda^{n-2k}/n!,$$

and this can always be evaluated to machine accuracy for any k and λ. Accuracy loss occurs for increasing m, since (2) is a sum of terms of similar orders of magnitude but of alternating sign.

4. Direct comparisons between solutions

The right-hand sides of (2) and (4) are of course equal. A more explicit comparison can be made by expanding the incomplete gamma function in powers of λ as per

$$\gamma(2k, \lambda)/\lambda^{2k} = \sum_{n=0}^{\infty}\frac{(-1)^n\lambda^n}{n!(n+2k)}.$$

Substituting in (2) and reordering the sums, we find

$$\begin{aligned} J(2m, n) &= \sum_{k=1}^{m}\frac{(4k-1)!!}{(2k)!(n+2k)}\binom{-2k-1/2}{m-k} \\ &= \sum_{k=1}^{m}\frac{1}{2^{2k}(n+2k)}\binom{4k}{2k}\binom{-2k-1/2}{m-k}, \end{aligned}$$

and a number of other rearrangements of terms. In the particular case $n = 0$, we have the very pleasing relationship,

$$\binom{2m}{m}\sum_{n=1}^{2m}\frac{1}{n} = \sum_{k=1}^{m}\frac{(-1)^{k+1}}{2k}\binom{2m+2k}{m+k}\binom{m+k}{2k}.$$

For completeness, it should be pointed out that the generating function approach can be applied directly to the integral $J(2m, 0)$ to provide yet another finite sum expression for this integral. The generating function for this case is

$$\frac{1}{\sqrt{1+t^2}}\ln\left[\frac{2(1+t^2)(t-1+\sqrt{1+t^2})}{t(1-t+\sqrt{1+t^2})}\right] = \sum_{n=0}^{\infty}J(n, 0)t^n.$$

The left-hand side can be expanded in powers of t to give the required expressions—again having distinct character for n even and odd. The formulae are rather long and so are not presented here.

5. Alternate computational procedure

Yet another expansion for the right-hand side of (1) was suggested by a referee. By differentiating both sides with respect to λ we obtain

$$V_{2m}'(\lambda) = -\int_0^1 e^{-\lambda x}[P_{2m}(x) - P_{2m}(0)]\,dx.$$

By direct substitution from an integral in Gradshtein and Ryzhik [2, page 92], we obtain

$$V_{2m}'(\lambda) = \frac{e^{-\lambda}}{\lambda}[P_{2m}(1) - P_{2m}(0)] + \frac{1}{\lambda}\sum_{k=0}^{2m}\frac{(-1)^k}{\lambda^k}[e^{-\lambda}P_{2m}^{(k)}(1) - P_{2m}^{(k)}(0)],$$

where $P_n^{(k)}$ is the kth derivative of that Legendre polynomial.

Integrating both sides over the interval λ to ∞, since $V_n(\infty) = 0$, and using the extended definition of the exponential integrals [1, page 228],

$$E_n(z) = z^{n-1}\int_z^\infty \frac{e^{-t}}{t^n}\,dt,$$

we obtain the finite sum

$$V_{2m}(\lambda) = [P_{2m}(0) - P_{2m}(1)]E_1(\lambda) - \sum_{k=1}^{2m}\frac{(-1)^k}{\lambda^k}\left[E_{k+1}(\lambda)P_{2m}^{(k)}(1) - \frac{1}{k}P_{2m}^{(k)}(0)\right].$$

The terms in this expression can be readily evaluated, the exponential integrals using the relevant IMSL routine, and the derivatives from the formula relating the derivatives of the Legendre polynomial with the Jacobi polynomials (see e.g., [1, page 779]). Unfortunately the expression is similar to (2), in that it involves sums of terms of similar orders of magnitude but of alternating sign, and so loses accuracy with increasing m.

An alternate approach, that turns out to be suitable for numerical evaluation of $V_{2m}(\lambda)$ over the entire range of values of λ and m, is based upon the well-known recurrence relation for Legendre polynomials. With this it is possible to express the integral (1) in terms of a pair of non-homogeneous recurrence relations. If we define $K_m(\lambda) = (4m+1)V_{2m}(\lambda)$ and $L_m(\lambda) = \lambda\int_0^1 e^{-\lambda x}P_m(x)\,dx - P_m(0)$, then

$$K_{m+1} = \frac{(4m+5)(4m+3)}{(2m+2)\lambda}L_{2m+1} - \frac{(4m+5)(2m+1)}{(4m+1)(2m+2)}K_m$$

and

$$L_{m+1} = L_{m-1} + \frac{(2m+1)}{\lambda} L_m + \frac{(2m+1)}{\lambda} P_m(0).$$

An excellent technique which evaluates the functions of the second recurrence relation to machine accuracy is Olver's method [3]. This rewrites the recurrence relation as a triple of recurrence relations, two of which are evaluated forwards to an index greater than the desired m, the number of additional steps required for a given accuracy being determined as part of the procedure. The third relation is then evaluated by backward recurrence. The functions of the recurrence relation for K_m can then be simply evaluated by forward recurrence.

The integrals are evaluated by this method to machine accuracy. The choice of the power series method for small λ is because of its faster speed in this regime.

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References

[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions*, (U.S. Nat. Bur. of Standards, Washington, 1964).

[2] I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series, and Products*, (Academic Press, 1980).

[3] F. W. J. Olver, *J. of Res.* NBS-B **17**B (1967)111.