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EXISTENCE AND BIFURCATION RESULTS FOR FOURTH-ORDER ELLIPTIC EQUATIONS INVOLVING TWO CRITICAL SOBOLEV EXPONENTS

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Abstract. Let Ω be a smooth bounded domain in R^N, with $N \geq 5$. We provide existence and bifurcation results for the elliptic fourth-order equation $\Delta^2 u - \Delta_p u = f(\lambda, x, u)$ in Ω, under the Dirichlet boundary conditions $u = 0$ and $\nabla u = 0$. Here λ is a positive real number, $1 < p \leq 2^{\#}$ and $f(., ., u)$ has a subcritical or a critical growth s, $1 < s \leq 2^*$, where $2^* := \frac{2N}{N-4}$ and $2^{\#} := \frac{2N}{N-2}$. Our approach is variational, and it is based on the mountain-pass theorem, the Ekeland variational principle and the concentration-compactness principle.

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1. Introduction. An approach for confronting second-order critical semilinear elliptic equations in a bounded domain Ω in R^N was introduced in **[2]**, where it was shown that the Palais-Smale compactness condition holds for certain levels of the associated functional. Therefore, under the appropriate assumptions, the mountain-pass theorem could be applied to yield a solution to the critical problem.

The existence of solutions of fourth-order critical elliptic problems can also be proved by using this approach, see **[4, 5, 8, 11, 15]** and the references therein.

In this paper, we study problems of the form

$$\left.\begin{array}{l}\Delta^2 u - \Delta_p u = f(\lambda, x, u) \text{ in } \Omega,\\ u = 0,\ \nabla u = 0 \text{ on } \partial\Omega,\end{array}\right\} \qquad (1)$$

where Ω is a smooth bounded domain in R^N, with $N \geq 5$, $\Delta^2 u$ is the biharmonic operator, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplace operator, $f : R \times \Omega \times R \to R$ is a function with either subcritical or critical growth in the third variable and λ is a positive real number.

Problem (1) has not been addressed in such a general context before. A similar problem was examined by **[6]**, **[12]** and **[16]**, who studied not the difference, but the sum of the biharmonic and the $p-$Laplace operator for the case $p = 2$ and with Navier boundary conditions.

Owing to the presence of the biharmonic and $p-$Laplace operators in the equation, two critical exponents could appear: the critical exponent $2^* := \frac{2N}{N-4}$ for the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^q(\Omega)$ and the critical exponent $2^\# := \frac{2N}{N-2}$ for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$. Our purpose is to provide solutions for the subcritical and critical cases, which arise as s, the growth of f in the third variable, varies between 1 and 2^* and p varies between 1 and $2^\#$. These solutions will be found as the critical points of the Frechet differentiable energy functional given by

$$\Phi_\lambda(u) := \frac{1}{2}\int_\Omega (\Delta u)^2 dx + \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega \int_0^u f(\lambda, x, s)\, ds\, dx,$$

which is defined on the Sobolev space $E := H_0^2(\Omega)$ endowed with the equivalent norm

$$||u||_E^2 = \int_\Omega (\Delta u)^2.$$

We now present our results. In Section 2, we examine the subcritical case where $f(\lambda, x, u) = \lambda |u|^{s-2}u$, $1 < p < 2^\#$ and $1 < s < 2^*$, and prove the following:

THEOREM 1. *Let* $1 < p < 2^\#$.

(i) *Suppose that* $2 < s < 2^*$. *Then if* $p < s$, *(1) admits a solution for every* $\lambda > 0$, *while if* $s \leq p$, *there exists* $\lambda_0 > 0$ *such that (1) admits a solution for every* $\lambda > \lambda_0$.
(ii) *Suppose that* $1 < s < 2$. *Then if* $s < p$, *(1) admits a solution for every* $\lambda > 0$, *while if* $p < s$, *there exists* $\lambda_0 > 0$ *such that (1) admits a solution for every* $\lambda > \lambda_0$.
(iii) *If* $\lambda > \lambda_1$, $s = 2$ *and* $2 < p < 2^\#$, *then (1) admits a solution.*

Here λ_1 denotes the first eigenvalue of Δ^2 with Dirichlet boundary conditions.

In Section 3, we examine the subcritical case for s and the critical case $p = 2^\#$. We show the following:

THEOREM 2. *(i) If* $p = 2^\#$ *and* $2 < s < 2^\#$, *then there exists* $\widehat{\lambda} > 0$ *such that (1) admits a nontrivial solution for every* $\lambda > \widehat{\lambda}$.

(ii) If $p = 2^\#$ *and* $2^\# < s < 2^*$, *then (1) admits a solution for every* $\lambda > 0$.

In Section 4, in an effort to extend our results to the critical case $s = 2^*$, we assume that $f(\lambda, x, u) = \lambda |u|^{2^*-2}u + g(x)$, where $g : \Omega \to R$ is a nontrivial continuous function, and in this situation, we obtain:

THEOREM 3. *If* $||g||_{\frac{2N}{N+4}}$ *is small enough, then (18) admits a solution.*

Here, p is restricted in the interval $(1, 2^\#)$, and it is an open question whether there is a solution if $p = 2^\#$.

Finally, in Section 5, we study the bifurcation properties for the problem

$$\left.\begin{array}{l} \Delta^2 u - \Delta_p u = \lambda u + h(x, \lambda)|u|^{2^*-2}u \text{ in } \Omega, \\ u = 0,\ \ \nabla u = 0 \text{ on } \partial\Omega, \end{array}\right\} \tag{2}$$

where $1 < p < 2^\#$, and we have the following:

THEOREM 4. *Equation* (2) *admits a continuum* C *of nontrivial solutions* $(\lambda, u) \subseteq R \times E$ *bifurcating from* $(\lambda_1, 0)$, *which meets the boundary of* $[\lambda_1 - d, \lambda_1 + d] \times B(0, \rho_0)$.

2. The subcritical case. In this section, we assume that $f(\lambda, x, u) = \lambda|u|^{s-2}u$, $1 < p < 2^{\#}$ and $1 < s < 2^*$.

LEMMA 5. *Suppose that one of the following statements holds:*

(i) $1 < p < 2^{\#}$, $s \in (1, 2^*)\backslash\{2\}$ *and* $\lambda > 0$.

(ii) $s = 2$, $2 < p < 2^{\#}$ *and* $\lambda > 0$.

(iii) $s = 2$, $1 < p \leq 2$ *and* $\lambda < \lambda_1$.

Then $\Phi_\lambda(.)$ *satisfies the Palais-Smale condition.*

Proof. Assume first that $2 \leq p < 2^{\#}$. Let $\{u_n\}_{n\in N}$ be a Palais-Smale sequence, that is,

(i) $\Phi_\lambda(u_n)$ is bounded and

(ii) $\Phi'_\lambda(u_n) \to 0$.

From (i), there exists $M > 0$ such that

$$-M \leq \frac{1}{2}\int_\Omega (\Delta u_n)^2 + \frac{1}{p}\int_\Omega |\nabla u_n|^p - \frac{\lambda}{s}\int_\Omega |u_n|^s \leq M, \tag{3}$$

while (ii) implies that

$$\int_\Omega (\Delta u_n)^2 + \int_\Omega |\nabla u_n|^p - \lambda\int_\Omega |u_n|^s = o_n(1)\,\|u_n\|_E\,. \tag{4}$$

Multiplying (4) by $-1/a$, $a > 0$, and adding memberwise to (3), we obtain

$$-M - o_n(1)\,\|u_n\|_E$$

$$\leq \left(\frac{1}{2} - \frac{1}{a}\right)\int_\Omega (\Delta u_n)^2 + \left(\frac{1}{p} - \frac{1}{a}\right)\int_\Omega |\nabla u_n|^p + \lambda\left(\frac{1}{a} - \frac{1}{s}\right)\int_\Omega |u_n|^s$$

$$\leq M - o_n(1)\,\|u_n\|_E\,. \tag{5}$$

By taking $a = p$ in (5), the boundedness of $\|u_n\|_E$ is straightforward for the case $p \leq s$. For $s < p$, we take $a > p$ and exploit the embeddings $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ and $(L^p(\Omega))^N \hookrightarrow (L^2(\Omega))^N$ to get

$$\left(\frac{1}{2} - \frac{1}{a}\right)\int_\Omega (\Delta u_n)^2 + \left(\frac{1}{p} - \frac{1}{a}\right)\int_\Omega |\nabla u_n|^p + \lambda c\left(\frac{1}{a} - \frac{1}{s}\right)\left(\int_\Omega |\nabla u_n|^p\right)^{\frac{s}{p}}$$

$$\leq M - o_n(1)\,\|u_n\|_E\,,$$

from where we obtain once more the desired boundedness. Obvious modifications of the same idea yields boundedness for the rest of the cases.

Thus, we may assume that, up to a subsequence, $u_n \to u$ weakly in E. From the Sobolev embedding, we obtain that

$$\left.\begin{array}{l}\Delta u_n \to \Delta u \text{ weakly in } L^2(\Omega),\\ u_n \to u \text{ in } L^s(\Omega) \text{ and}\\ \nabla u_n \to \nabla u \text{ in } (L^p(\Omega))^N.\end{array}\right\} \tag{6}$$

By (4), $\Phi'_\lambda(u_n)(u_n) \to 0$, that is,

$$\int_\Omega (\Delta u_n)^2 + \int_\Omega |\nabla u_n|^p - \lambda \int_\Omega |u_n|^s \to 0,$$

and so

$$\int_\Omega (\Delta u_n)^2 \to \lambda \int_\Omega |u|^s - \int_\Omega |\nabla u|^p . \tag{7}$$

On the other hand, since $\Phi'_\lambda(u_n)(u) \to 0$,

$$\int_\Omega (\Delta u_n)(\Delta u) + \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla u - \lambda \int_\Omega |u_n|^{s-2} u_n u \to 0. \tag{8}$$

Combining (6)–(8), we conclude that

$$\int_\Omega (\Delta u)^2 = \lambda \int_\Omega |u|^s - \int_\Omega |\nabla u|^p .$$

Consequently, $\|u_n\|_E \to \|u\|_E$. The uniform convexity of E implies that $u_n \to u$ in E. □

Proof of Theorem 1. (i) Assume first that $2 \le p < s$. By the Sobolev embedding, if $\|u\|_E$ is sufficiently small, then

$$\Phi_\lambda(u) \ge \frac{1}{2}\int_\Omega (\Delta u)^2 + \frac{1}{p}\int_\Omega |\nabla u|^p - d\left(\int_\Omega (\Delta u)^2\right)^{\frac{s}{2}} > \delta \tag{9}$$

for some $d, \delta > 0$. Note that for $u \neq 0$,

$$\Phi_\lambda(tu) = \frac{t^2}{2}\int_\Omega (\Delta u)^2 + \frac{t^p}{p}\int_\Omega |\nabla u|^p - \frac{\lambda t^s}{s}\int_\Omega |u|^s \to -\infty$$

as $t \to \infty$. Applying the mountain-pass theorem we get a solution to (1).

Suppose next that $2 < s \le p$. We define

$$\lambda_0 := \inf_{u \in E\setminus\{0\}} \frac{\frac{1}{2}\int_\Omega (\Delta u)^2 + \frac{1}{p}\int_\Omega |\nabla u|^p}{\frac{1}{s}\int_\Omega |u|^s}. \tag{10}$$

The continuity of the embedding $H_0^1(\Omega) \hookrightarrow L^t(\Omega)$, $t \in (1, 2^{\#}]$ implies that for every $u \in E\backslash\{0\}$,

$$\frac{\frac{1}{2}\int_\Omega (\Delta u)^2 + \frac{1}{p}\int_\Omega |\nabla u|^p}{\frac{1}{s}\int_\Omega |u|^s} \geq \frac{c_1\left(\int_\Omega |\nabla u|^p\right)^{\frac{2}{p}} + \frac{1}{p}\int_\Omega |\nabla u|^p}{c_2\left(\int_\Omega |\nabla u|^p\right)^{\frac{s}{p}}}$$
$$= \frac{c_1}{c_2}\left(\int_\Omega |\nabla u|^p\right)^{\frac{2-s}{p}} + \frac{1}{pc_2}\left(\int_\Omega |\nabla u|^p\right)^{\frac{p-s}{p}} > \eta \tag{11}$$

for some $\eta, c_1, c_2 > 0$. Thus, $\lambda_0 > 0$. Consequently, if $\lambda > \lambda_0$, there exists $u_\lambda \in E\backslash\{0\}$ such that

$$\frac{1}{2}\int_\Omega (\Delta u_\lambda)^2 + \frac{1}{p}\int_\Omega |\nabla u_\lambda|^p < \frac{\lambda}{s}\int_\Omega |u_\lambda|^s \tag{12}$$

and so $\Phi_\lambda(u_\lambda) < 0$. Since (9) guarantees that $\Phi_\lambda(.)$ is positive close to the origin, the mountain-pass theorem provides a solution to (1).

Now let $1 < p < 2$. In view of the embedding $E \hookrightarrow L^s(\Omega)$, we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\int_\Omega (\Delta u)^2 - d\left(\int_\Omega (\Delta u)^2\right)^{\frac{s}{2}} \tag{13}$$

for some $d > 0$, which implies that $\Phi_\lambda(.)$ is positive near the origin. Since $\lim_{t\to+\infty}\Phi_\lambda(tu) = -\infty$, the mountain-pass theorem provides a solution to (1).

(ii) Assume first that $s < p$. In view of the embedding $E \hookrightarrow L^s(\Omega)$, we have

$$\Phi_\lambda(u) \geq d\left(\int_\Omega |u|^s\right)^{\frac{2}{s}} - \frac{\lambda}{s}\int_\Omega |u|^s$$

for some $d > 0$ and so $\Phi_\lambda(.)$ is bounded below. Since $\Phi_\lambda(.)$ satisfies the Palais-Smale condition, Ekeland's variational principle [**9**] provides a solution to (1), which is nontrivial because $\Phi_\lambda(.)$ assumes negative values near the origin.

Let now $1 < p \leq s < 2$. Then $\Phi_\lambda(.)$ satisfies the Palais-Smale condition and is bounded below. If $\lambda > \lambda_0$, in view of (11) and (12), $\Phi_\lambda(.)$ assumes negative values and so Ekeland's variational principle provides a nontrivial solution to (1).

(iii) By exploiting the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, we get

$$\begin{aligned}\Phi_\lambda(u) &= \frac{1}{2}\int_\Omega (\Delta u)^2 + \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{2}\int_\Omega |u|^2 \\ &\geq \frac{1}{2}(\lambda_1 - \lambda)\int_\Omega |u|^2 + \frac{1}{p}\int_\Omega |\nabla u|^p \\ &\geq \frac{1}{2}(\lambda_1 - \lambda)\int_\Omega |u|^2 + d\left(\int_\Omega |u|^2\right)^{\frac{p}{2}}\end{aligned}$$

for some $d > 0$. Thus, $\Phi_\lambda(.)$ is bounded below. Also, for an eigenfunction u_1 corresponding to λ_1 and $t > 0$ sufficiently small,

$$\Phi_\lambda(tu_1) = \frac{t^2}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\int_\Omega (\Delta u_1)^2 + \frac{t^p}{p}\int_\Omega |\nabla u_1|^p < 0.$$

Since $\Phi_\lambda(.)$ also satisfies the Palais-Smale condition, Ekeland's variational principle provides a solution to (1). □

3. The critical case $p = 2^\#$.

Proof of Theorem 2. (i) Let $p_n \in (s, 2^\#)$, with $p_n \to 2^\#$. Theorem 1 guarantees that there exists $\lambda_n > 0$ such that (1) admits a solution for every $\lambda > \lambda_n$. The Sobolev embedding implies that the sequence $\{\lambda_n\}_{n\in N}$ is bounded. Define $\widehat{\lambda} := \sup_{n\to+\infty} \lambda_n$. Thus, for $\lambda > \widehat{\lambda}$, there exists $u_n \in E$ such that

$$\frac{1}{2}\int_\Omega (\Delta u_n)^2 + \frac{1}{p_n}\int_\Omega |\nabla u_n|^{p_n} = \frac{\lambda}{s}\int_\Omega |u_n|^s . \tag{14}$$

The embeddings $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ and $L^{p_n}(\Omega) \hookrightarrow L^2(\Omega)$ imply that

$$\|u\|_{L^s(\Omega)} \le c\|\nabla u\|_{L^2(\Omega)} \text{ and } \|\nabla u\|_{L^2(\Omega)} \le c_n\|\nabla u\|_{L^{p_n}(\Omega)},$$

where $\{c_n\}_{n\in N}$ is a bounded sequence. Thus,

$$\|u\|_{L^s(\Omega)} \le d\|\nabla u\|_{L^{p_n}(\Omega)} \tag{15}$$

for some $d > 0$. Combining (14) and (15), we see that $\|\nabla u_n\|_{L^{p_n}(\Omega)}$, $n \in N$, is bounded. By (14), we conclude that the sequence $\{\|u_n\|_E\}_{n\in N}$ is bounded. By passing to a subsequence, if necessary, we may assume that $u_n \to u$ weakly in E. Thus, for $\psi \in C_0^\infty(\Omega)$ and $\lambda > \widehat{\lambda}$, we have

$$\int_\Omega \Delta u_n \Delta\psi + \int_\Omega |\nabla u_n|^{p_n-2}\nabla u_n \nabla\psi = \lambda\int_\Omega |u_n|^{s-2} u_n\psi$$

for every $n \in N$. It is clear that

$$\int_\Omega \Delta u_n\Delta\psi \to \int_\Omega \Delta u\Delta\psi,$$

$$\int_\Omega |u_n|^{s-2} u_n\psi \to \int_\Omega |u|^{s-2} u\psi,$$

while Theorem IV.9 in [**1**] yields

$$\int_\Omega |\nabla u_n|^{p_n-2}\nabla u_n\nabla\psi \to \int_\Omega |\nabla u|^{2^\#-2}\nabla u\nabla\psi.$$

Thus,

$$\int_\Omega \Delta u\Delta\psi + \int_\Omega |\nabla u|^{2^\#-2}\nabla u\nabla\psi = \lambda\int_\Omega |u|^{s-2} u\psi,$$

that is, u is a solution to (1), with $p = 2^\#$. We show that $u \ne 0$. Indeed, if we assume that $u_n \to 0$ in E, then for the sequence $v_n := \frac{u_n}{\|u_n\|_E}$, $n \in N$, we would have

$$1 = \int_\Omega (\Delta v_n)^2 = \lambda\|u_n\|_E^{s-2}\int_\Omega |v_n|^s - \|u_n\|_E^{p_n-2}\int_\Omega |\nabla v_n|^{p_n} \to 0,$$

a contradiction.
(ii) Assume that E is supplied with the norm

$$|||u||| = \left(\int_\Omega (\Delta u)^2\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla u|^{2^\#}\right)^{\frac{1}{2^\#}}.$$

We show that $\Phi_\lambda(.)$ satisfies the Palais-Smale condition. Let $\{u_n\}_{n\in N}$ be a Palais-Smale sequence. Working as in Lemma 5, we see that $\{u_n\}_{n\in N}$ is bounded in E with respect to the norm $|||.|||$. Therefore, by passing to a subsequence, if necessary, we may assume that $u_n \to u$ weakly in E and $W_0^{1,2^\#}(\Omega)$. Since $\Phi'_\lambda(u_n)(u_n) \to 0$ and $\Phi'_\lambda(u_n)(u) \to 0$, we have

$$\int_\Omega (\Delta u_n)^2 + \int_\Omega |\nabla u_n|^{2^\#} \to \lambda \int_\Omega |u|^s \tag{16}$$

and

$$\int_\Omega (\Delta u_n)(\Delta u) + \int_\Omega |\nabla u_n|^{2^\#-2} \nabla u_n \nabla u \to \lambda \int_\Omega |u|^s. \tag{17}$$

Note that $\nabla u_n \to \nabla u$ in $L^{2^\#-2}(\Omega)$ and $\Delta u_n \to \Delta u$ weakly, so (17) yields

$$\int_\Omega (\Delta u)^2 + \int_\Omega |\nabla u|^{2^\#} = \lambda \int_\Omega |u|^s,$$

and this fact combined with (16) shows that $u_n \to u$ in E and $W^{1,2^\#}(\Omega)$. By (13), $\Phi_\lambda(.)$ is positive near the origin. Since $\lim_{t\to+\infty} \Phi_\lambda(tu) = -\infty$, the mountain-pass theorem provides a solution to (1). □

4. The critical case $s = 2^*$. In this section, we study the nonhomogeneous equation

$$\Delta^2 u - \Delta_p u = \lambda |u|^{2^*-2} u + g \text{ in } \Omega \tag{18}$$

subject to the Dirichlet boundary conditions, where $g : \Omega \to R$ is a nontrivial continuous function and $\lambda > 0$. We follow the approach of Guedda [**11**].

The energy functional associated to (18) is

$$\Psi_\lambda(u) := \frac{1}{2}\int_\Omega (\Delta u)^2 dx + \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{2^*}\int_\Omega |u|^{2^*} dx - \int_\Omega gu. \tag{19}$$

Let $S := \inf\{\|u\|_E^2 : \|u\|^{2^*} = 1\}$ be the best constant in the Sobolev inclusion $H_0^2(\Omega) \subset L^{2^*}(\Omega)$. By Theorem 2.1 in [**8**], S is attained by the functions u_ε given by

$$u_\varepsilon(x) := K_N \left(\frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2}\right)^{\frac{N-4}{2}}, \tag{20}$$

where

$$K_N := [(N-4)(N-2)N(N+2)]^{\frac{N-4}{8}}$$

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for any $\varepsilon > 0$ and $x_0 \in R^N$. Furthermore, the functions u_ε, with $x_0 = 0$, are the only positive spherically symmetric solutions of the equation

$$\Delta^2 u = u^{\frac{N+4}{N-4}} \text{ in } R^N,$$

which are decreasing in $|x|$.

LEMMA 6. *Suppose that* $1 < p < 2^\#$. *Then* $\Psi_\lambda(.)$ *satisfies a local Palais-Smale condition in the strip* $\left(-\infty, \frac{2\lambda}{N}\left(\frac{S}{\lambda}\right)^{\frac{N}{4}} - K\right)$, *where*

$$K := \frac{(2^* - 1)(2^\# - 1)^\eta \|g\|_\eta^\eta}{\lambda^{\eta-1}(2^* - 2^\#)^{\eta-1} 2^* 2^\#} \text{ and } \eta := \frac{2N}{N+4}. \tag{21}$$

Proof. Assume that $\lim_{n\to+\infty} \Psi_\lambda(u_n) = \alpha < \frac{2\lambda}{N}\left(\frac{S}{\lambda}\right)^{\frac{N}{4}} - K$ and $\Psi'_\lambda(u_n) \to 0$ in E^*. Then,

$$\frac{1}{2}\int_\Omega (\Delta u_n)^2 + \frac{1}{p}\int_\Omega |\nabla u_n|^p - \frac{\lambda}{2^*}\int_\Omega |u_n|^{2^*} - \int_\Omega g u_n = \alpha + o_n(1) \tag{22}$$

and

$$\int_\Omega (\Delta u_n)^2 + \int_\Omega |\nabla u_n|^p - \lambda \int_\Omega |u_n|^{2^*} - \int_\Omega g u_n = o_n(1)\|u_n\|_E. \tag{23}$$

Combining (22) and (23), we get

$$\left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega (\Delta u_n)^2 + \left(\frac{1}{p} - \frac{1}{2^*}\right)\int_\Omega |\nabla u_n|^p - \left(1 - \frac{1}{p}\right)\int_\Omega g u_n$$
$$= \alpha + o_n(1) + o_n(1)\|u_n\|_E,$$

which implies that $\{u_n\}_{n\in N}$ is bounded in E. By passing to a subsequence, if necessary, we have that $u_n \to u$ weakly in E. In view of the Sobolev embedding and the concentration-compactness principle **[13]**,

$$\left.\begin{array}{c} u_n \to u \text{ in } L^2(\Omega) \text{ and a.e. in } \overline{\Omega}, \\ \nabla u_n \to \nabla u \text{ in } L^q(\Omega)^N,\ 1 < q < 2^\#, \text{ and a.e. in } \overline{\Omega}, \\ |u_n|^{2^*} \to \nu = |u|^{2^*} + \sum_{j\in J} \nu_j \delta_{x_j} \text{ in the w}^*\text{- sense}, \\ (\Delta u_n)^2 \to \mu \geq (\Delta u)^2 + \sum_{j\in J} \mu_j \delta_{x_j} \text{ in the w}^*\text{- sense}, \\ S\nu_j^{\frac{2}{2^*}} \leq \mu_j, \end{array}\right\} \tag{24}$$

where J is a finite set and $x_j \in \overline{\Omega}$. We show that $\nu_j = \mu_j = 0$ for every $j \in J$. For a fixed $j \in J$ and $\varepsilon > 0$, let $\varphi \in C_0^\infty(R^N)$ such that

$$\left.\begin{array}{c} 0 \leq \varphi \leq 1,\ \varphi = 1 \text{ on } B(x_j, \varepsilon),\ \varphi = 0 \text{ on } R^N \backslash B(x_j, 2\varepsilon), \\ |\nabla\varphi| \leq \frac{2}{\varepsilon} \text{ and } |\Delta\varphi| \leq \frac{2}{\varepsilon^2}. \end{array}\right\} \tag{25}$$

By hypothesis,

$$\Psi'_\lambda(u_n)(u_n \varphi \chi_{\overline{\Omega}}) \to 0 \text{ as } n \to \infty,$$

that is,

$$\int_{B(x_j,2\varepsilon)\cap\Omega} (\Delta u_n)\Delta(u_n\varphi) + \int_{B(x_j,2\varepsilon)\cap\Omega} |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n\varphi)$$
$$- \int_{B(x_j,2\varepsilon)\cap\Omega} g u_n \varphi - \lambda \int_{B(x_j,2\varepsilon)\cap\Omega} |u_n|^{2^*}\varphi \to 0.$$

In view of (24) and (25),

$$\begin{aligned}&\int_{B(x_j,2\varepsilon)\cap\Omega}(\Delta u_n)\Delta(u_n\varphi)+\int_{B(x_j,2\varepsilon)\cap\Omega}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n\varphi)\\&\quad-\int_{B(x_j,2\varepsilon)\cap\Omega}gu_n\varphi\to\lambda\int_{B(x_j,2\varepsilon)\cap\Omega}\varphi\,d\nu,\end{aligned}\tag{26}$$

as $n\to+\infty$. Since

$$\begin{aligned}&\int_{B(x_j,2\varepsilon)\cap\Omega}|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n\varphi)\\&\quad=\int_{B(x_j,2\varepsilon)\cap\Omega}|\nabla u_n|^{p}\,\varphi+\int_{B(x_j,2\varepsilon)\cap\Omega}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi u_n\\&\quad\to\int_{B(x_j,2\varepsilon)\cap\Omega}|\nabla u|^{p}\,\varphi+\int_{B(x_j,2\varepsilon)\cap\Omega}|\nabla u|^{p-2}\nabla u\nabla\varphi u,\end{aligned}$$

(26) becomes

$$\begin{aligned}&\lim_{n\to\infty}\int_{B(x_j,2\varepsilon)\cap\Omega}(\Delta u_n)\,\Delta\,(u_n\varphi)\\&\quad=\int_{B(x_j,2\varepsilon)\cap\Omega}\varphi d\nu-\int_{B(x_j,2\varepsilon)\cap\Omega}|\nabla u|^{p}\,\varphi-\int_{B(x_j,2\varepsilon)\cap\Omega}|\nabla u|^{p-2}\nabla u\nabla\varphi u\\&\qquad-\int_{B(x_j,2\varepsilon)\cap\Omega}gu_n\varphi\to\lambda\nu_j,\end{aligned}\tag{27}$$

as $\varepsilon\to 0$. Also,

$$\begin{aligned}&\int_{B(x_j,2\varepsilon)\cap\Omega}(\Delta u_n)(\Delta u_n\varphi)=\int_{B(x_j,2\varepsilon)\cap\Omega}(\Delta u_n)^2\varphi\\&\quad+\int_{B(x_j,2\varepsilon)\cap\Omega}(\Delta u_n)(\Delta\varphi)u_n+2\int_{B(x_j,2\varepsilon)\cap\Omega}(\Delta u_n)(\nabla u_n\nabla\varphi).\end{aligned}\tag{28}$$

But

$$\lim_{n\to+\infty}\int_{B(x_j,2\varepsilon)\cap\Omega}(\Delta u_n)^2\varphi\to\int_{B(x_j,2\varepsilon)\cap\Omega}\varphi\,d\mu\geq\mu_j,\tag{29}$$

as $\varepsilon\to 0$,

$$\begin{aligned}&\lim_{n\to+\infty}\left|\int_{B(x_j,2\varepsilon)\cap\Omega}(\Delta u_n)(\Delta\varphi)u_n\right|\\&\quad\leq\lim_{n\to+\infty}\left[\left(\int_{B(x_j,2\varepsilon)\cap\Omega}|\Delta u_n|^2\right)^{\frac12}\left(\int_{B(x_j,2\varepsilon)\cap\Omega}|\Delta\varphi|^2|u_n|^2\right)^{\frac12}\right]\end{aligned}$$

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$$\begin{aligned}
&\le c_1 \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\Delta\varphi|^2 |u|^2 \right)^{\frac{1}{2}} \\
&\le c_1 \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\Delta\varphi|^{\frac{N}{2}} \right)^{\frac{2}{N}} \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\Delta\varphi|^2 |u|^2 \right)^{\frac{1}{2}} \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |u|^{2^*} \right)^{\frac{1}{2^*}} \\
&\le c_2 \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |u|^{2^*} \right)^{\frac{1}{2^*}} \to 0, \qquad (30)
\end{aligned}$$

as $\varepsilon \to 0$, and

$$\begin{aligned}
&\lim_{n\to+\infty} \left| \int_{B(x_j,2\varepsilon)\cap\Omega} (\Delta u_n)(\nabla u_n \nabla\varphi) \right| \\
&\quad \le \lim_{n\to+\infty} \left[\left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\Delta u_n|^2 \right)^{\frac{1}{2}} \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\nabla\varphi|^2 |\nabla u_n|^2 \right)^{\frac{1}{2}} \right] \\
&\quad \le c_3 \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\nabla\varphi|^2 |\nabla u|^2 \right)^{\frac{1}{2}} \\
&\quad \le c_3 \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\nabla\varphi|^N \right)^{\frac{1}{N}} \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\nabla u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \\
&\quad \le c_4 \left(\int_{B(x_j,2\varepsilon)\cap\Omega} |\nabla u|^{\frac{2N}{N-2}} \right)^{\frac{N-2}{2N}} \to 0, \qquad (31)
\end{aligned}$$

as $\varepsilon \to 0$. Combining (27)–(31), we obtain $\mu_j \le \lambda\nu_j$. By (24), $S\nu_j^{2/2^*} \le \lambda\nu_j$, which implies that either $\nu_j = 0$ or $\nu_j \ge \left(\frac{S}{\lambda}\right)^{N/4}$. If we assume that $\nu_j \ge \left(\frac{S}{\lambda}\right)^{N/4}$, then

$$\begin{aligned}
\alpha &= \lim_{n\to+\infty} \left[\Psi_\lambda(u_n) - \frac{1}{2^\#} \Psi'_\lambda(u_n) u_n \right] \\
&= \lim_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{2^\#} \right) \int_\Omega (\Delta u_n)^2 + \left(\frac{1}{p} - \frac{1}{2^\#} \right) \int_\Omega |\nabla u_n|^p + \lambda \left(\frac{1}{2^\#} - \frac{1}{2^*} \right) \right. \\
&\qquad \left. \times \int_\Omega |u_n|^{2^*} \right] - \left(1 - \frac{1}{2^\#} \right) \int_\Omega gu \\
&\ge \left(\frac{1}{2} - \frac{1}{2^\#} \right) \int_\Omega (\Delta u)^2 + \left(\frac{1}{2} - \frac{1}{2^\#} \right) \mu_j + \lambda \left(\frac{1}{2^\#} - \frac{1}{2^*} \right) \int_\Omega |u|^{2^*}
\end{aligned}$$

$$+\lambda\left(\frac{1}{2^{\#}}-\frac{1}{2^{*}}\right)\nu_j-\left(1-\frac{1}{2^{\#}}\right)||g||_\eta\left(\int_\Omega|u|^{2^*}\right)^{\frac{1}{2^*}}$$

$$\geq\frac{2\lambda}{N}\left(\frac{S}{\lambda}\right)^{\frac{N}{4}}+\lambda\left(\frac{1}{2^{\#}}-\frac{1}{2^{*}}\right)\int_\Omega|u|^{2^*}-\left(1-\frac{1}{2^{\#}}\right)||g||_\eta\left(\int_\Omega|u|^{2^*}\right)^{\frac{1}{2^*}}.$$

Let $z(x):=\lambda(\frac{1}{2^{\#}}-\frac{1}{2^*})x-(1-\frac{1}{2^{\#}})||g||_\eta x^{1/2^*}$. Since the minimum value of $z(x)$ for positive x is $-K$, we get a contradiction. Thus, $\nu_j=0$ for every $j\in J$. Consequently, $u_n\to u$ in $L^{2^*}(\Omega)$. Exploiting the complete continuity of the inverse biharmonic operator, we can now show that $u_n\to u$ in E. □

Working as in Lemma 3.1 in [**11**], we have

LEMMA 7. *There exist constants $r,\delta>0$ such that if $\|g\|_\eta<\delta$, then $\Psi_\lambda(u)>0$ for all $\|u\|_E=r$.*

Proof. By the Hölder and the Sobolev inequalities, we have that

$$\Psi_\lambda(u)\geq\frac{1}{2}\int_\Omega(\Delta u)^2\,dx-\frac{\lambda}{2^*}\int_\Omega|u|^{2^*}dx-\|g\|_\eta\|u\|_{2^*}$$
$$\geq\frac{1}{2}\int_\Omega(\Delta u)^2\,dx-\frac{\lambda}{2^*S^{2^*/2}}\left(\int_\Omega(\Delta u)^2\,dx\right)^{\frac{2^*}{2}}-\|g\|_\eta S^{\frac{1}{2}}\left(\int_\Omega(\Delta u)^2\,dx\right)^{\frac{1}{2}}.$$

Define $k(x):=\frac{1}{2}x^2-\frac{\lambda}{2^*}S^{-2^*/2}x^{2^*}-\|g\|_\eta S^{1/2}x$, $x>0$. It is easy to see that there exists $\delta>0$ such that if $0<\|g\|_\eta<\delta$, then $k(.)$ has a positive maximum attained at a point $r=r(\|g\|_\eta)>0$. Consequently, $\Psi_\lambda(u)>0$ for every $u\in E$, with $\|u\|_E=r$. □

Proof of Theorem 3. Without loss of generality, we may assume that $0\in\Omega$ and $g(0)>0$. By taking $\varepsilon>0$ small enough, we have that

$$\int_\Omega gu_\varepsilon>0,$$

where u_ε is defined by (20) with $x_0=0$. Equation (19) implies that $\Psi_\lambda(tu_\varepsilon)<0$ for small $t>0$. Thus,

$$\inf\{\Psi_\lambda(u):\|u\|_E\leq r\}<0.$$

We now choose g so that $0<||g||_\eta<\delta$ and $\frac{2\lambda}{N}\left(\frac{S}{\lambda}\right)^{N/4}-K\geq 0$ (see (21)). An application of the Ekeland variational principle provides a solution to (18).

5. Bifurcation from the principal eigenvalue. Let $\varepsilon>0$ and $\gamma\in(0,1)$. We say that Ω is ε-close in $C^{4,\gamma}$-sense to the unit ball $B(0,1)$ if there exists a surjective mapping $g\in C^{4,\gamma}(\overline{B}(0,1),\overline{\Omega})$ such that

$$\|g-Id\|_{C^{4,\gamma}(\overline{B}(0,1),\overline{\Omega})}\leq\varepsilon.$$

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THEOREM 8. *There is* $\varepsilon_{2,N} > 0$ *such that if* Ω *is* ε*-close in the* $C^{4,\gamma}$*-sense to* $B(0,1)$*, with* $\varepsilon < \varepsilon_{2,N}$*, then the eigenfunction* $\varphi_{1,\Omega}(.)$ *for the first eigenvalue* λ_1 *of*

$$\left.\begin{array}{c}\Delta^2\varphi = \lambda\varphi \text{ in } \Omega,\\ u = 0,\ \nabla u = 0 \text{ on } \partial\Omega\end{array}\right\}$$

is unique up to normalization and there exists $c > 0$ *such that* $\varphi_{1,\Omega}(x) \geq cd(x, \partial\Omega)^2$.

For more details, we refer to [**10**].

We assume that our perturbation term h satisfies the following:
(h) $h : \overline{\Omega} \times [\lambda_1 - d, \lambda_1 + d] \to R$ is continuous with $h_\infty = \sup\{|h(x,\lambda)| : (x,\lambda) \in \overline{\Omega} \times [\lambda_1 - d, \lambda_1 + d]\}$ and

$$\int_\Omega h(x,\lambda_1)\varphi_{1,\Omega}^{2^*}(x)dx \neq 0.$$

DEFINITION 9. Let $\mathcal{F} : X \to X^*$ be an operator on the real reflexive Banach space X. The operator $\mathcal{F}$ is said to satisfy the local (S^+) property on the set $G \subseteq X$ if any sequence $\{x_n\}_{n\in N}$ in G with $x_n \to x$ weakly in X and $limsup_{n\to+\infty} \langle \mathcal{F}(x_n), x_n - x\rangle \leq 0$ satisfies $x_n \to x$ strongly in X.

We define the operators $J, S, H_\lambda : E \to R$ with the use of the duality pairing in E :

$$(J(u), v) = \int_\Omega \Delta u \Delta v,$$

$$(S(u), v) = \int_\Omega uv$$

and

$$(H_\lambda(u), v) = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega h(x,\lambda)|u|^{2^*-2}uv.$$

It is clear that $u \in E$ is a (weak) solution to (2) if and only if u solves the operator equation:

$$N_\lambda(u) := J(u) + \lambda S(u) - H_\lambda(u) = 0.$$

LEMMA 10. *Suppose that* $\rho_0 < \min\{1, h_\infty^{-(N-4)/8} S^{N/8}\}$*. Then,* $N_\lambda(.)$ *satisfies the local* (S^+) *property in* $B(0,\rho_0)$.

Proof. Let $\{u_n\}_{n\in N}$ be a sequence in $B(0,\rho_0)$. By passing to a subsequence, if necessary, we may assume that $u_n \to u_0$ weakly in E. Furthermore, let

$$\limsup_{n\to+\infty} N_\lambda(u_n)(u_n - u_0) \leq 0,$$

that is,

$$\limsup_{n\to+\infty}\left\{\int_\Omega \Delta u_n \Delta(u_n - u_0) + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_0) - \lambda\int_\Omega u_n(u_n - u_0) - \int_\Omega h(x,\lambda)|u_n|^{2^*-2}u_n(u_n - u_0)\right\} \leq 0. \tag{32}$$

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Note that, by (24),

$$\int_\Omega \Delta u_n \Delta u_0 \to \int_\Omega (\Delta u_0)^2, \tag{33}$$

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_0) \to 0, \tag{34}$$

$$\int_\Omega u_n(u_n - u_0) \to 0 \tag{35}$$

and

$$\int_\Omega h(x,\lambda)|u_n|^{2^*} \to \int_\Omega h(x,\lambda)|u_0|^{2^*} + \int_\Omega h(x,\lambda) d\,\widetilde{\nu}, \tag{36}$$

where $\widetilde{\nu} = \sum_{j\in J} \nu_j \delta_{x_j}$. Since the sequence $\{|u_n|^{2^*-2}u_n\}_{n\in N}$ is bounded in $(L^{2^*}(\Omega))'$, we have that, up to a subsequence, $|u_n|^{2^*-2}u_n \to |u_0|^{2^*-2}u_0$ weakly in $(L^{2^*}(\Omega))'$. Thus,

$$\int_\Omega h(x,\lambda)|u_n|^{2^*-2}u_n u_0 \to \int_\Omega h(x,\lambda)|u_0|^{2^*}. \tag{37}$$

In view of hypothesis (h), (24) and (33)–(37), (32) yields

$$\widetilde{\mu}(\overline{\Omega}) \le h_\infty \widetilde{\nu}(\overline{\Omega}),$$

where $\widetilde{\mu} = \sum_{j\in J} \mu_j \delta_{x_j}$, and by exploiting (24) again, we get

$$\widetilde{\mu}(\overline{\Omega}) \le h_\infty S^{-\frac{2^*}{2}} \widetilde{\mu}(\overline{\Omega})^{\frac{2^*}{2}}.$$

Consequently, $\widetilde{\mu}(\overline{\Omega}) = 0$ or $h_\infty^{-(N-4)/4} S^{N/4} \le \widetilde{\mu}(\overline{\Omega})$. If $h_\infty^{-(N-4)/4} S^{N/4} \le \widetilde{\mu}(\overline{\Omega})$, then, since $||u_n||_E < \rho_0$, we should have $\widetilde{\mu}(\overline{\Omega}) < \rho_0^2 < h_\infty^{-(N-4)/4} S^{N/4}$, a contradiction. Consequently, $\widetilde{\mu} = 0$. In view of (24) and the strict convexity of E, we get that $u_n \to u$ in E. □

In view of Lemma 10 and Theorem 1.6 in [**7**], the degree $Deg(N_\lambda, D, 0)$ is well defined for all open, bounded and nonempty sets $D \subset B(0, \rho_0)$ whenever $0 \notin N_\lambda(\partial D)$. Define

$$\widetilde{N}_\lambda(u) := J(u) + \lambda S(u).$$

The degree $Deg(\widetilde{N}_\lambda, B(0,\rho), 0)$, for any $0 < \rho < \rho_0$, is also well defined for $\lambda \in (\lambda_1 - d, \lambda_1 + d)$, $\lambda \neq \lambda_1$,

$$Deg(\widetilde{N}_\lambda, B(0,\rho), 0) = 1,\ \ \lambda \in (\lambda_1 - d, 0)$$

and

$$Deg(\widetilde{N}_\lambda, B(0,\rho), 0) = -1,\ \ \lambda \in (0, \lambda_1 + d).$$

For more details, we refer to [**3**, **7**].

The proof of the following lemma follows as an easy combination of Hölder's inequality with the Sobolev embeddings and it is omitted.

LEMMA 11. *The operator* $H_\lambda(.)$ *satisfies*

$$\lim_{\|u\|_E \to 0} \frac{\|H_\lambda(u)\|_{E^*}}{\|u\|_E} = 0$$

uniformly for λ *in a bounded subset of* R.

By exploiting the previous lemma and the homotopy invariance property of the degree, we get that for every $\lambda \in (\lambda_1 - d, \lambda_1 + d)$, $\lambda \neq \lambda_1$, there exists $\rho > 0$ such that

$$Deg(N_\lambda, B(0, \rho), 0) = 1, \ \lambda \in (\lambda_1 - d, 0)$$

and

$$Deg(N_\lambda, B(0, \rho), 0) = -1, \ \lambda \in (0, \lambda_1 + d).$$

Note that the index of the isolated zero of N_λ changes by magnitude 2 when λ crosses λ_1, so working as in Theorem 1.3 and Corollary 1.12 in **[14]** we get

THEOREM 12. *Equation* (2) *admits a continuum* C *of nontrivial solutions* $(\lambda, u) \subseteq R \times E$ *bifurcating from* $(\lambda_1, 0)$*, which meets the boundary of* $[\lambda_1 - d, \lambda_1 + d] \times B(0, \rho_0)$.

REMARK 13. Most of the above results can be extended to the case of the equation $\Delta^2 u + \Delta_p u = \lambda |u|^{s-2} u$ with Dirichlet boundary conditions.

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