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SYSTEMS OF BRIOT-BOUQUET EQUATIONS WITH ANALYTIC SOLUTIONS

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ABSTRACT. In this note we use functional analysis arguments to prove the existence of families of analytic solutions for the singular system of complex ordinary differential equations $zW' = h(z, W)$.

Introduction. We consider the singular system of complex ordinary differential equations $zW' = h(z, W)$ where h is a holomorphic complex-valued vector function of (z, W), the z-space has one complex dimension and the W-space is complex n-dimensional. We prove the existence of families of analytic solutions indexed by the independent eigenvectors of the matrix $h_W(0, 0)$ corresponding to the integer eigenvalue m.

Such systems were first studied by C. C. A. Briot and J. C. Bouquet [1] for the one-dimensional case. One can show that in this case the equation has a one-parameter family of analytic solutions [6, p. 116]. Further results appear in [4] where the approach taken has been to obtain formal power series solutions and then prove convergence. An alternative approach was taken in [3] using a contraction mapping to obtain the existence of families of analytic solutions in the n-dimensional case depending on solutions of an arbitrary polynomial equation. We obtain similar, though more explicit results using a direct functional analysis method.

Main Results. Consider the system of Briot-Bouquet equations

$$(1) \qquad zW' = h(z, W), \qquad h(0, 0) = 0 \qquad \left(' = \frac{d}{dz}\right)$$

where z is a complex variable, W an n-vector and $h(z, W)$ is a map of $\mathbb{C}^n \to \mathbb{C}^n$ whose components are holomorphic functions of (z, W). We wish to find a nontrivial solution of Eq. (1) such that each of its components is an analytic function.

To this end we let $h_W(z, W)$ denote the Jacobian matrix of $h(z, W)$ with respect to W. Suppose that $h_W(0, 0)$ has a positive integer eigenvalue and let m

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be the largest positive integer eigenvalue of $h_W(0, 0)$. We shall also assume that

$$h(z, 0) = o(|z|^m).$$

With the above assumptions we can show

THEOREM A. *There exists a neighborhood N of 0 in $\mathbb{C}^n$ such that for each eigenvector $C \in N$ of $h_W(0, 0)$ corresponding to m there is an analytic solution*

$$W(z) = Cz^m + \sum_{i=m+1}^{\infty} C_i(C)z^i, \qquad 0 \le |z| \le 1$$

of Eq. (1).

Proof. Let $A = h_W(0, 0)$ and $H(z, W) = h(z, W) - AW$. Then let B_m be the space of functions which are analytic in the circle $|z| < 1$, continuous on $|z| \le 1$ and have power series expansions of the form $\sum_{i=m+1}^{\infty} C_i z^i$. We will define $B_m^{(1)}$ to be the subspace of B_m consisting of those functions whose derivatives are also continuous on $|z| \le 1$. Then set

$$\mathcal{B}_m = \{(f_1(z), \ldots, f_n(z)) \mid f_i \in B_m, i = 1, \ldots, n\}$$

and

$$\mathcal{B}_m^{(1)} = \{(f_1(z), \ldots, f_n(z)) \mid f_i \in B_m^{(1)}, i = 1, \ldots, n\}$$

with the norms

$$\|f\|_{\mathcal{B}_\mathcal{M}} = \sup_{\substack{|z|\le 1 \\ 1\le i\le n}} |f_i(z)| \quad \text{and} \quad \|f\|_{\mathcal{B}_\mathcal{M}^{(1)}} = \sup_{\substack{|z|\le 1 \\ 1\le i\le n}} |f_i'(z)|.$$

It is readily verified that the operator $T: \mathcal{B}_m^{(1)} \to \mathcal{B}_m$ defined by

$$TW = zW'(z) - AW(z)$$

is a 1-1 map. Moreover

$$H(z, Cz^m + W(z)) : C^n \times \mathcal{B}_m^{(1)} \to \mathcal{B}_m.$$

Thus to solve Eq. (1) it suffices to solve the equation

$$zW'(z) = AW(z) + H(z, Cz^m + W(z))$$

for C an eigenvector of A corresponding to m and $W \in \mathcal{B}_m^{(1)}$.

As $f(z) = \int_0^z f'(w)\, dw$ for $f \in B_m^{(1)}$ it follows from the Arzela-Ascoli theorem that the injection map of $\mathcal{B}_m^{(1)} \to \mathcal{B}_m$ is compact. Moreover it is clear that the operator defined by $zW'(z)$ is a bounded 1-1 map of $\mathcal{B}_m^{(1)}$ onto $\mathcal{B}_m$ and is thus invertible. Hence by the stability theorem for Fredholm operators [5, p. 238] it follows that T is a linear homeomorphism of $\mathcal{B}_m^{(1)}$ onto $\mathcal{B}_m$. Therefore by the implicit function theorem [2, p. 265] for each sufficiently small eigenvector C there exists a unique $W_C(z)$ such that $zW_C'(z) = AW_C(z) + H(z, Cz^m + W_C(z))$. Thus our theorem follows.

When the nonlinear part of $h(z, W)$ is of sufficiently high order one need not assume m is a maximal integer eigenvalue to assure the existence of analytic solutions. Indeed, suppose now that m is a positive integer eigenvalue of $h_W(0, 0)$ and k is the largest positive integer such that km is an eigenvalue of $h_W(0, 0)$. Then we can readily show

THEOREM B. *Let m and k be as above and suppose that*

$$h(z, W) - h_W(0, 0)W = o(|z|^{km} + |W|^{km}).$$

Then there exists a neighborhood N of 0 in $\mathbb{C}^n$ such that for each eigenvector $C \in N$ of $h_W(0, 0)$ corresponding to m there is an analytic solution

$$W(z) = Cz^m + \sum_{i=km+1}^{\infty} C_i(C)z^i \qquad 0 \le |z| \le 1$$

of Eq. (1).

Proof. It is clear that

$$h\left(z, Cz^m + \sum_{i=km+1}^{\infty} C_i z^i\right) - ACz^m = \sum_{i=km+1}^{\infty} B_i z^i.$$

Hence if we replace B_m by B_{km} and repeat the arguments used in proving Theorem *A* our Theorem follows.

If $h(z, W)$ does not depend on z, so that Eq. (1) has the form

$$zW' = h(W), \qquad h(0) = 0 \tag{2}$$

then a stronger result obtains. Suppose that m is a positive integer eigenvalue of $h_W(0)$ and no other positive integer multiple of m is an eigenvalue of $h_W(0)$. Then

THEOREM C. *There exists a neighbourhood N of 0 in C^n such that for each eigenvector $C \in N$ of A corresponding to m there is an analytic solution*

$$W(z) = Cz^m + \sum_{i=2}^{\infty} C_{mi}(C)z^{mi} \qquad 0 \le |z| \le 1.$$

of Eq. (2).

Proof. The proof is almost the same as that of Theorem A. Just take B_m to be the space of analytic functions with power series of the form $\sum_{i=2}^{\infty} C_{mi} z^{mi}$.

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